



A New Analytic Study of Fundamental Filter in Topological Vector Space via Di-Proximit Structure

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ABSTRACT: This article introduces the concept of Di-proximit space and examines several results in this new context. We provide new definitions of the terms Di-proximit space, Di-proximit topological vector space, found relation between tower and Di-proximit. We provide new definitions of the terms cluster point in Di-proximit topological vector space, Di – Cauchy filter, Di-convergent filter in Di-proximit topological vector space and the relationship between Di – Cauchy filter and Di-convergent filter in Di-proximit topological vector space.

Keywords: Approach space, di-proximit space, topological vector space, filter, Di-Cauchy filter.

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1. Introduction

In modern functional analysis, the concept of the proximit space is one of the most important concepts. Recent research has looked at applications in many different areas of mathematics to see how space theory techniques affect quantum field theory. The study of convergence in generalized structures has played a fundamental role in the development of modern topology and functional analysis. In the Kelley [1] Cauchy filter generalize Cauchy sequences and fundamental role in the study of completeness in uniform spaces and topological vector spaces. the classical book by Dugundji [4] provides a systematic and rigorous treatment of general topology, including topological and metric spaces, convergence and continuous mappings. In addition, the contributions of Bourbaki [8] in general topology provided further refinement of convergence concepts and their connection with algebraic and geometric structures. The two do not have to be brought together over a meaningful set of all point distances by this point-based distance. In the Hogbe –Nlend [3] filter generalize convergence beyond sequences and form the basis of approach spaces. Lowen defines approach spaces, which were first introduced in 1989 [5].

Approach spaces can be perceived generally by using Lowen's research [10]. The study is based on the description of convergence developed in Narici and Beckenstein [11] and the metric framework for topological vector spaces presented by Schaefer and Wolff [17]. In this work [12] Lowen adopt this viewpoint of convergence and focus on filter convergence. In this direction, Abbas and Hussein [13] introduced new results in the theory of Normed Approach Space, where the classical normed space structure is generalized through the approach framework. Their work investigates convergence, Cauchy sequences. In another direction, Neamah and Hussein [14,15] introduced t^ω - Normed approach space, presenting notions of t^ω - convergence and t^ω - Cauchy sequences and establishing the concept of t^ω - Banach Spaces. These contributions expanded the analytical tools for studying convergence in generalized metric frameworks. Furthermore, the dissertation of Kadhim and Hussein [7,6] offered New Results of Banach Algebra Via Proximit Structure and Approach Structure, introducing several new algebraic convergence

concepts and demonstrating their applications with in functional analysis. Similarly, Saeed and Hussein [16] developed fundamental results in Normed Approach Spaces Via β - Approach Structure, showing that every β -Normed Approach Structure induces a normed structure but not necessarily vice versa. Their work also provided examples of β -Banach spaces and explored completeness under the β - Approach. The works of Kream and Hussein [9] introduced Random Approach Normed Spaces and Random Approach Spaces, establishing new types of convergence such as δ_R - Cauchy and δ_R - convergent sequences, and linking these structures with classical Banach spaces. Abed and Hussein [2] describes of \mathfrak{S} -proximit space for nonempty sets. The relationship between approach space and \mathfrak{S} -proximit space has been clarified , we also discussed the definitions of Cauchy sequence, cluster point, \mathfrak{S} -convergent sequence,

2. Preliminaries

Definition 2.1. [10] Let \mathfrak{X} be a non-empty set. A function $\mathfrak{S}: \mathfrak{X} \times 2^{\mathfrak{X}} \rightarrow [0, \infty]$ define on \mathfrak{X} is said to be approach distance on \mathfrak{X} if satisfy the following properties:

$$A_1 : \forall n \in \mathfrak{X} : \mathfrak{S}(n, \{n\}) = 0.$$

$$A_2 : \forall n \in \mathfrak{X} : \mathfrak{S}(n, \emptyset) = \infty.$$

$$A_3 : \forall n \in \mathfrak{X} : \forall \mathcal{Q}, \mathcal{V} \in 2^{\mathfrak{X}} : \mathfrak{S}(n, \mathcal{Q} \cup \mathcal{V}) = \min\{\mathfrak{S}(n, \mathcal{Q}), \mathfrak{S}(n, \mathcal{V})\}.$$

$$A_4 : \forall n \in \mathfrak{X} : \forall \mathcal{Q} \in 2^{\mathfrak{X}}, \forall \epsilon \in [0, \infty] : \mathfrak{S}(n, \mathcal{Q}) \leq \mathfrak{S}(n, \mathcal{Q}^\epsilon) + \epsilon. \text{ For each } \epsilon \in [0, \infty] \text{ and } \mathcal{Q}^\epsilon = \{n \in \mathfrak{X} | \mathfrak{S}(n, \mathcal{Q}) \leq \epsilon\}.$$

A pair $(\mathfrak{X}, \mathfrak{S})$ where \mathfrak{S} is a distance is called an approach space and denoted by app-spaces.

Definition 2.2. [10] A function $I_\epsilon: 2^{\mathfrak{X}} \rightarrow 2^{\mathfrak{X}}$ is called a tower on \mathfrak{X} if under the following condition:
For all $\mathcal{Q}, \mathcal{V} \in 2^{\mathfrak{X}}$ and $\epsilon, \delta \in R^+$ we have

$$T_1 : \mathcal{Q} \subset I_\epsilon(\mathcal{Q}).$$

$$T_2 : I_\epsilon(\emptyset) = \emptyset.$$

$$T_3 : I_\epsilon(\mathcal{Q} \cup \mathcal{V}) = I_\epsilon(\mathcal{Q}) \cup I_\epsilon(\mathcal{V}).$$

$$T_4 : I_\epsilon(I_\delta(\mathcal{Q})) \subseteq I_{\epsilon+\delta}(\mathcal{Q}).$$

$$T_5 : I_\epsilon(\mathcal{Q}) = \bigcap_{\epsilon < \delta} I_\delta(\mathcal{Q}).$$

With note that from (T_3) and (T_5) we have for each $\mathcal{Q} \subset \mathcal{V} \subset \mathfrak{X}$ and for each $\alpha \leq \beta \in R^+ : I_\alpha(\mathcal{Q}) \subset I_\beta(\mathcal{V})$.

Definition 2.3. [5] An operator $cl: 2^{\mathfrak{X}} \rightarrow 2^{\mathfrak{X}}$ that determines the qualities of a pre-topology on a set \mathfrak{X} for all $\mathcal{Q}, \mathcal{V} \in 2^{\mathfrak{X}}$ is determined by an operator that determines the properties of an operator on a set

$$C_1 : \mathcal{Q} \subseteq cl(\mathcal{Q}),$$

$$C_2 : cl(\emptyset) = \emptyset,$$

$$C_3 : cl(\mathcal{Q} \cup \mathcal{B}) = cl(\mathcal{Q}) \cup cl(\mathcal{V}).$$

This operator is called a pre-topological closure operator. A set \mathfrak{X} with a pre-topology τ is called a pre-topological space, written as (\mathfrak{X}, τ) where τ is a family of open sets and cl_τ denotes the associated closure operator.

Definition 2.4. [8] Let \mathfrak{X} to be non-empty set, and \mathcal{F} is a collection of subsets of \mathfrak{X} is called a filter on \mathfrak{X} if it possesses the following properties:

1. $\emptyset \notin \mathcal{F}$
2. If $\acute{A}, \acute{E} \in \mathcal{F}$, then $\acute{A} \cap \acute{E} \in \mathcal{F}$
3. If $\acute{A} \in \mathcal{F}$ and $\acute{A} \subseteq \acute{E}$, then $\acute{E} \in \mathcal{F}$

Example 2.5. Let's $\mathfrak{X} = \{1, 2, 3\}$ then

1. $\mathcal{F}_1 = \{\mathfrak{X}, \{1, 2\}\}$ is a filter on 1.
2. $\mathcal{F}_2 = \{\mathfrak{X}, \{1\}, \{1, 2\}, \{1, 3\}\}$ is a filter on 2.
3. $\mathcal{F}_3 = \{\mathfrak{X}, \{1, 2\}, \{2, 3\}\}$ is not filter on 3.

Definition 2.6. [8] In metric space $(\mathfrak{X}, \mathfrak{Q})$, a filter \mathcal{F} is Cauchy filter ,if for each $\epsilon > 0$ there exists $\mathfrak{B} \in \mathcal{F}$, $\mathfrak{Q}(n, m) < \epsilon$. For all $n, m \in \mathfrak{B}$.

3. Di-Proximit Spaces

Definition 3.1. Let \mathfrak{X} be a non – empty set. A function $\hat{J}: 2^{\mathfrak{X}} \times 2^{\mathfrak{X}} \rightarrow [0, \infty]$ define on \mathfrak{X} is said to be Di-functional if for all $\mathcal{Q}, \mathcal{V}, \mathcal{C} \in 2^{\mathfrak{X}}$ meets the following properties:

- \mathcal{P}_1 : If $\hat{J}(\mathcal{Q}, \mathcal{V}) = 0 \leftrightarrow \mathcal{Q} \cap \mathcal{V} \neq \emptyset$.
- \mathcal{P}_2 : If $\mathcal{V} = \emptyset$, then $\hat{J}(\mathcal{Q}, \mathcal{V}) = \infty$.
- \mathcal{P}_3 : $\hat{J}(\mathcal{Q}, \mathcal{V} \cup \mathcal{C}) = \min \left\{ \hat{J}(\mathcal{Q}, \mathcal{V}), \hat{J}(\mathcal{Q}, \mathcal{C}) \right\}$
- \mathcal{P}_4 : If $\mathcal{V} \subseteq \mathcal{C}$, then $\hat{J}(\mathcal{Q}, \mathcal{C}) \leq \hat{J}(\mathcal{Q}, \mathcal{V})$
- \mathcal{P}_5 : $\hat{J}(\mathcal{Q}, \mathcal{V}) \leq \hat{J}(\mathcal{Q}^\epsilon, \mathcal{V}^\gamma) + \epsilon + \gamma$.

The pair (\mathfrak{X}, \hat{J}) is said to be Di-proximit space. For every $\mathcal{Q} \in 2^{\mathfrak{X}}$ and $\epsilon, \gamma \in [0, \infty]$, we write:

$$I_\epsilon(\mathcal{Q}) = \left\{ n \in \mathfrak{X} \mid \hat{J}(\{n\}, \mathcal{Q}) \leq \epsilon \right\}.$$

Theorem 3.2. If $(I_\epsilon)_{\epsilon \in R^+}$ is a tower on \mathfrak{X} , then the function define as:

$$\hat{J}(\mathcal{Q}, \mathcal{V}) = \min\{\inf\{\epsilon \in R^+ : \mathcal{Q} \subseteq I_\epsilon(\mathcal{V})\}, \inf\{\delta \in R^+ : \mathcal{V} \subseteq I_\delta(\mathcal{Q})\}\}$$

for all $\mathcal{Q}, \mathcal{V} \subseteq \mathfrak{X}$ is Di-proximit on \mathfrak{X} .

Proof: $\mathcal{P}_1 \Rightarrow$ Let $\mathcal{Q}, \mathcal{V} \in 2^{\mathfrak{X}}$, if $\hat{J}(\mathcal{Q}, \mathcal{V}) = 0$, then

$$\begin{aligned} \min\{\inf\{\epsilon \in R^+ : \mathcal{Q} \subseteq I_\epsilon(\mathcal{V})\}, \inf\{\delta \in R^+ : \mathcal{V} \subseteq I_\delta(\mathcal{Q})\}\} &= 0. \\ \{\inf\{\epsilon \in R^+ : \mathcal{Q} \subseteq I_\epsilon(\mathcal{V})\}, \inf\{\delta \in R^+ : \mathcal{V} \subseteq I_\delta(\mathcal{Q})\}\} &= 0. \end{aligned}$$

Hence

$$\inf\{\epsilon \in R^+ : \mathcal{Q} \subseteq I_\epsilon(\mathcal{V})\} = 0 \& \inf\{\delta \in R^+ : \mathcal{V} \subseteq I_\delta(\mathcal{Q})\} = 0.$$

Since

$$\mathcal{Q} \subseteq I_\epsilon(\mathcal{Q}) \& \mathcal{V} \subseteq I_\delta(\mathcal{V}).$$

Then

$$\mathcal{Q} \cap \mathcal{V} \neq \emptyset$$

\Leftarrow) Let $\mathcal{Q}, \mathcal{V} \in 2^{\mathfrak{X}}$, $\mathcal{Q} \cap \mathcal{V} \neq \emptyset$, $n \in \mathcal{Q} \cap \mathcal{V}$, then $n \in \mathcal{Q}$ and $n \in \mathcal{V}$.

\mathcal{P}_2 : Let $\mathcal{Q}, \mathcal{V} \in 2^{\mathfrak{X}}$, if $\mathcal{V} = \emptyset$, then

$$\begin{aligned} \hat{J}(\mathcal{Q}, \mathcal{V}) &= \min\{\inf\{\epsilon \in R^+ : \mathcal{Q} \subseteq I_\epsilon(\mathcal{V})\}, \inf\{\delta \in R^+ : \mathcal{V} \subseteq I_\delta(\mathcal{Q})\}\} \\ &= \min\{\inf\{\epsilon \in R^+ : \mathcal{Q} \subseteq I_\epsilon(\emptyset)\}, \inf\{\delta \in R^+ : \emptyset \subseteq I_\delta(\mathcal{Q})\}\} = \infty. \end{aligned}$$

\mathcal{P}_3 : Let $\mathcal{Q}, \mathcal{V}, \mathcal{C} \in 2^{\mathfrak{X}}$. Then

$$\begin{aligned} \hat{J}(\mathcal{Q}, \mathcal{V} \cup \mathcal{C}) &= \min\{\inf\{\epsilon \in \mathbb{R}^+ : \mathcal{Q} \subseteq \mathbf{I}_\epsilon(\mathcal{V} \cup \mathcal{C})\}, \inf\{\delta \in \mathbb{R}^+ : \mathcal{V} \cup \mathcal{C} \subseteq \mathbf{I}_\delta(\mathcal{Q})\}\} \\ &= \min\{\inf\{\epsilon \in \mathbb{R}^+ : \mathcal{Q} \subseteq (\mathbf{I}_\epsilon(\mathcal{V}) \cup \mathbf{I}_\epsilon(\mathcal{C}))\}, \inf\{\{\delta \in \mathbb{R}^+ : \mathcal{V} \subseteq \mathbf{I}_\delta(\mathcal{Q}) \vee \mathcal{C} \subseteq \mathbf{I}_\delta(\mathcal{Q})\}\}\} \\ &= \min\{\min\{\inf\{\epsilon \in \mathbb{R}^+ : \mathcal{Q} \subseteq \mathbf{I}_\epsilon(\mathcal{V})\}, \inf\{\delta \in \mathbb{R}^+ : \mathcal{V} \subseteq \mathbf{I}_\delta(\mathcal{Q})\}\}, \\ &\quad \min\{\inf\{\epsilon \in \mathbb{R}^+ : \mathcal{Q} \subseteq \mathbf{I}_\epsilon(\mathcal{C})\}, \inf\{\delta \in \mathbb{R}^+ : \mathcal{C} \subseteq \mathbf{I}_\delta(\mathcal{Q})\}\}\} \\ &= \min\{\hat{J}(\mathcal{Q}, \mathcal{V}), \hat{J}(\mathcal{Q}, \mathcal{C})\}. \end{aligned}$$

\mathcal{P}_4 : If $\mathcal{V} \subseteq \mathcal{C}$, by condition (\mathcal{P}_3) then

$$\begin{aligned} \mathcal{V} \cup \mathcal{C} &= \mathcal{C}, \hat{J}(\mathcal{Q}, \mathcal{C}) = \hat{J}(\mathcal{Q}, \mathcal{V} \cup \mathcal{C}) = \min\{\hat{J}(\mathcal{Q}, \mathcal{V}), \hat{J}(\mathcal{Q}, \mathcal{C})\} \\ &\leq \hat{J}(\mathcal{Q}, \mathcal{V}) \end{aligned}$$

\mathcal{P}_5 : Let $\mathcal{Q}, \mathcal{V} \in 2^{\mathfrak{X}}$ and $\epsilon, \alpha \in [0, \infty]$. Then

$$\begin{aligned} \hat{J}(\mathcal{Q}, \mathcal{V}) &= \min\{\inf\{\epsilon \in \mathbb{R}^+ : \mathcal{Q} \subseteq \mathbf{I}_\epsilon(\mathcal{V})\}, \inf\{\delta \in \mathbb{R}^+ : \mathcal{V} \subseteq \mathbf{I}_\delta(\mathcal{Q})\}\} \\ &\leq \min\{\inf\{\epsilon \in \mathbb{R}^+ : \mathcal{Q}^\alpha \subseteq \mathbf{I}_\epsilon(\mathcal{V}^\epsilon)\}, \inf\{\delta \in \mathbb{R}^+ : \mathcal{V}^\epsilon \subseteq \mathbf{I}_\delta(\mathcal{Q}^\alpha)\}\} + \epsilon + \alpha \\ &\leq \hat{J}(\mathcal{Q}^\alpha, \mathcal{V}^\epsilon) + \epsilon + \alpha. \end{aligned}$$

□

Theorem 3.3. Let $\hat{J}: 2^{\mathfrak{X}} \times 2^{\mathfrak{X}} \rightarrow [0, \infty]$ is an Di-proximit space on \mathfrak{X} and for all $\epsilon, \epsilon_1, \epsilon_2 \in \mathbb{R}^+$. Then the family \mathbf{I}_ϵ defined by $\mathbf{I}_\epsilon(\mathcal{Q}) = \min_{\epsilon_1 < \epsilon < \epsilon_2} \{\mathcal{Q}^{\epsilon_1}, \mathcal{Q}^{\epsilon_2}\}$ is a tower on \mathfrak{X} .

Proof: T_1 : Let $\mathcal{Q} \in 2^{\mathfrak{X}}$. Suppose that $\mathfrak{r} \in \mathcal{Q}$, then $\mathfrak{r} \cap \mathcal{Q} \neq \emptyset$ and $\hat{J}(\{\mathfrak{r}\}, \mathcal{Q}) = 0 < \epsilon$. Since $\mathcal{Q} \subseteq \mathcal{Q}^{\epsilon_1}$ and $\mathcal{Q} \subseteq \mathcal{Q}^{\epsilon_2}$, then $\mathcal{Q} \subseteq \min\{\mathcal{Q}^{\epsilon_1}, \mathcal{Q}^{\epsilon_2}\}$ like that $\mathbf{I}_\epsilon(\mathcal{Q}) = \mathcal{Q}^\epsilon$. We have $\mathcal{Q} \subseteq \mathbf{I}_\epsilon(\mathcal{Q})$.

T_2 : Let $\epsilon \in \mathbb{R}^+, \mathcal{Q} \in 2^{\mathfrak{X}}$. If $\mathcal{Q} = \emptyset$, we get

$$\mathbf{I}_\epsilon(\mathcal{Q}) = \mathcal{Q}^\epsilon = \{\mathfrak{r} \in \mathfrak{X} : \hat{J}(\{\mathfrak{r}\}, \mathcal{Q}) \leq \epsilon\}.$$

But $\mathcal{Q} = \emptyset$, then $\hat{J}(\{\mathfrak{r}\}, \emptyset) = \infty$. Hence $\mathbf{I}_\epsilon(\emptyset) = \emptyset$.

T_3 : Let $\mathcal{Q}, \mathcal{V} \in 2^{\mathfrak{X}}$, let $\epsilon \in \mathbb{R}^+$.

Let $\mathfrak{r} \in \mathbf{I}_\epsilon(\mathcal{Q} \cup \mathcal{V})$, then $\mathfrak{r} \in (\mathcal{Q} \cup \mathcal{V})^\epsilon$. So that

$$\begin{aligned} (\mathcal{Q} \cup \mathcal{V})^\epsilon &= \{\mathfrak{r} \in \mathfrak{X} : \hat{J}(\{\mathfrak{r}\}, \mathcal{Q} \cup \mathcal{V}) \leq \epsilon\} \\ &= \{\mathfrak{r} \in \mathfrak{X} : \min\{\hat{J}(\{\mathfrak{r}\}, \mathcal{Q}), \hat{J}(\{\mathfrak{r}\}, \mathcal{V})\} \leq \epsilon\} \\ &= \{\mathfrak{r} \in \mathfrak{X} : \hat{J}(\{\mathfrak{r}\}, \mathcal{Q}) \leq \epsilon \vee \hat{J}(\{\mathfrak{r}\}, \mathcal{V}) \leq \epsilon\} \\ &= \{\mathfrak{r} \in \mathfrak{X} : \hat{J}(\{\mathfrak{r}\}, \mathcal{Q}) \leq \epsilon\} \vee \{\mathfrak{r} \in \mathfrak{X} : \hat{J}(\{\mathfrak{r}\}, \mathcal{V}) \leq \epsilon\} \\ &= \mathbf{I}_\epsilon(\mathcal{Q}) \cup \mathbf{I}_\epsilon(\mathcal{V}). \end{aligned}$$

T_4 : Let $\mathcal{Q} \in 2^{\mathfrak{X}}$, let $\epsilon, \gamma \in \mathbb{R}^+$. Assume $\mathfrak{r} \in \mathbf{I}_\epsilon(\mathbf{I}_\gamma(\mathcal{Q}))$, then $\mathfrak{r} \in ((\mathcal{Q})^{(\gamma)})^{(\epsilon)}$ like that

$$\begin{aligned} ((\mathcal{Q})^{(\gamma)})^{(\epsilon)} &= \{\mathfrak{r} \in \mathfrak{X} : \hat{J}(\{\mathfrak{r}\}, \mathcal{Q}^\gamma) \leq \epsilon\} \\ &= \{\mathfrak{r} \in \mathfrak{X} : \hat{J}(\{\mathfrak{r}\}, \mathcal{Q}) \leq \epsilon + \gamma\} \\ &= \mathbf{I}_{\epsilon+\gamma}(\mathcal{Q}). \end{aligned}$$

T₅: Let $\mathcal{Q} \in 2^{\mathfrak{X}}, \epsilon, \epsilon_1, \epsilon_2 \in \mathbb{R}^+$. Assume that $\mathfrak{r} \in I_\epsilon(\mathcal{Q})$, then $\mathfrak{r} \in \mathcal{Q}^\epsilon$ like that

$$\mathcal{Q}^\epsilon = \left\{ \mathfrak{r} \in \mathfrak{X} : \hat{J}(\{\mathfrak{r}\}, \mathcal{Q}) \leq \epsilon \right\}.$$

Since $\epsilon_1 < \epsilon < \epsilon_2$ and $I_\epsilon(\mathcal{Q}) = \min \{\mathcal{Q}^{\epsilon_1}, \mathcal{Q}^{\epsilon_2}\}$. Then $\mathfrak{r} \in \bigcap_{\epsilon < \epsilon_2} I_{\epsilon_2}(\mathcal{Q})$. \square

Example 3.4. Let $\mathfrak{X} = [0, \infty]$, define $\hat{J} : 2^{\mathfrak{X}} * 2^{\mathfrak{X}} \rightarrow [0, \infty]$ as follows:

$$\hat{J}(\mathcal{Q}, \mathcal{V}) = \begin{cases} 0 & \mathcal{Q}, \mathcal{V} \text{ are un bounded} \\ \infty & \mathcal{Q} \neq \emptyset \text{ and } \mathcal{V} = \emptyset \\ \inf_{n \in \mathcal{Q}, m \in \mathcal{V}} |n - m| & \mathcal{Q}, \mathcal{V} \text{ are bounded} \end{cases}$$

Then $(\mathfrak{X}, \hat{J}, I_\epsilon)$ is Di- proximit space.

Solve:

Let $\mathcal{Q}, \mathcal{V}, \mathcal{C} \in 2^{\mathfrak{X}}$ \mathcal{P}_1 : * If \mathcal{Q}, \mathcal{V} are un bounded, \Rightarrow Let $\hat{J}(\mathcal{Q}, \mathcal{V}) = 0$. Then $\mathcal{Q} = \mathcal{V}$ such that $\mathcal{Q} \neq \emptyset$ and $D \neq \emptyset$. Therefore $\mathcal{Q} \cap \mathcal{V} \neq \emptyset$

\Leftarrow) If $\mathcal{Q} \cap \mathcal{V} \neq \emptyset$, then $\hat{J}(\mathcal{Q}, \mathcal{V}) = 0$

* If \mathcal{Q}, \mathcal{V} are bounded,

\Rightarrow) then $\hat{J}(\mathcal{Q}, \mathcal{V}) = 0 = \inf_{n \in \mathcal{Q}, m \in \mathcal{V}} |n - m| = 0$, then $|n - m| = 0$, such that $n - m = 0$
 $n = m$ so that $n \in \mathcal{Q} \cap \mathcal{V}$. Then $\mathcal{Q} \cap \mathcal{V} \neq \emptyset$.

\Leftarrow) If $\mathcal{Q} \cap \mathcal{V} \neq \emptyset$, there exist $n \in \mathcal{Q} \cap \mathcal{V}$, then $\hat{J}(\mathcal{Q}, \mathcal{V}) = \inf_{n \in \mathcal{Q}, m \in \mathcal{V}} |n - m| = 0$

\mathcal{P}_2 : If $\mathcal{Q} \neq \emptyset, \mathcal{V} = \emptyset$, then $\hat{J}(\mathcal{Q}, \mathcal{V}) = \hat{J}(\mathcal{Q}, \emptyset) = \infty$

\mathcal{P}_3 : * If $\mathcal{Q}, \mathcal{V}, \mathcal{C}$ are unbounded, then

$$\hat{J}(\mathcal{Q}, \mathcal{V} \cup \mathcal{C}) = \inf_{n \in \mathcal{Q}, m \in \mathcal{V} \cup \mathcal{C}} |n - m| = 0 = \min \left\{ \hat{J}(\mathcal{Q}, \mathcal{V}), \hat{J}(\mathcal{Q}, \mathcal{C}) \right\}$$

* If $\mathcal{Q} \neq \emptyset, \mathcal{V} = \emptyset, \mathcal{C} = \emptyset$, then $\mathcal{V} \cup \mathcal{C} = \emptyset$

$$\hat{J}(\mathcal{Q}, \mathcal{V} \cup \mathcal{C}) = \inf_{n \in \mathcal{Q}, m \in \mathcal{V} \cup \mathcal{C}} |n - m| = \infty = \min \left\{ \hat{J}(\mathcal{Q}, \mathcal{V}), \hat{J}(\mathcal{Q}, \mathcal{C}) \right\}$$

* If \mathcal{Q}, \mathcal{V} and \mathcal{C} are bounded, then

$$\begin{aligned} \hat{J}(\mathcal{Q}, \mathcal{V} \cup \mathcal{C}) &= \inf_{n \in \mathcal{Q}, m \in \mathcal{V} \cup \mathcal{C}} |n - m| \\ &= \inf_{n \in \mathcal{Q}, m \in \mathcal{V} \vee m \in \mathcal{C}} |n - m| \\ &= \min \left\{ \inf_{n \in \mathcal{Q}, m \in \mathcal{V}} |n - m|, \inf_{n \in \mathcal{Q}, m \in \mathcal{C}} |n - m| \right\} \\ &= \min \left\{ \hat{J}(\mathcal{Q}, \mathcal{V}), \hat{J}(\mathcal{Q}, \mathcal{C}) \right\} \end{aligned}$$

\mathcal{P}_4 : If $\mathcal{V} \subseteq \mathcal{C}$, by condition (\mathcal{P}_3) then

$$\begin{aligned} \mathcal{V} \cup \mathcal{C} = \mathcal{C}, \hat{J}(\mathcal{Q}, \mathcal{C}) = \hat{J}(\mathcal{Q}, \mathcal{V} \cup \mathcal{C}) &= \min \left\{ \hat{J}(\mathcal{Q}, \mathcal{V}), \hat{J}(\mathcal{Q}, \mathcal{C}) \right\}. \\ &\leq \hat{J}(\mathcal{Q}, \mathcal{V}) \end{aligned}$$

\mathcal{P}_5 : * If \mathcal{Q}, \mathcal{V} are un bounded and $\epsilon, \alpha \in [0, \infty]$, then

$$\hat{J}(\mathcal{Q}, \mathcal{V}) = 0 \leq \hat{J}(\mathcal{Q}^\epsilon, \mathcal{V}^\alpha) + \epsilon + \alpha.$$

* If \mathcal{Q}, \mathcal{V} are bounded and $\alpha, \epsilon \in [0, \infty]$, then

$$\begin{aligned} \hat{J}(\mathcal{Q}, \mathcal{V}) &= \inf_{n \in \mathcal{Q}, m \in \mathcal{V}} |n - m| \\ &\leq \inf_{n \in \mathcal{Q}^\epsilon, m \in \mathcal{V}^\alpha} |n - m| + \epsilon + \alpha \text{ for all } (\mathcal{Q} \subseteq \mathcal{Q}^\epsilon, \mathcal{V} \subseteq \mathcal{V}^\alpha) \\ &= \hat{J}(\mathcal{Q}^\epsilon, \mathcal{V}^\alpha) + \epsilon + \alpha. \end{aligned}$$

* If $\mathcal{Q} \neq \emptyset, \mathcal{V} = \emptyset$ and $\epsilon, \alpha, \in [0, \infty]$, then

$$\hat{J}(\mathcal{Q}, \mathcal{V}) = \infty \leq \hat{J}(\mathcal{Q}^\epsilon, \mathcal{V}^\alpha) + \epsilon + \alpha.$$

Therefore $(\mathfrak{X}, \hat{J}, \mathbb{I}_\epsilon)$ is Di - proximit space.

Definition 3.5. A function $\psi : (\mathfrak{X}, \hat{J}_\mathfrak{X}, \mathbb{I}_\epsilon') \rightarrow (\mathbb{R}, \hat{J}_\mathbb{R}, \mathbb{I}_\epsilon'')$ such that $(\mathfrak{X}, \hat{J}_\mathfrak{X}, \mathbb{I}_\epsilon')$ and $(\mathbb{R}, \hat{J}_\mathbb{R}, \mathbb{I}_\epsilon'')$ are Di-proximit space is called Di-contraction if $\psi(\mathbb{I}_\epsilon''(\mathcal{Y})) \subseteq \mathbb{I}_\epsilon''(\psi(\mathcal{Y}))$ for all $\mathcal{Y} \subseteq \mathfrak{X}$ and $\epsilon \in \mathbb{R}^+$.

Definition 3.6. A Di-proximit topological vector space (Di-p.t.v.s for short) is Di-proximit vector space \mathfrak{X} (over \mathbb{F}) together Di-proximit topology τ on \mathfrak{X} such that the maps $(n, m) \rightarrow n + m$ and $(\zeta, n) \rightarrow \zeta n$ are contraction from $\mathfrak{X} \times \mathfrak{X} \rightarrow \mathfrak{X}$ and $\mathbb{F} \times \mathfrak{X} \rightarrow \mathfrak{X}$.

4. Convergence in Di-Proximit Topological Vector Space

Definition 4.1. A set $\hat{E} \in 2^\mathfrak{X}$ is claimed to be cluster point in Di-p.t.v.s $(\mathfrak{X}, \hat{J}_\tau, \mathbb{I}_\epsilon, +, \cdot)$. If a disjoint filter exists $\{\mathbb{F}_c\}_{c=1}^\infty$ of \mathfrak{X} for all $\hat{A}_c \in \{\mathbb{F}_c\}_{c=1}^\infty$ such that $\inf_{c \in \mathbb{Z}^+} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0$ and $\inf_{c \in \mathbb{Z}^+, \epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$, which is expressed by $\{\hat{A}_c\}_{c=1}^\infty \rightarrow \hat{E}$. We represented the set of all cluster points in Di-p.t.v.s by $\hat{\Omega}(\mathfrak{X})$.

Definition 4.2. A filter $\{\mathbb{F}_c\}_{c=1}^\infty$ of \mathfrak{X} is referred to as a Cauchy filter in Di-p.t.v.s (Di- Cauchy), if each of the cluster point $\hat{E}_s, \lim_{c, s \rightarrow \infty} \inf_{\hat{A}_c, \hat{E}_s \in \{\mathbb{F}_c\}_{c=1}^\infty} \hat{J}_\tau(\hat{A}_c, \hat{E}_s) = 0$ and $\lim_{c, s \rightarrow \infty} \inf_{\epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}_s$, for all $\hat{A}_c \in \{\mathbb{F}_c\}_{c=1}^\infty$. A filter $\{\mathbb{F}_c\}_{c=1}^\infty$ in \mathfrak{X} is claimed to be Di-convergent filter in Di-p.t.v.s if for each $\hat{E} \in \hat{\Omega}(\mathfrak{X})$ like that $\hat{J}_\tau(\hat{A}_c, \hat{E}) = 0$ and $\mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$, for all $\hat{A}_c \in \{\mathbb{F}_c\}_{c=1}^\infty$ and $c \in \mathbb{Z}^+$.

Proposition 4.3. Let's $(\mathfrak{X}, \hat{J}_\tau, \mathbb{I}_\epsilon, +, \cdot)$ be Di-p.t.v.s, hence the following are equivalents:

1. Di-convergent filter in Di-p.t.v.s.
2. $\lim_{c \rightarrow \infty} \inf_{\hat{A}_c, \hat{E} \in \{\mathbb{F}_c\}_{c=1}^\infty} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0, \lim_{c \rightarrow \infty} \inf_{\epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}, \lim_{c \rightarrow \infty} \sup_{\hat{A}_c, \hat{E} \in \{\mathbb{F}_c\}_{c=1}^\infty} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0$ and $\lim_{c \rightarrow \infty} \sup_{\epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$.

Proof: Let's $\{\mathbb{F}_c\}_{c=1}^\infty$ be Di-convergent filter in Di-p.t.v.s.

Hence for all $\hat{E} \in \hat{\Omega}(\mathfrak{X})$ and for all $\hat{A}_c \in \{\mathbb{F}_c\}_{c=1}^\infty$ like that $\hat{J}_\tau(\hat{A}_c, \hat{E}) = 0$ and $\mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$, so $\inf_{c \in \mathbb{Z}^+} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0, \inf_{c \in \mathbb{Z}^+, \epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$ and $\sup_{c \in \mathbb{Z}^+} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0, \sup_{c \in \mathbb{Z}^+, \epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$. Thus $\lim_{c \rightarrow \infty} \inf_{\hat{A}_c, \hat{E} \in \{\mathbb{F}_c\}_{c=1}^\infty} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0, \lim_{c \rightarrow \infty} \inf_{\epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}, \lim_{c \rightarrow \infty} \sup_{\hat{A}_c, \hat{E} \in \{\mathbb{F}_c\}_{c=1}^\infty} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0$ and $\lim_{c \rightarrow \infty} \sup_{\epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$. For all $\hat{A}_c \in \{\mathbb{F}_c\}_{c=1}^\infty$.

On the other hand, if condition(2) is met that's $\lim_{c \rightarrow \infty} \inf_{\hat{A}_c, \hat{E} \in \{\mathbb{F}_c\}_{c=1}^\infty} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0, \lim_{c \rightarrow \infty} \inf_{\epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}, \lim_{c \rightarrow \infty} \sup_{\hat{A}_c, \hat{E} \in \{\mathbb{F}_c\}_{c=1}^\infty} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0$ and $\lim_{c \rightarrow \infty} \sup_{\epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$, for all $\hat{A}_c \in \{\mathbb{F}_c\}_{c=1}^\infty$. Afterward \hat{E} is cluster point, that's $\inf_{c \in \mathbb{Z}^+} \hat{J}_\tau(\hat{A}_c, \hat{E}) = 0$ and $\inf_{c \in \mathbb{Z}^+, \epsilon \in \mathbb{R}^+} \mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$. So that $\hat{J}_\tau(\hat{A}_c, \hat{E}) = 0$ and $\mathbb{I}_\epsilon(\hat{A}_c) \subseteq \hat{E}$. So $\{\mathbb{F}_c\}_{c=1}^\infty$ is Di-convergent filter in Di-p.t.v.s. \square

Remark 4.4. All Di-convergent filter are Di-Cauchy filter (Cauchy filter Di-p.t.v.s), but the reverse not true.

Example 4.5. Let's $\mathfrak{X} = \mathfrak{R}/\{0\}$ and $\hat{J}_\tau : 2^\mathfrak{X} \times 2^\mathfrak{X} \rightarrow [0, \infty]$ defined as

$$\hat{J}_\tau(\hat{A}_c, \hat{E}) = \begin{cases} \infty & \hat{A}_c \neq \emptyset \text{ and } \hat{E} = \emptyset \\ \inf_{n \in \hat{A}_c, m \in \hat{E}} |n - m| & \hat{A}_c \neq \emptyset \text{ and } \hat{E} \neq \emptyset \end{cases}$$

$$\mathbb{F}_c = \left\{ \hat{A} \subseteq \mathfrak{R}/\{0\} : \exists n \in \mathbb{Z}^+, \left(\frac{-1}{n}, \frac{1}{n} \right) \subseteq \hat{A} \right\}$$

be Di- Cauchy filter proximit topological vector space not Di-convergent filter in Di-p.t.v.s.

Proposition 4.6. *The following are equivalents if \mathfrak{X} is a Di-proximit metrizable topological vector space:*

1. $\{F_c\}_{c=1}^{\infty}$ is Di-convergent filter in Di-p.t.v.s.
2. $\sup_{\hat{E} \in \hat{\Omega}(\mathfrak{X})} \hat{J}_{\supset}(\hat{A}_c, \hat{E}) = 0$. For all $\hat{A}_c \in \{F_c\}_{c=1}^{\infty}$.

Proof: Suppose that $\{F_c\}_{c=1}^{\infty}$ is Di-convergent filter in Di-p.t.v.s.

Then for each $\hat{E} \in \hat{\Omega}(\mathfrak{X})$ like that $\hat{J}_{\tau}(\hat{A}_c, \hat{E}) = 0$ and $I_{\epsilon}(\hat{A}_c) \subseteq \hat{E}$.

$$\begin{aligned} \hat{J}_{\tau}(\hat{A}_c, \hat{E}) &= \inf_{m_c \in \hat{A}_c} \inf_{m \in \hat{E}} \supset(m_c, m) = 0 \\ &= \sup_{\hat{E} \in \hat{\Omega}(\mathfrak{X})} \inf_{m_c \in \hat{A}_c} \inf_{m \in \hat{E}} \supset(m_c, m) = 0 \\ &= \sup_{\hat{E} \in \hat{\Omega}(\mathfrak{X})} \hat{J}_{\supset}(\hat{A}_c, \hat{E}) = 0. \end{aligned}$$

On the other hand, if condition(2) is met that's

$$\sup_{\hat{E} \in \hat{\Omega}(\mathfrak{X})} \hat{J}_{\supset}(\hat{A}_c, \hat{E}) = \sup_{\hat{E} \in \hat{\Omega}(\mathfrak{X})} \inf_{m_c \in \hat{A}_c} \inf_{m \in \hat{E}} \supset(m_c, m) = 0$$

For $m_c \in \hat{A}_c$, $m \in \hat{E}$ then $\hat{J}_{\supset}(\hat{A}_c, \hat{E}) = \hat{J}_{\tau}(\hat{A}_c, \hat{E}) = 0$, and $I_{\epsilon}(\hat{A}_c) \subseteq \hat{E}$. □

5. Conclusion

In this paper, we derived a more refined definition of the Di-proximit and relationship with tower . We proved every Di-convergence filter are Di-Cauchy filter of Di-proximit topological vector space.

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