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Fixed-Points of $(\dot{\alpha}, \dot{\beta})$-$(\dot{\psi}, \dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty Type Contraction Maps in Strong b-Metric Spaces with Applications

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ABSTRACT: This paper introduces the novel notion of an $(\dot{\alpha}, \dot{\beta})$-$(\dot{\psi}, \dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction applicable to both single-valued and multi-valued mappings within a strong b-metric space. We derive multiple fixed-point theorems for this contraction class. These outcomes generalize several known fixed-point results in the literature, notably including Geraghty's theorem. Supporting examples and applications for integral equations and functional equations that are arise in dynamic programming are provided to demonstrate the efficacy of the results.

Keywords: Fixed-points, strong b-metric spaces, multi-valued maps, integral equations.

Contents

1. Introduction

In nonlinear analysis, fixed-point theory serves as a fundamental mathematical tool for addressing diverse challenges. It finds widespread application in disciplines such as computer science, economics, engineering, chemistry, biology, and medical sciences. Contemporary studies highlight the efficacy of fixed-point methods in proving the existence and uniqueness of solutions for intricate mathematical frameworks. These approaches have been effectively utilized for generalized coupled systems involving finite-delay fractional differential equations, nonexpansive mappings within Banach spaces, and chaotic dynamical systems like Chua's attractor, through the use of contraction principles, Schauder's fixed-point theorem, and various iterative techniques. By offering a robust analytical structure, fixed-point theory guarantees theoretical resolvability while facilitating real-world implementations in fields such as control systems, predator–prey interactions, nonlinear integral equations, and the examination of complex chaotic phenomena (see e.g., [5,19,40]). The Banach contraction principle stands out as a pivotal result in the realm of metric fixed-point theory, attracting substantial attention from researchers owing to its broad real-world utility (see, e.g., [6,7,13,17,18,25]). Geraghty [16] advanced this theorem by incorporating a supporting function to broaden its scope. Nadler [29] further generalized the principle to encompass multi-valued functions, thereby enriching the landscape of metric fixed-point results. Later developments extended the idea of Banach contractions via multi-valued operators and measures of noncompactness (consult [12,36,38,39] for additional insights). Khojasteh et al. [23] initially defined the $\mathcal{Z}$-contraction concept through a family of control functions known as simulation functions, thereby offering an extended form of the Banach contraction principle. Olgun et al. [30] established fixed-point theorems for this generalized $\mathcal{Z}$-contraction. Chandok et al. [11] merged simulation functions with $\mathcal{C}$-class functions to prove the existence and uniqueness of coincidence points, thus extending the findings of [23,30].

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Jleli et al. [17] proposed the idea of α-admissibility to broaden the Banach contraction principle. Karapınar [21] further advanced the work of Jleli et al. [17] and Khojasteh et al. [23] by defining α-admissible $\mathcal{Z}$-contractions. Patel [31] established fixed-point results for multivalued contractions in α-complete metric spaces by employing generalized simulation functions. In the realm of medicine, El Mamounia et al. [28] developed a fractional-order framework to model COVID-19 transmission. Utilizing Krasnoselskii's fixed-point theorem alongside Banach's contraction principle, they demonstrated the existence and uniqueness of solutions, while also verifying the stability of disease-free and endemic equilibrium points. These findings underscore the role of fixed-point theory in delivering a solid mathematical basis for analyzing epidemic patterns.

In computer science, Enescu et al. [15] introduced a multimodal normalizing flow technique for generating and classifying images. A distinctive aspect of their method is that model parameters emerge as an interpretable fixed-point solution to the optimization objective. This strategy enhances the learning of multimodal distributions, circumvents challenges associated with gradient descent fine-tuning, and illustrates the value of fixed-point techniques in advancing contemporary machine learning systems. From a theoretical standpoint, Younis et al. [41] investigated best proximity points for proximal contractions within extended b-metric spaces. Their findings broaden and refine prior results in proximity theory, yielding fresh perspectives on coincidence points for multi-valued mappings. This research exemplifies how extensions of fixed-point theorems continue to deepen nonlinear analysis and supply practical reference points for diverse applications.

Collectively, these recent advances—spanning economics, medicine, computer science, and pure mathematics—underscore the broad utility and critical importance of fixed-point theory in tackling practical challenges. The concept of a strong b-metric space was first presented by Kirk and Shahzad in 2014. They refined the relaxed triangle inequality characteristic of b-metric spaces, imposing a stricter condition that endows the space with various topological features absent in conventional b-metric spaces.

Definition 1.1 **([24])** Let $\mathcal{X} \neq \emptyset$ be a set and $s \geq 1$. Suppose $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a distance function satisfying for each $\omega_1, \omega_2, \omega_3 \in \mathcal{X}$:

($\mathcal{X}_1$) $d(\omega_1, \omega_2) = 0$ iff. $\omega_1 = \omega_2$;

($\mathcal{X}_2$) $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$;

($\mathcal{X}_3$) $d(\omega_1, \omega_3) \leq sd(\omega_1, \omega_2) + d(\omega_2, \omega_3)$.

Then the pair $(\mathcal{X}, d)$ is called a strong b-metric space ($\mathcal{S}b\mathcal{MS}$).

Substituting the condition ($\mathcal{X}_3$) in Definition 1.1 with the inequality $d(\omega_1, \omega_3) \leq s[d(\omega_1, \omega_2)+d(\omega_2, \omega_3)]$ transforms the pair $(\mathcal{X}, d)$ into a standard b-metric space.

Example 1.1 **([3])** Let $\mathcal{X} = \{1, 2, 3\}, d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be defined by $d(1, 1) = d(2, 2) = d(3, 3) = 0, d(1, 2) = d(2, 1) = 2, d(2, 3) = d(3, 2) = 1, d(1, 3) = d(3, 1) = 6$. Then $(\mathcal{X}, d)$ is $\mathcal{S}b\mathcal{MS}$ with $s = 4$.

Example 1.2 **([1])** Let $\mathcal{X} = \mathbb{R}$, and $d(\omega_1, \omega_2) = \max\{|\omega_1 - \omega_2|, 2|\omega_1 - \omega_2| - 1\}$, for all $\omega_1, \omega_2 \in \mathcal{X}$, then $(\mathcal{X}, d)$ is $\mathcal{S}b\mathcal{MS}$ with $s = 2$ but not a metric space.

Example 1.3 **([1,14])** Let $\mathcal{X} = [1, \infty)]$, and $d(\omega_1, \omega_2) = (\omega_1 - \omega_2)^4$, for all $\omega_1, \omega_2 \in \mathcal{X}$, then d is continuous b-metric with $s = 8$ but not a $\mathcal{S}b\mathcal{MS}$.

In 2015, Khojasteh et al. [23] introduced a family $\mathcal{Z}$ of mappings $\dot{\zeta} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ that satisfy the following properties:

($\dot{\zeta}_1$) $\dot{\zeta}(0, 0) = 0$;

($\dot{\zeta}_2$) $\dot{\zeta}(\rho, \theta) < \theta - \rho$ for all $\rho, \theta > 0$;

($\dot{\zeta}_3$) If $\{\rho_n\}, \{\theta_n\} \subset [0, \infty)$ are sequences $\ni \lim\limits_{n\to\infty} \rho_n = \lim\limits_{n\to\infty} \theta_n = l > 0$, then $\limsup\limits_{n\to\infty} \dot{\zeta}(\rho_n, \theta_n) < 0$.

These mappings are referred to as simulation functions. The function $\dot{\zeta}$ is employed to define the notion of $\mathcal{Z}$-contraction, which extends the classical Banach contraction principle and unifies several known contractive conditions involving terms of the form $d(\mathcal{T}\nu,\mathcal{T}\mu)$ and $d(\nu,\mu)$. Khojasteh et al. [23] proved the following theorem.

Theorem 1.1 ([23]) *Let $(\mathcal{X},d)$ be a complete metric space and let $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be a self-mapping such that $\zeta\big(d(\mathcal{T}\nu,\mathcal{T}\mu),d(\nu,\mu)\big)\geq 0$ for all $\nu,\mu\in\mathcal{X}$, where $\dot{\zeta}\in\mathcal{Z}$. Then $\mathcal{T}$ has a unique fixed-point $\nu^*\in\mathcal{X}$. Moreover, for any initial point $\nu_0\in\mathcal{X}$, the Picard iteration sequence defined by $\nu_n=\mathcal{T}\nu_{n-1}$ for all $n\in\mathbb{N}$, converges to ν^*.*

Karapınar [21] obtained fixed-point theorems in complete metric spaces by proposing a novel contractive condition that incorporates an admissible mapping within the framework of a simulation function. Subsequently, Hakan et al. [37] developed a fixed-point result by introducing a generalized simulation function defined on quasi-metric spaces.

Roldán-López-de-Hierro et al. [33] refined the original concept of simulation function by substituting condition $(\dot{\zeta}_3)$ with a stronger variant, denoted $(\dot{\zeta}_3')$, stated as follows: if $\{\rho_n\},\{\theta_n\}\subset[0,\infty)$ are sequences satisfying
$\lim_{n\to\infty}\rho_n=\lim_{n\to\infty}\theta_n=l>0$ and $\rho_n<\theta_n$ for all n, then $\limsup_{n\to\infty}\zeta(\rho_n,\theta_n)<0$.

A function $\dot{\zeta}$ satisfying $(\dot{\zeta}_1)$, $(\dot{\zeta}_2)$, and $(\dot{\zeta}_3')$ is termed a simulation function in the sense of Roldán-López-de-Hierro and belongs to the family Ω.

Definition 1.2 ([4]) A function $\mathcal{G}:(\mathbb{R}_+\cup\{0\})^2\to\mathbb{R}$ is said to belong to $\mathcal{C}$-class if it is continuous and satisfies:

(i) $\mathcal{G}(\theta,\varrho)\leq\theta$ for all $\theta,\varrho\in\mathbb{R}_+\cup\{0\}$;

(ii) $\mathcal{G}(\theta,\varrho)=\theta$ implies $\theta=0$ or $\varrho=0$, for every $\theta,\varrho\in\mathbb{R}_+\cup\{0\}$.

Definition 1.3 ([26]) A function $\mathcal{G}:(\mathbb{R}_+\cup\{0\})^2\to\mathbb{R}$ is said to satisfy property $(\mathcal{C}_{\mathcal{G}})$ if there exists a constant $\mathcal{C}_G\geq 0$ such that

$(\mathcal{G}_1)$ $\mathcal{G}(\theta,\varrho)>\mathcal{C}_G$ implies $\theta>\varrho$,

$(\mathcal{G}_2)$ $\mathcal{G}(\varrho,\varrho)\leq\mathcal{C}_G$ for all $\theta,\varrho\in\mathbb{R}_+\cup\{0\}$.

Definition 1.4 A function $\dot{\zeta}_{\mathcal{G}}:\mathbb{R}_+\cup\{0\}\times\mathbb{R}_+\cup\{0\}\to\mathbb{R}$ is called a $\mathcal{C}_{\mathcal{G}}$-simulation function if there exists a $\mathcal{C}$-class function $\mathcal{G}:(\mathbb{R}_+\cup\{0\})^2\to\mathbb{R}$ possessing property $(\mathcal{C}_{\mathcal{G}})$ with associated constant $\mathcal{C}_{\mathcal{G}}\geq 0$, and the following conditions hold:

(a) $\dot{\zeta}_{\mathcal{G}}(\varrho,\theta)<\mathcal{G}(\theta,\varrho)$ for all $\varrho,\theta>0$;

(b) If $\{\varrho_n\}$ and $\{\theta_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty}\varrho_n=\lim_{n\to\infty}\theta_n=l>0 \text{ and } \varrho_n<\theta_n \text{ for all } n, \text{ then } \limsup_{n\to\infty}\dot{\zeta}_{\mathcal{G}}(k\varrho_n,\theta_n)<\mathcal{C}_{\mathcal{G}}, \text{ for any } k>1.$$

From now on, let $\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$ stand for the collection of all $\mathcal{C}_{\mathcal{G}}$-simulation functions. Note that every simulation function belongs to $\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$.

Example 1.4 ([26]) Here are some instances of $\mathcal{C}_{\mathcal{G}}$-simulation mappings:

- $\dot{\zeta}_{\mathcal{G}}(\lambda,\eta)=\mu\eta-\lambda$, where $\mu\in(0,1)$,
- $\dot{\zeta}_{\mathcal{G}}(\lambda,\eta)=\frac{\eta}{1+\eta}-\lambda$,

Definition 1.5 Let $(\mathcal{X},d)$ be a metric space, let $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be a single-valued mapping, and $\dot{\alpha},\dot{\beta}:\mathcal{X}\to[0,+\infty)$ be two functions. Then $\mathcal{T}$ is said to be

$(\mathcal{D}_1)$ [34] *$\dot{\alpha}$-admissible* if $\dot{\alpha}(\varpi,\omega)\geq 1\Rightarrow\dot{\alpha}(\mathcal{T}\varpi,\mathcal{T}\omega)\geq 1$ for all $\varpi,\omega\in\mathcal{X}$.

$(\mathcal{D}_2)$ [22] *triangular $\dot{\alpha}$-admissible* if

i. $\mathcal{T}$ is $\dot{\alpha}$-admissible, and
ii. whenever $\dot{\alpha}(\varpi_1, \varpi_3) \geq 1$ and $\dot{\alpha}(\varpi_3, \varpi_2) \geq 1$, it follows that $\dot{\alpha}(\varpi_1, \varpi_2) \geq 1$,

for all $\varpi_1, \varpi_2, \varpi_3 \in \mathcal{X}$.

$(\mathcal{D}_3)$ [32] *$\dot{\alpha}$-orbital admissible* if $\dot{\alpha}(\varpi, \mathcal{T}\varpi) \geq 1 \Rightarrow \dot{\alpha}(\mathcal{T}\varpi, \mathcal{T}^2\varpi) \geq 1$ for all $\varpi \in \mathcal{X}$.

$(\mathcal{D}_4)$ [32] *triangular $\dot{\alpha}$-orbital admissible* if

i. $\mathcal{T}$ is $\dot{\alpha}$-orbital admissible, and
ii. whenever $\dot{\alpha}(\varpi, \omega) \geq 1$ and $\dot{\alpha}(\omega, \mathcal{T}\omega) \geq 1$, it follows that $\dot{\alpha}(\varpi, \mathcal{T}\omega) \geq 1$, for all $\varpi, \omega \in \mathcal{X}$.

$(\mathcal{D}_5)$ [2] *cyclic $(\dot{\alpha}, \dot{\beta})$-admissible* if

i. for some $\varpi \in \mathcal{X}$ with $\dot{\alpha}(\varpi) \geq 1$, we have $\dot{\beta}(\mathcal{T}\varpi) \geq 1$;
ii. for some $\varpi \in \mathcal{X}$ with $\dot{\beta}(\varpi) \geq 1$, we have $\dot{\alpha}(\mathcal{T}\varpi) \geq 1$.

Remark 1.1 ([32]) Every $\dot{\alpha}$-admissible mapping is orbital $\dot{\alpha}$-admissible, and every triangular $\dot{\alpha}$-admissible mapping is triangular $\dot{\alpha}$-orbital admissible.

Throughout this work, we denote by $\mathcal{CB}(\mathcal{X})$ the collection of all nonempty closed and bounded subsets of the metric space $(\mathcal{X}, d)$. The Pompeiu–Hausdorff metric is denoted by $\mathcal{H}$, defined for any $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{X})$ as

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{a\in\mathcal{A}} \mathcal{D}(a, \mathcal{B}), \sup_{b\in\mathcal{B}} \mathcal{D}(b, \mathcal{A})\right\},$$

where, for any point $\varpi_0 \in \mathcal{X}$ and subset $\mathcal{S} \subseteq \mathcal{X}$,

$$\mathcal{D}(\varpi_0, \mathcal{S}) = \inf_{s\in\mathcal{S}} d(\varpi_0, s).$$

Definition 1.6 ([27]) A multi-valued mapping $\mathcal{T}$ is said to be $\dot{\alpha}$-admissible if, for any ϖ_1 and ϖ_2 satisfying $\dot{\alpha}(\varpi_1, \varpi_2) \geq 1$, it holds that $\dot{\alpha}(\varpi_2, \varpi_3) \geq 1$ for every $\varpi_3 \in \mathcal{T}\varpi_2$.

Definition 1.7 ([12]) Let $\mathcal{X}$ be a nonempty set and $\mathcal{T} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ a multi-valued mapping. The mapping $\mathcal{T}$ is said to be cyclic multi-valued $(\dot{\alpha}, \dot{\beta})$-admissible if the following conditions hold:
(i) for every $\varpi_1, \varpi_2 \in \mathcal{X}$, if $\dot{\alpha}(\varpi_1) \geq 1$, then $\dot{\beta}(u) \geq 1$ for all $u \in \mathcal{T}\varpi_1$;
(ii) for every $\varpi_1, \varpi_2 \in \mathcal{X}$, if $\dot{\beta}(\varpi_2) \geq 1$, then $\dot{\alpha}(v) \geq 1$ for all $v \in \mathcal{T}\varpi_2$.

Subsequent sections include several established results. Some serve as supporting lemmas for proving our primary theorems, while others provide reference points for comparing our contributions.

Proposition 1.1 ([24]) *Let $(\varpi_n)_{n\in\mathbb{N}}$ be a sequence in a $\mathcal{SbMS}$ $(\mathcal{X}, d)$ with coefficient $s \geq 1$. If $\sum_{n=1}^{\infty} d(\varpi_n, \varpi_{n+1}) < +\infty$, then $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Lemma 1.1 ([24]) *Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an $\dot{\alpha}$-orbital admissible mapping. Suppose there exists $\varpi_1 \in \mathcal{X}$ such that $\dot{\alpha}(\varpi_1, \mathcal{T}\varpi_1) \geq 1$. Define the sequence $(\varpi_n)_{n\in\mathbb{N}}$ by $\varpi_{n+1} = \mathcal{T}\varpi_n$ for all $n \in \mathbb{N}$. Then, $\dot{\alpha}(\varpi_n, \varpi_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $m > n$.*

Lemma 1.2 *Suppose $(\mathcal{X}, d)$ is a $\mathcal{SbMS}$ with coefficient $s \geq 1$ and $\{\varpi_n\}$ is a sequence in $\mathcal{X}$ such that $d(\varpi_n, \varpi_{n+1}) \to 0$ as $n \to \infty$. If $\{\varpi_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that for every $k > 0$, corresponds to m_k, we can take n_k which is smallest such that $d(\varpi_{m_k}, \varpi_{n_k}) \geq \epsilon, d(\varpi_{m_k}, \varpi_{n_k-1}) < \epsilon$ and*

(i) $\lim_{k\to\infty} d(\varpi_{m_k}, \varpi_{n_k}) = \epsilon$; *(ii)* $\lim_{k\to\infty} d(\varpi_{m_k+1}, \varpi_{n_k}) = \epsilon$;

(iii) $\lim_{k\to\infty} d(\varpi_{m_k}, \varpi_{n_k+1}) = \epsilon$; *(iv)* $\lim_{k\to\infty} d(\varpi_{m_k+1}, \varpi_{n_k+1}) = \epsilon$.

$\mathcal{SbMS}$ occupy an intermediate position between b-metric spaces and standard metric spaces. This positioning highlights their importance, as numerous established fixed-point theorems that apply in $\mathcal{SbMS}$ fail to hold in general b-metric spaces (see, e.g., [1]).

2. Fixed-points of $(\dot\alpha,\dot\beta)$-$(\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_\mathcal{G}}$-Geraghty type contraction maps

This section introduces the notion of an $(\dot\alpha,\dot\beta)$-$(\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_\mathcal{G}}$-Geraghty type contraction mapping within a $\mathcal{S}b\mathcal{MS}$, applicable to both single-valued and multi-valued mappings. Subsequently, we establish and examine the existence of fixed-points for such mappings. The class $\mathfrak{F}$ comprises all functions $\Lambda:[0,\infty)\to[0,1)$ with the property that $\Lambda(t)$ can approach 1 if and only if t approaches 0. The class $\dot\Psi$ consists of all continuous non-decreasing functions $\dot\psi:[0,\infty)\to[0,\infty)$ such that $\dot\psi^{-1}(0)=0$(*i.e.*, $\dot\psi(t)>0$ for every $t>0$ and $\dot\psi(0)=0$).

2.1. For single-valued mappings

Definition 2.1 Let $(\mathcal{X},d)$ be a $\mathcal{S}b\mathcal{MS}$ and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be a map, and let $\dot\alpha,\dot\beta:\mathcal{X}\to[0,\infty)$ be two functions. We say that $\mathcal{T}$ is an $(\dot\alpha,\dot\beta)$-$(\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_\mathcal{G}}$-Geraghty type contraction mapping with respect to the $\mathcal{C}_\mathcal{G}$-simulation function $\dot\zeta$ if for $\varpi,\omega\in\mathcal{X}$, and there exist $\dot\psi,\dot\phi\in\dot\Psi$, $\Lambda\in\mathfrak{F}$ and $\dot L\ge 0$ such that

$$\dot\alpha(\varpi)\dot\beta(\omega)\ge 1\Rightarrow\dot\zeta\left(\dot\psi(d(\mathcal{T}\varpi,\mathcal{T}\omega)),\Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega))+\dot L\dot\phi(\mathcal{N}(\varpi,\omega))\right)\ge\mathcal{C}_\mathcal{G},\tag{2.1}$$

where $\mathcal{M}(\varpi,\omega)=\max\{d(\varpi,\omega),d(\varpi,\mathcal{T}\varpi),d(\omega,\mathcal{T}\omega),\frac{d(\varpi,\mathcal{T}\omega)+d(\omega,\mathcal{T}\varpi)}{2s}\}$, and
$\mathcal{N}(\varpi,\omega)=\min\{d(\varpi,\mathcal{T}\varpi),d(\omega,\mathcal{T}\omega),d(\varpi,\mathcal{T}\omega),d(\omega,\mathcal{T}\varpi)\}$.

Theorem 2.1 *Let $(\mathcal{X},d)$ be a complete $\mathcal{S}b\mathcal{MS}$ and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be an $(\dot\alpha,\dot\beta)$-$(\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_\mathcal{G}}$-Geraghty type contraction mapping and also the following conditions hold:*

(a_{1_i}) *$\mathcal{T}$ is a cyclic $(\dot\alpha,\dot\beta)$-admissible mapping;*

$(a_{1_{ii}})$ *there exists $\varpi_0\in\mathcal{X}$ such that $\dot\alpha(\varpi_0)\ge 1$ and $\dot\beta(\varpi_0)\ge 1$;*

$(a_{1_{iii}})$ *$\mathcal{T}$ is continuous.*

Then $\mathcal{T}$ has a fixed-point in $\mathcal{X}$.

Proof: According to condition $(a_{1_{ii}})$ of the theorem, there exists an element $\varpi_0\in\mathcal{X}$ such that $\dot\alpha(\varpi_0)\ge 1$ and $\dot\beta(\varpi_0)\ge 1$. Construct an iterative sequence $(\varpi_n)\subseteq\mathcal{X}$ by defining $\varpi_{n+1}=\mathcal{T}\varpi_n$ for all $n\in\mathbb{N}\cup\{0\}$. If there exists some $n\in\mathbb{N}$ such that $\varpi_{n-1}=\varpi_n$, then the proof is complete, as ϖ_n is a fixed-point of $\mathcal{T}$. Otherwise, we assume that $\varpi_n\ne\varpi_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$.
Given that $\dot\alpha(\varpi_0)\ge 1$ and $\mathcal{T}$ is a cyclic $(\dot\alpha,\dot\beta)$-admissible map, we obtain $\dot\beta(\varpi_1)=\dot\beta(\mathcal{T}\varpi_0)\ge 1$ and $\dot\alpha(\varpi_2)=\dot\alpha(\mathcal{T}\varpi_1)\ge 1$. Proceeding iteratively, we find that $\dot\alpha(\varpi_{2k})\ge 1$ and $\dot\beta(\varpi_{2k+1})\ge 1$ for all $k\in\mathbb{N}\cup\{0\}$. Similarly, starting from $\dot\beta(\varpi_0)\ge 1$ and using the cyclic $(\dot\alpha,\dot\beta)$-admissible property of $\mathcal{T}$, we get $\dot\alpha(\varpi_1)=\dot\alpha(\mathcal{T}\varpi_0)\ge 1$, which implies $\dot\beta(\varpi_2)=\dot\beta(\mathcal{T}\varpi_1)\ge 1$. Continuing this process, we have $\dot\beta(\varpi_{2k})\ge 1$ and $\dot\alpha(\varpi_{2k+1})\ge 1$ for all $k\in\mathbb{N}\cup\{0\}$. Combining these results, it follows that $\dot\alpha(\varpi_n)\ge 1$ and $\dot\beta(\varpi_n)\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$. We now show that $\lim\limits_{n\to\infty}d(\varpi_n,\varpi_{n+1})=0$.
Since $\dot\alpha(\varpi_n)\dot\beta(\varpi_{n+1})\ge 1$ and from the inequality (2.1), we get

$$\begin{aligned}
&\dot\zeta(\dot\psi(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_{n+1})),\Lambda(\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1})))\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1}))+\dot L\dot\phi(\mathcal{N}(\varpi_n,\varpi_{n+1})))\ge\mathcal{C}_\mathcal{G}\\
\Rightarrow\ &\mathcal{C}_\mathcal{G}\le\dot\zeta(\dot\psi(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_{n+1})),\Lambda(\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1})))\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1}))+\dot L\dot\phi(\mathcal{N}(\varpi_n,\varpi_{n+1})))\\
&\quad<\mathcal{G}(\Lambda(\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1})))\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1}))+\dot L\dot\phi(\mathcal{N}(\varpi_n,\varpi_{n+1})),\dot\psi(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_{n+1}))).
\end{aligned}$$

By applying $(\mathcal{G}_1)$, we obtain that

$$\dot\psi(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_{n+1}))<\Lambda(\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1})))\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1}))+\dot L\dot\phi(\mathcal{N}(\varpi_n,\varpi_{n+1})).$$

Therefore

$$\begin{aligned}
\dot\psi(d(\varpi_{n+1},\varpi_{n+2}))&=\dot\psi(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_{n+1}))\\
&<\Lambda(\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1})))\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1}))+\dot L\dot\phi(\mathcal{N}(\varpi_n,\varpi_{n+1}))\\
&<\dot\psi(\mathcal{M}(\varpi_n,\varpi_{n+1}))+\dot L\dot\phi(\mathcal{N}(\varpi_n,\varpi_{n+1})),
\end{aligned}\tag{2.2}$$

where

$$\begin{aligned}\mathcal{M}(\varpi_n,\varpi_{n+1}) &= \max\{d(\varpi_n,\varpi_{n+1}), d(\varpi_n,\mathcal{T}\varpi_n), d(\varpi_{n+1},\mathcal{T}\varpi_{n+1}), \tfrac{d(\varpi_n,\mathcal{T}\varpi_{n+1})+d(\varpi_{n+1},\mathcal{T}\varpi_n)}{2s}\}\\ &\le \max\{d(\varpi_n,\varpi_{n+1}), d(\varpi_{n+1},\varpi_{n+2}), \tfrac{d(\varpi_n,\varpi_{n+2})}{2s}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(\varpi_n,\varpi_{n+1}) &= \min\{d(\varpi_n,\mathcal{T}\varpi_n), d(\varpi_{n+1},\mathcal{T}\varpi_{n+1}), d(\varpi_n,\mathcal{T}\varpi_{n+1}), d(\varpi_{n+1},\mathcal{T}\varpi_n)\}\\ &\le \min\{d(\varpi_n,\varpi_{n+1}), d(\varpi_{n+1},\varpi_{n+2}), d(\varpi_n,\varpi_{n+2}), d(\varpi_{n+1},\varpi_{n+1})\} = 0.\end{aligned}$$

Since $\mathcal{X}$ is a $\mathcal{S}b\mathcal{MS}$, we have

$$\begin{aligned}\tfrac{d(\varpi_n,\varpi_{n+2})}{2s} \le \tfrac{sd(\varpi_n,\varpi_{n+1})+d(\varpi_{n+1},\varpi_{n+2})}{2s} &\le \tfrac{d(\varpi_n,\varpi_{n+1})+d(\varpi_{n+1},\varpi_{n+2})}{2}\\ &\le \max\{d(\varpi_n,\varpi_{n+1}), d(\varpi_{n+1},\varpi_{n+2})\}.\end{aligned}$$

If $\max\{d(\varpi_n,\varpi_{n+1}), d(\varpi_{n+1},\varpi_{n+2})\} = d(\varpi_{n+1},\varpi_{n+2})$, then by (2.2) we have

$$\dot{\psi}(d(\varpi_{n+1},\varpi_{n+2})) < \dot{\psi}(\mathcal{M}(\varpi_n,\varpi_{n+1})) \le \dot{\psi}(d(\varpi_{n+1},\varpi_{n+2})),$$

which is a contradiction.
Hence, for all $n \in \mathbb{N}\cup\{0\}$, $d(\varpi_{n+1},\varpi_{n+2}) < d(\varpi_n,\varpi_{n+1})$. From (2.2) we have

$$\dot{\psi}(d(\varpi_{n+1},\varpi_{n+2})) \le \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_{n+1})))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_{n+1})) < \dot{\psi}(d(\varpi_n,\varpi_{n+1})). \tag{2.3}$$

Therefore, $(d(\varpi_n,\varpi_{n+1}))_{n\in\mathbb{N}}$ is a decreasing sequence of non-negative reals. As a consequence, there exists $r \ge 0$ such that $\lim\limits_{n\to\infty} d(\varpi_n,\varpi_{n+1}) = r$. Suppose $r > 0$; as $n \to \infty$ in (2.3) and by the continuity of $\dot{\psi}$, we get

$$\lim_{n\to\infty} \dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_{n+1})) = \dot{\psi}(r) \text{ and, } \lim_{n\to\infty} \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_{n+1})))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_{n+1})) = \dot{\psi}(r).$$

Using (2.1) and (b) of Definition 1.4, we get

$$\begin{aligned}\mathcal{C}_\mathcal{G} &\le \limsup_{n\to\infty} \dot{\zeta}\left(\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_{n+1})), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_{n+1})))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_{n+1})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi_{n+1}))\right)\\ &< \mathcal{C}_\mathcal{G},\end{aligned}$$

this results in a contradiction. Consequently, the limit as n tends to infinity of $d(\varpi_n,\varpi_{n+1})$ equals zero. Let us assume that the sequence (ϖ_n) fails to be Cauchy. Since $\dot{\alpha}(\varpi_{m_k}) \ge 1$ and $\dot{\beta}(\varpi_{n_k}) \ge 1$, it follows that $\dot{\alpha}(\varpi_{m_k})\dot{\beta}(\varpi_{n_k}) \ge 1$. From equation (2.1), we arrive at the following

$$\begin{aligned}&\dot{\zeta}(\dot{\psi}(d(\mathcal{T}\varpi_{m_k},\mathcal{T}\varpi_{n_k})), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})))\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_{m_k},\varpi_{n_k}))) \ge \mathcal{C}_\mathcal{G}\\ \Rightarrow\ &\mathcal{C}_\mathcal{G} \le \dot{\zeta}(\dot{\psi}(d(\mathcal{T}\varpi_{m_k},\mathcal{T}\varpi_{n_k})), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})))\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_{m_k},\varpi_{n_k})))\\ &\quad < \mathcal{G}(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})))\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_{m_k},\varpi_{n_k})), \dot{\psi}(d(\mathcal{T}\varpi_{m_k},\mathcal{T}\varpi_{n_k}))).\end{aligned}$$

By applying $(\mathcal{G}_1)$, we obtain that

$$\dot{\psi}(d(\mathcal{T}\varpi_{m_k},\mathcal{T}\varpi_{n_k})) < \Lambda(\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})))\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_{m_k},\varpi_{n_k})).$$

Hence

$$\begin{aligned}\dot{\psi}(d(\varpi_{m_k+1},\varpi_{n_k+1})) &= \dot{\psi}(d(\mathcal{T}\varpi_{m_k},\mathcal{T}\varpi_{n_k}))\\ &< \Lambda(\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})))\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_{m_k},\varpi_{n_k}))\\ &< \dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_{m_k},\varpi_{n_k})),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\mathcal{M}(\varpi_{m_k},\varpi_{n_k}) &= \max\{d(\varpi_{m_k},\varpi_{n_k}), d(\varpi_{m_k},\mathcal{T}\varpi_{m_k}), d(\varpi_{n_k},\mathcal{T}\varpi_{n_k}), \tfrac{d(\varpi_{m_k},\mathcal{T}\varpi_{n_k})+d(\varpi_{n_k},\mathcal{T}\varpi_{m_k})}{2s}\}\\ &= \max\{d(\varpi_{m_k},\varpi_{n_k}), d(\varpi_{m_k},\varpi_{m_k+1}), d(\varpi_{n_k},\varpi_{n_k+1}), \tfrac{d(\varpi_{m_k},\varpi_{n_k+1})+d(\varpi_{n_k},\varpi_{m_k+1})}{2s}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(\varpi_{m_k},\varpi_{n_k}) &= \min\{d(\varpi_{m_k},\mathcal{T}\varpi_{m_k}),d(\varpi_{n_k},\mathcal{T}\varpi_{n_k}),d(\varpi_{m_k},\mathcal{T}\varpi_{n_k}),d(\varpi_{n_k},\mathcal{T}\varpi_{m_k})\}\\ &= \min\{d(\varpi_{m_k},\varpi_{m_k+1}),d(\varpi_{n_k},\varpi_{n_k+1}),d(\varpi_{m_k},\varpi_{n_k+1}),d(\varpi_{n_k},\varpi_{m_k+1})\}.\end{aligned}$$

Applying the Lemma 1.2 and considering the limit as k approaches infinity in (2.4), we obtain

$$\lim_{k\to\infty}\mathcal{M}(\varpi_{m_k},\varpi_{n_k})=\epsilon \quad\text{and}\quad \lim_{k\to\infty}\mathcal{N}(\varpi_{m_k},\varpi_{n_k})=0.$$

Taking $k\to\infty$ and using the property $\dot{\psi}$ in (2.4), we get

$$\begin{aligned}&\lim_{k\to\infty}\dot{\psi}(d(\mathcal{T}\varpi_{m_k},\mathcal{T}\varpi_{n_k}))=\dot{\psi}(\epsilon)\text{ and,}\\ &\lim_{k\to\infty}\Lambda(\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})))\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k}))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_{m_k},\varpi_{n_k}))=\dot{\psi}(\epsilon).\end{aligned}$$

Therefore using (b) of Definition 1.4 and (2.1), we obtain

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leq \lim_{k\to\infty}\dot{\zeta}\left(\dot{\psi}(d(\mathcal{T}\varpi_{m_k},\mathcal{T}\varpi_{n_k})),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k})))\dot{\psi}(\mathcal{M}(\varpi_{m_k},\varpi_{n_k}))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_{m_k},\varpi_{n_k}))\right)\\ &< \mathcal{C}_{\mathcal{G}},\end{aligned}$$

a contradiction. Therefore $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, by the completeness of $\mathcal{X}$, $(\varpi_n)_{n\in\mathbb{N}}$ converges to some $\varpi^*\in\mathcal{X}$. By continuity of $\mathcal{T}$, we have

$$d(\varpi^*,\mathcal{T}\varpi^*)=d(\lim_{n\to\infty}\varpi_{n+1},\mathcal{T}\varpi^*)=d(\lim_{n\to\infty}\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)=0.$$

Hence $\varpi^*=\mathcal{T}\varpi^*$. This completes the proof. □

Theorem 2.2 *In Theorem 2.1, replace* $(a_{1_{iii}})$ *with* $(a^*_{1_{iii}})$*: If* $(\varpi_n)\subseteq\mathcal{X}$ *converges to* ϖ^* *as* $n\to\infty$ *and* $\dot{\beta}(\varpi_n)\geq 1$ *for all* n*, then* $\dot{\beta}(\varpi^*)\geq 1$*. Consequently,* $\mathcal{T}$ *possesses a fixed-point.*

Proof: Suppose that $d(\varpi^*,\mathcal{T}\varpi^*)>0$. By the condition $(a^*_{1_{iii}})$ of the theorem $\dot{\beta}(\varpi^*)\geq 1$. Since $\dot{\alpha}(\varpi_n)\dot{\beta}(\varpi^*)\geq 1$ for all $n\geq 0$. As $(\mathcal{X},d)$ is a $\mathcal{S}b\mathcal{M}\mathcal{S}$, we have

$$\begin{aligned}d(\varpi^*,\mathcal{T}\varpi^*) &\leq sd(\varpi^*,\varpi_{n+1})+d(\varpi_{n+1},\mathcal{T}\varpi^*)\\ &= sd(\varpi^*,\varpi_{n+1})+d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*).\end{aligned}$$

Taking as $n\to\infty$ and using the property of $\dot{\psi}$, we have

$$\dot{\psi}(d(\varpi^*,\mathcal{T}\varpi^*))\leq\lim_{n\to\infty}\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)). \tag{2.5}$$

By taking $\varpi=\varpi_n,\omega=\varpi^*$ in (2.1), and using $\mathcal{T}$ is an $(\dot{\alpha},\dot{\beta})$-$(\dot{\psi},\dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping, we obtain

$$\begin{aligned}&\dot{\zeta}(\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)))\geq\mathcal{C}_{\mathcal{G}}\\ \Rightarrow\ &\mathcal{C}_{\mathcal{G}}\leq\dot{\zeta}(\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)))\\ &\quad<\mathcal{G}(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)),\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))).\end{aligned}$$

By utilizing $(\mathcal{G}_1)$, we derive that

$$\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))<\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)). \tag{2.6}$$

By applying inequalities (2.5) and (2.6), we derive

$$\begin{aligned}\dot{\psi}(d(\varpi^*,\mathcal{T}\varpi^*)) &\leq\lim_{n\to\infty}\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))\\ &<\lim_{n\to\infty}\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\lim_{n\to\infty}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)).\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}\mathcal{M}(\varpi_n,\varpi^*) &= \max\{d(\varpi_n,\varpi^*), d(\varpi_n,\mathcal{T}\varpi_n), d(\varpi^*,\mathcal{T}\varpi^*), \tfrac{d(\varpi_n,\mathcal{T}\varpi^*)+d(\varpi^*,\mathcal{T}\varpi_n)}{2s}\}\\ &= \max\{d(\varpi_n,\varpi^*), d(\varpi_n,\varpi_{n+1}), d(\varpi^*,\mathcal{T}\varpi^*), \tfrac{d(\varpi_n,\mathcal{T}\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(\varpi_n,\varpi^*) &= \min\{d(\varpi_n,\mathcal{T}\varpi_n), d(\varpi^*,\mathcal{T}\varpi^*), d(\varpi_n,\mathcal{T}\varpi^*), d(\varpi^*,\mathcal{T}\varpi_n)\}\\ &= \min\{d(\varpi_n,\varpi_{n+1}), d(\varpi^*,\mathcal{T}\varpi^*), d(\varpi_n,\mathcal{T}\varpi^*), d(\varpi^*,\varpi_{n+1})\}.\end{aligned}$$

This implies that $\lim\limits_{n\to\infty}\mathcal{M}(\varpi_n,\varpi^*) = d(\varpi^*,\mathcal{T}\varpi^*)$ and $\lim\limits_{n\to\infty}\mathcal{N}(\varpi_n,\varpi^*) = 0$. Hence, by (2.7), we get

$$\begin{aligned}&\lim_{n\to\infty}\dot\psi(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)) = \dot\psi(d(\varpi^*,\mathcal{T}\varpi^*)) \text{ and}\\ &\lim_{n\to\infty}\Big[\Lambda(\dot\psi(\mathcal{M}(\varpi_n,\varpi^*)))\dot\psi(\mathcal{M}(\varpi_n,\varpi^*)) + \dot L\dot\phi(\mathcal{N}(\varpi_n,\varpi^*))\Big] = \dot\psi(d(\varpi^*,\mathcal{T}\varpi^*)),\end{aligned}$$

Hence by (b) of Definition 1.4 and (2.1), we derive

$$\begin{aligned}\mathcal{C}_\mathcal{G} &\le \limsup_{n\to\infty}\dot\zeta\left(\dot\psi(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)), \Lambda(\dot\psi(\mathcal{M}(\varpi_n,\varpi^*)))\dot\psi(\mathcal{M}(\varpi_n,\varpi^*)) + \dot L\dot\phi(\mathcal{N}(\varpi_n,\varpi^*))\right)\\ &< \mathcal{C}_\mathcal{G},\end{aligned}$$

which means that $d(\varpi^*,\mathcal{T}\varpi^*) = 0$, and that ϖ^* is a fixed-point of $\mathcal{T}$. □

Corollary 2.1 Let $(\mathcal{X}, d)$ be a $\mathcal{SbMS}$ with $s \ge 1$ and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be a mapping such that for all $\varpi_1,\varpi_2\in\mathcal{X}$,

$$d(\mathcal{T}\varpi_1,\mathcal{T}\varpi_2) \le \Lambda(d(\varpi_1,\varpi_2))\cdot d(\varpi_1,\varpi_2),$$

where $\Lambda\in\mathfrak{F}$. Then $\mathcal{T}$ has a fixed-point.

Proof: Take $\dot\alpha(\varpi)\dot\beta(\omega) = 1$ for all $\varpi,\omega\in\mathcal{X}, \dot\psi(t) = t > 0, \mathcal{C}_\mathcal{G} = 0, \dot L = 0$ and $\mathcal{M}(\varpi,\omega) = d(\varpi,\omega)$ in Theorem 2.1. □

Definition 2.2 Let $(\mathcal{X}, d)$ be a $\mathcal{SbMS}$ and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be a map, and let $\dot\alpha:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ be a function. We say that $\mathcal{T}$ is an $(\dot\alpha,\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_\mathcal{G}}$-Geraghty type contraction mapping with respect to the $\mathcal{C}_\mathcal{G}$-simulation function $\dot\zeta$ if for $\varpi,\omega\in\mathcal{X}$, and there exist $\dot\psi,\dot\phi\in\dot\Psi$, $\Lambda\in\mathfrak{F}$ and $\dot L\ge 0$ such that

$$\dot\zeta\left(\dot\alpha(\varpi,\omega)\dot\psi(d(\mathcal{T}\varpi,\mathcal{T}\omega)), \Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot L\dot\phi(\mathcal{N}(\varpi,\omega))\right) \ge \mathcal{C}_\mathcal{G}, \tag{2.8}$$

where $\mathcal{M}(\varpi,\omega) = \max\{d(\varpi,\omega), d(\varpi,\mathcal{T}\varpi), d(\omega,\mathcal{T}\omega), \frac{d(\varpi,\mathcal{T}\omega)+d(\omega,\mathcal{T}\varpi)}{2s}\}$, and $\mathcal{N}(\varpi,\omega) = \min\{d(\varpi,\mathcal{T}\varpi), d(\omega,\mathcal{T}\omega), d(\varpi,\mathcal{T}\omega), d(\omega,\mathcal{T}\varpi)\}$.

Remark 2.1 The $(\dot\alpha,\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_\mathcal{G}}$-Geraghty type contraction mapping outlined in Definition 2.2 serves as an extension of the criteria introduced for metric spaces ([20], Definition 3.1).

Theorem 2.3 *Let $(\mathcal{X}, d)$ be a complete $\mathcal{SbMS}$ and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be an $(\dot\alpha,\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_\mathcal{G}}$-Geraghty type contraction mapping and also the following conditions hold:*

(a_{2_i}) *$\mathcal{T}$ is a triangular $\dot\alpha$-orbital admissible mapping;*

$(a_{2_{ii}})$ *there exists $\varpi_0\in\mathcal{X}$ such that $\dot\alpha(\varpi_0,\mathcal{T}\varpi_0)\ge 1$;*

$(a_{2_{iii}})$ *$\mathcal{T}$ is continuous.*

Then $\mathcal{T}$ has a fixed-point in $\mathcal{X}$.

Proof: According to condition $(a_{2_{ii}})$ of the theorem, there exists $\varpi_0\in\mathcal{X}$ such that $\dot\alpha(\varpi_0,\mathcal{T}\varpi_0)\ge 1$. We construct an iterative sequence $(\varpi_n)_{n\in\mathbb{N}}\subseteq\mathcal{X}$ by setting $\varpi_n = \mathcal{T}\varpi_{n-1}$ for every $n\in\mathbb{N}$. If $\varpi_{n-1} = \varpi_n$ for some $n\in\mathbb{N}$, the proof is complete, as ϖ_n serves as a fixed-point of $\mathcal{T}$. Thus, we suppose that

$\varpi_n \neq \varpi_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Given that $\mathcal{T}$ is $\dot{\alpha}$-admissible and $\dot{\alpha}(\varpi_0, \mathcal{T}\varpi_0) \geq 1$, it follows that $\dot{\alpha}(\mathcal{T}\varpi_0, \mathcal{T}^2\varpi_0) = \dot{\alpha}(\varpi_1, \varpi_2) \geq 1$. By iterating this argument, we conclude that $\dot{\alpha}(\varpi_n, \varpi_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Since $\mathcal{T}$ is an $(\dot{\alpha}, \dot{\psi}, \dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping, we have for all $n \in \mathbb{N}$

$$\begin{aligned}
&\dot{\zeta}(\dot{\alpha}(\varpi_n, \varpi_{n+1})\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_{n+1})), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_{n+1}))) \geq \mathcal{C}_{\mathcal{G}} \\
&\Rightarrow \mathcal{C}_{\mathcal{G}} \leq \dot{\zeta}(\dot{\alpha}(\varpi_n, \varpi_{n+1})\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_{n+1})), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_{n+1}))) \\
&\qquad < \mathcal{G}(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_{n+1})), \dot{\alpha}(\varpi_n, \varpi_{n+1})\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_{n+1}))).
\end{aligned}$$

By applying $(\mathcal{G}_1)$, we obtain that

$$\dot{\alpha}(\varpi_n, \varpi_{n+1})\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_{n+1})) < \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_{n+1})).$$

Therefore

$$\begin{aligned}
\dot{\psi}(d(\varpi_{n+1}, \varpi_{n+2})) &\leq \dot{\alpha}(\varpi_n, \varpi_{n+1})\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_{n+1})) \\
&< \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_{n+1})) \\
&< \dot{\psi}(\mathcal{M}(\varpi_n, \varpi_{n+1})) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_{n+1})),
\end{aligned} \tag{2.9}$$

Following the approach of Theorem 2.1, we get

$$\mathcal{M}(\varpi_n, \varpi_{n+1}) = d(\varpi_n, \varpi_{n+1}) \text{ and } \mathcal{N}(\varpi_n, \varpi_{n+1}) = 0.$$

From (2.9), we get

$$\begin{aligned}
\dot{\psi}(d(\varpi_{n+1}, \varpi_{n+2})) &< \Lambda(\dot{\psi}(d(\varpi_n, \varpi_{n+1})))\dot{\psi}(d(\varpi_n, \varpi_{n+1})) \\
&< \dot{\psi}(d(\varpi_n, \varpi_{n+1})),
\end{aligned} \tag{2.10}$$

Since for all $n \in \mathbb{N}, d(\varpi_{n+1}, \varpi_{n+2}) < d(\varpi_n, \varpi_{n+1})$, then $(d(\varpi_n, \varpi_{n+1}))_{n\in\mathbb{N}}$ is a non-negative and decreasing sequence. As a consequence, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(\varpi_n, \varpi_{n+1}) = r$. Suppose $r > 0$; by continuity of $\dot{\psi}$ and inequality (2.10), we have

$$1 = \lim_{n\to\infty} \frac{\dot{\psi}(d(\varpi_{n+1}, \varpi_{n+2}))}{\dot{\psi}(d(\varpi_n, \varpi_{n+1}))} \leq \lim_{n\to\infty} \Lambda(\dot{\psi}(d(\varpi_n, \varpi_{n+1}))) \leq 1.$$

This implies that $\lim\limits_{n\to\infty} \Lambda(\dot{\psi}(d(\varpi_n, \varpi_{n+1}))) = 1$. Given that Λ belongs to the family $\mathfrak{F}$, it follows that $\lim\limits_{n\to\infty} \dot{\psi}(d(\varpi_n, \varpi_{n+1})) = 0$. By the characteristics of the function $\dot{\psi}$, we deduce that $\lim_{n\to\infty} d(\varpi_n, \varpi_{n+1}) = 0$. Assuming $r = 0$ leads to a contradiction with the condition $r > 0$. Therefore,

$$\lim_{n\to\infty} d(\varpi_n, \varpi_{n+1}) = 0. \tag{2.11}$$

We now show that $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, suppose that $\lim\limits_{n,m\to\infty} d(\varpi_n, \varpi_m) = \epsilon > 0$. For $n < m$, we have

$$d(\varpi_n, \varpi_m) \leq sd(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \varpi_{m+1}) + sd(\varpi_m, \varpi_{m+1}). \tag{2.12}$$

Equation (2.8) yields

$$\begin{aligned}
&\dot{\zeta}(\dot{\alpha}(\varpi_n, \varpi_m)\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_m)), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_m)))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_m)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_m))) \geq \mathcal{C}_{\mathcal{G}} \\
&\Rightarrow \mathcal{C}_{\mathcal{G}} \leq \dot{\zeta}(\dot{\alpha}(\varpi_n, \varpi_m)\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_m)), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_m)))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_m)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_m))) \\
&\qquad < \mathcal{G}(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_m)))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_m)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_m)), \dot{\alpha}(\varpi_n, \varpi_m)\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_m))).
\end{aligned}$$

By applying $(\mathcal{G}_1)$, we obtain that

$$\dot{\alpha}(\varpi_n, \varpi_m)\dot{\psi}(d(\mathcal{T}\varpi_n, \mathcal{T}\varpi_m)) < \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_m)))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi_m)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi_m)).$$

Therefore

$$\begin{aligned}\dot{\psi}(d(\varpi_{n+1},\varpi_{m+1})) &\leq \dot{\alpha}(\varpi_n,\varpi_m)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_m))\\ &< \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_m)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_m)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi_m))\\ &< \dot{\psi}(\mathcal{M}(\varpi_n,\varpi_m)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi_m)),\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}\mathcal{M}(\varpi_n,\varpi_m) &= \max\{d(\varpi_n,\varpi_m), d(\varpi_n,\mathcal{T}\varpi_n), d(\varpi_m,\mathcal{T}\varpi_m), \tfrac{d(\varpi_n,\mathcal{T}\varpi_m)+d(\varpi_m,\mathcal{T}\varpi_n)}{2s}\}\\ &= \max\{d(\varpi_n,\varpi_m), d(\varpi_n,\varpi_{n+1}), d(\varpi_m,\varpi_{m+1}), \tfrac{d(\varpi_n,\varpi_{m+1})+d(\varpi_m,\varpi_{n+1})}{2s}\}\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\mathcal{N}(\varpi_n,\varpi_m) &= \min\{d(\varpi_n,\mathcal{T}\varpi_n), d(\varpi_m,\mathcal{T}\varpi_m), d(\varpi_n,\mathcal{T}\varpi_m), d(\varpi_m,\mathcal{T}\varpi_n)\}\\ &= \min\{d(\varpi_n,\varpi_{n+1}), d(\varpi_m,\varpi_{m+1}), d(\varpi_n,\varpi_{m+1}), d(\varpi_m,\varpi_{n+1})\}.\end{aligned} \tag{2.15}$$

Since

$$\begin{aligned}\tfrac{d(\varpi_n,\varpi_{m+1})+d(\varpi_m,\varpi_{n+1})}{2s} &\leq \tfrac{sd(\varpi_n,\varpi_m)+d(\varpi_m,\varpi_{m+1})+sd(\varpi_m,\varpi_n)+d(\varpi_n,\varpi_{n+1})}{2s}\\ &= d(\varpi_n,\varpi_m) + \tfrac{d(\varpi_m,\varpi_{m+1})+d(\varpi_n,\varpi_{n+1})}{2s}.\end{aligned} \tag{2.16}$$

From (2.11), (2.14)-(2.16), we get

$$\lim_{n\to\infty} \mathcal{M}(\varpi_n,\varpi_m) = \lim_{n\to\infty} d(\varpi_n,\varpi_m) \tag{2.17}$$

and

$$\lim_{n\to\infty} \mathcal{N}(\varpi_n,\varpi_m) = 0. \tag{2.18}$$

By the continuity of $\dot{\psi}$, from (2.11) and (2.12), we have

$$\lim_{n\to\infty} \dot{\psi}(d(\varpi_n,\varpi_m)) \leq \lim_{n\to\infty} \dot{\psi}(d(\varpi_{n+1},\varpi_{m+1})). \tag{2.19}$$

Combine the inequalities (2.13), (2.17)-(2.19), we derive

$$\begin{aligned}\lim_{n\to\infty} \dot{\psi}(d(\varpi_n,\varpi_m)) &\leq \lim_{n\to\infty} \dot{\psi}(d(\varpi_{n+1},\varpi_{m+1}))\\ &\leq \lim_{n\to\infty} \dot{\alpha}(\varpi_n,\varpi_m)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi_m))\\ &\leq \lim_{n\to\infty} (\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_m)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi_m)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi_m)))\\ &\leq \lim_{n\to\infty} \dot{\psi}(d(\varpi_n,\varpi_m))\end{aligned}$$

As by the assumption $\lim_{n\to\infty} d(\varpi_n,\varpi_m) = \epsilon > 0$, we have

$$1 \leq \lim_{n\to\infty} \Lambda(\dot{\psi}(d(\varpi_n,\varpi_m))) \leq 1 \text{ which implies that } \lim_{n\to\infty} \Lambda(\dot{\psi}(d(\varpi_n,\varpi_m))) = 1.$$

Thus, $\lim_{n\to\infty} \dot{\psi}(d(\varpi_n,\varpi_m)) = 0$, and that $\lim_{n\to\infty} d(\varpi_n,\varpi_m) = 0$, it is a contradiction. Hence $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete, there exists $\varpi^* \in \mathcal{X}$ such that $\lim_{n\to\infty} \varpi_n = \varpi^*$. By the continuity of $\mathcal{T}$, we have

$$d(\varpi^*,\mathcal{T}\varpi^*) = d(\lim_{n\to\infty} \varpi_{n+1},\mathcal{T}\varpi^*) = d(\lim_{n\to\infty} \mathcal{T}\varpi_n,\mathcal{T}\varpi^*) = 0.$$

Therefore $\varpi^* = \mathcal{T}\varpi^*$. □

Theorem 2.4 *In Theorem 2.3, replace* $(a_{2_{iii}})$ *with* $(a^*_{2_{iii}})$*: If* $(\varpi_n) \subseteq \mathcal{X}$ *converges to* ϖ^* *as* $n \to \infty$ *and* $\dot{\alpha}(\varpi_n,\varpi_{n+1}) \geq 1$ *for all* n*, then* $\dot{\alpha}(\varpi_n,\varpi^*) \geq 1$*. Consequently,* $\mathcal{T}$ *possesses a fixed-point.*

Proof: As noted in the proof of Theorem 2.3, the sequence $(\varpi_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence in $\mathcal{X}$ and converges to a point $\varpi^* \in \mathcal{X}$. Given that $\dot{\alpha}(\varpi_n,\varpi_{n+1}) \geq 1$, applying condition $(a^*_{2_{iii}})$ yields

$$\alpha(\varpi_n,\varpi^*) \geq 1,\ \forall n$$

and since $(\mathcal{X},d)$ is a $\mathcal{S}b\mathcal{MS}$ we have

$$d(\varpi^*,\mathcal{T}\varpi^*)\leq sd(\varpi^*,\varpi_{n+1})+d(\varpi_{n+1},\mathcal{T}\varpi^*).$$

Assume $d(\varpi^*,\mathcal{T}\varpi^*)>0$. Taking as $n\to\infty$

$$d(\varpi^*,\mathcal{T}\varpi^*)\leq\lim_{n\to\infty}d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*).$$

As $\dot{\psi}\in\dot{\Psi}$, we have

$$\dot{\psi}(d(\varpi^*,\mathcal{T}\varpi^*))\leq\lim_{n\to\infty}\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)).\tag{2.20}$$

By taking $\varpi=\varpi_n,\omega=\varpi^*$ in (2.8), we obtain

$$\begin{aligned}&\dot{\zeta}(\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)))\geq\mathcal{C}_{\mathcal{G}}\\ \Rightarrow&\,\mathcal{C}_{\mathcal{G}}\leq\dot{\zeta}(\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)))\\ &\quad<\mathcal{G}(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)),\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))).\end{aligned}$$

By utilizing $(\mathcal{G}_1)$, we derive that

$$\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))<\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)).\tag{2.21}$$

By applying inequalities (2.20) and (2.21), we derive

$$\begin{aligned}\dot{\psi}(d(\varpi^*,\mathcal{T}\varpi^*))&\leq\lim_{n\to\infty}\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))\\&\leq\lim_{n\to\infty}\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\lim_{n\to\infty}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)).\end{aligned}\tag{2.22}$$

Employing an analogous reasoning as that presented in Theorem 2.2, we derive that $\lim_{n\to\infty}\mathcal{M}(\varpi_n,\varpi^*)=d(\varpi^*,\mathcal{T}\varpi^*)$ and $\lim_{n\to\infty}\mathcal{N}(\varpi_n,\varpi^*)=0$. Hence, by (2.22), we get

$$\begin{aligned}&\lim_{n\to\infty}\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))=\dot{\psi}(d(\varpi^*,\mathcal{T}\varpi^*))\text{ and}\\&\lim_{n\to\infty}\Big[\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*))\Big]=\dot{\psi}(d(\varpi^*,\mathcal{T}\varpi^*)).\end{aligned}$$

Applying (2.8) and (b) of Definition 1.4, we get

$$\begin{aligned}\mathcal{C}_{\mathcal{G}}&\leq\limsup_{n\to\infty}\dot{\zeta}\left(\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(d(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*))\right)\\&<\mathcal{C}_{\mathcal{G}},\end{aligned}$$

which means that $d(\varpi^*,\mathcal{T}\varpi^*)=0$, and that ϖ^* is a fixed-point of $\mathcal{T}$. □

As an illustration of Theorem 2.1, consider the following construction.

Example 2.1 Let $\mathcal{X}=\mathbb{R}$, we define $d:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ by $d(\varpi,\omega)=\max\{|\varpi-\omega|,2|\varpi-\omega|-1\}$ for all $\varpi,\omega\in\mathcal{X}$ then $(\mathcal{X},d)$ is a complete $\mathcal{S}b\mathcal{MS}$ with coefficient $s=2$. We define $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ by

$$\mathcal{T}(\varpi)=\begin{cases}2\varpi-\frac{7}{4}, & \varpi>1,\\ \frac{2}{3}\varpi, & 0\leq\varpi\leq1,\\ 0, & \varpi<0.\end{cases}$$

Let $\dot{\alpha},\dot{\beta}:\mathcal{X}\to[0,\infty)$ be respectively defined as follows:

$$\dot{\alpha}(\varpi)=\begin{cases}\frac{2}{1+\varpi}, & 0\leq\varpi\leq1,\\ 0, & \text{otherwise,}\end{cases}\quad\text{and }\dot{\beta}(\varpi)=\begin{cases}\frac{3}{2+\varpi}, & 0\leq\varpi\leq1,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\dot{\psi},\dot{\phi}:[0,\infty)\to[0,\infty)$ be respectively defined as follows:

$$\dot{\psi}(\omega)=\begin{cases}\frac{\omega}{2}, & 0\leq\omega\leq1,\\ \frac{1}{2}, & \text{otherwise,}\end{cases}\quad\text{and }\dot{\phi}(\omega)=\begin{cases}\frac{\omega}{3}, & 0\leq\omega\leq1,\\ \frac{1}{3}, & \text{otherwise.}\end{cases}$$

Clearly that $\mathcal{T}$ is continuous.

Since $\dot\alpha(\varpi) \ge 1$ if and only if $\varpi \in [0,1]$, we have $\dot\beta(\mathcal{T}(\varpi)) = \frac{3}{2+\mathcal{T}(\varpi)} = \frac{3}{2+\frac{2\varpi}{3}} \ge 1$, and also $\varpi \in \mathcal{X}$, $\dot\beta(\varpi) \ge 1$ if and only if $\varpi \in [0,1]$, we have $\dot\alpha(\mathcal{T}(\varpi)) = \frac{2}{1+\mathcal{T}(\varpi)} = \frac{2}{1+\frac{2\varpi}{3}} \ge 1$. As a result $\mathcal{T}$ is a cyclic $(\dot\alpha,\dot\beta)$-admissible mapping. Clearly $\dot\psi,\dot\phi \in \dot\Psi$. The functions defined as $\dot\zeta(\varpi,\omega) = \frac{9}{10}\omega - \varpi$, $\mathcal{G}(\omega,\varpi) = \omega - \varpi$, and $\Lambda(l) = \frac{1}{1+l}$ for all $l > 0, \varpi, \omega \in [0,\infty)$ are specified mappings. It is evident that Λ, $\mathcal{G}$, and $\dot\zeta$ correspond to a Geraghty function, a $\mathcal{C}$-function, and a $\mathcal{C}_{\mathcal{G}}$-simulation function, respectively.
Here, it is sufficient to check the inequality (2.1) only for $\varpi,\omega \in [0,1]$. Without loss of generality, we assume $\varpi,\omega \in [0,1]$ with $\varpi \ge \omega$. Then we have

$$\begin{aligned}
\dot\psi(d(\mathcal{T}\varpi,\mathcal{T}\omega)) &= \tfrac{1}{3}(\varpi-\omega)\\
&\le \tfrac{9}{10}\left[\tfrac{1}{1+(\frac{\varpi-\omega}{2})}(\tfrac{\varpi-\omega}{2}) + \dot L\tfrac{1}{3}\min\{\tfrac{\varpi}{3},\tfrac{\omega}{3},(\varpi-\tfrac{2}{3}\omega),|\varpi-\tfrac{2}{3}\omega|\}\right]\\
&\le \tfrac{9}{10}\left[\tfrac{1}{1+\frac{\mathcal{M}(\varpi,\omega)}{2}}\tfrac{\mathcal{M}(\varpi,\omega)}{2} + \dot L\tfrac{1}{3}\min\{d(\varpi,\mathcal{T}\varpi),d(\omega,\mathcal{T}\omega),d(\varpi,\mathcal{T}\omega),d(\omega,\mathcal{T}\varpi)\}\right]\\
&= \tfrac{9}{10}\left[\Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot L\dot\phi(\mathcal{N}(\varpi,\omega))\right]
\end{aligned}$$

where $\mathcal{M}(\varpi,\omega) = \max\{d(\varpi,\omega), d(\varpi,\mathcal{T}\varpi), d(\omega,\mathcal{T}\omega), \frac{d(\varpi,\mathcal{T}\omega)+d(\omega,\mathcal{T}\varpi)}{2s}\}$.
So,

$$\begin{aligned}
&\dot\zeta(d(\mathcal{T}\varpi,\mathcal{T}\omega)),\Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot L\dot\phi(\mathcal{N}(\varpi,\omega)))\\
&= \tfrac{9}{10}\left[\Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot L\dot\phi(\mathcal{N}(\varpi,\omega))\right] - d(\mathcal{T}\varpi,\mathcal{T}\omega))\\
&= \tfrac{9}{10}\left[\left(\frac{\frac{\mathcal{M}(\varpi,\omega)}{2}}{1+\frac{\mathcal{M}(\varpi,\omega)}{2}}\right)\frac{\mathcal{M}(\varpi,\omega)}{2} + \dot L\dot\phi(\mathcal{N}(\varpi,\omega))\right] - d(\mathcal{T}\varpi,\mathcal{T}\omega))
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
&\mathcal{G}(\Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot L\dot\phi(\mathcal{N}(\varpi,\omega)),\mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega)))\\
&= \Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot L\dot\phi(\mathcal{N}(\varpi,\omega)) - \mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega))\\
&= \left(\frac{\frac{\mathcal{M}(\varpi,\omega)}{2}}{1+\frac{\mathcal{M}(\varpi,\omega)}{2}}\right)\frac{\mathcal{M}(\varpi,\omega)}{2} + \dot L\dot\phi(\mathcal{N}(\varpi,\omega)) - \mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega))
\end{aligned} \tag{2.24}$$

Based on equations (2.23) and (2.24), we deduce

$$\begin{aligned}
0 &< \dot\zeta(d(\mathcal{T}\varpi,\mathcal{T}\omega)),\Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot L\dot\phi(\mathcal{N}(\varpi,\omega)))\\
&< \mathcal{G}(\Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot L\dot\phi(\mathcal{N}(\varpi,\omega)), d(\mathcal{T}\varpi,\mathcal{T}\omega)))
\end{aligned} \tag{2.25}$$

By applying inequality (2.25) and Definition 2.1, it is evident that the mapping $\mathcal{T}$ constitutes an $(\dot\alpha,\dot\beta)$-$(\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping with $\mathcal{C}_{\mathcal{G}} = 0$. Hence, the conditions of Theorem 2.1 are satisfied, confirming that $\mathcal{T}$ has fixed-points in $\mathcal{X}$.

Remark 2.2 Corollary 2.1 cannot be applied since

$$d(\mathcal{T}0,\mathcal{T}2) = 2\Big|0 - \frac{9}{4}\Big| - 1 \nleq \Lambda(d(0,2))\cdot d(0,2) = \frac{3}{4}.$$

2.2. For multi-valued mappings

In this section we study the case $s > 1$. We denote by $\mathfrak{F}_b$ the family of all functions $\Lambda : [0,\infty) \to [0,\frac{1}{s})$ possessing the property that whenever a sequence $\{\varpi_n\}$ satisfies $\Lambda(\varpi_n) = \frac{1}{s}$ as $n \to \infty$, it necessarily follows that $\varpi_n \to 0$. Let $\dot\Psi_b$ stand for the collection of all continuous non-decreasing functions $\dot\psi : [0,\infty) \to [0,\infty)$ such that $\dot\psi(0) = 0, \dot\psi(t) > 0$ for all $t > 0$, and $\dot\psi(ct) \le c\dot\psi(t)$ for every $c > 1$ and $t \ge 0$.

With these classes at hand, we introduce the concept of a multi-valued $(\dot\alpha,\dot\beta)$-$(\dot\psi,\dot\phi)$-Geraghty type contraction.

Definition 2.3 Let $(\mathcal{X},d)$ be a $\mathcal{S}b\mathcal{MS}$ and $\mathcal{T} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ be a map, and let $\dot\alpha,\dot\beta : \mathcal{X} \to [0,\infty)$ be two functions. We say that $\mathcal{T}$ is a multi-valued $(\dot\alpha,\dot\beta)$-$(\dot\psi,\dot\phi)$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping with respect to the $\mathcal{C}_{\mathcal{G}}$-simulation function $\dot{\zeta}$ if for $\varpi,\omega\in\mathcal{X}$, and there exist $\dot{\psi},\dot{\phi}\in\dot{\Psi}_b$, $\Lambda\in\mathfrak{F}_b$ and $\dot{L}\geq 0$ such that

$$\dot{\alpha}(\varpi)\dot{\beta}(\omega)\geq 1\Rightarrow\dot{\zeta}(\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi,\omega)))\dot{\psi}(\mathcal{M}(\varpi,\omega))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega)))\geq\mathcal{C}_{\mathcal{G}},\tag{2.26}$$

where $\mathcal{M}(\varpi,\omega)=\max\{d(\varpi,\omega),\mathcal{D}(\varpi,\mathcal{T}\varpi),\mathcal{D}(\omega,\mathcal{T}\omega),\frac{\mathcal{D}(\varpi,\mathcal{T}\omega)+\mathcal{D}(\omega,\mathcal{T}\varpi)}{2s}\}$, and
$\mathcal{N}(\varpi,\omega)=\min\{\mathcal{D}(\varpi,\mathcal{T}\varpi),\mathcal{D}(\omega,\mathcal{T}\omega),\mathcal{D}(\varpi,\mathcal{T}\omega),\mathcal{D}(\omega,\mathcal{T}\varpi)\}$.

Theorem 2.5 *Let (X,d) be a complete $\mathcal{S}b\mathcal{M}\mathcal{S}$ and $\mathcal{T}:\mathcal{X}\to\mathcal{CB}(\mathcal{X})$ be a multi-valued $(\dot{\alpha},\dot{\beta})$-$(\dot{\psi},\dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping and also the following conditions hold:*

(a_{3_i}) *$\mathcal{T}$ is a cyclic multi-valued $(\dot{\alpha},\dot{\beta})$-admissible mapping;*

($a_{3_{ii}}$) *there exists $\varpi_0\in\mathcal{X}$ such that $\dot{\alpha}(\varpi_0)\geq 1$ and $\dot{\beta}(\varpi_0)\geq 1$;*

($a_{3_{iii}}$) *$\mathcal{T}$ is continuous.*

Then $\mathcal{T}$ has a fixed-point in $\mathcal{X}$.

Proof: According to condition $(a_{3_{ii}})$ of the theorem, there exists $\varpi_0\in\mathcal{X}$ such that $\dot{\alpha}(\varpi_0)\geq 1$ and $\dot{\beta}(\varpi_0)\geq 1$. Choose $\varpi_1\in\mathcal{T}\varpi_0$. By condition (a_{3_i}), we have $\dot{\beta}(\varpi_1)\geq 1$. Select $\varpi_2\in\mathcal{T}\varpi_1$; then condition (a_{3_i}) implies $\dot{\alpha}(\varpi_2)\geq 1$. Take $\varpi_3\in\mathcal{T}\varpi_2$, so $\dot{\beta}(\varpi_3)\geq 1$ by (a_{3_i}). Iterating this construction yields a sequence (ϖ_n) in $\mathcal{X}$ satisfying

$$\varpi_{n+1}\in\mathcal{T}\varpi_n\quad\text{with}\quad\dot{\alpha}(\varpi_{2n})\geq 1\quad\text{and}\quad\dot{\beta}(\varpi_{2n+1})\geq 1.\tag{2.27}$$

Since $\mathcal{T}$ is a cyclic multi-valued $(\dot{\alpha},\dot{\beta})$-admissible mapping and $\dot{\beta}(\varpi_0)\geq 1$, for the sequence (ϖ_n), we get

$$\dot{\alpha}(\varpi_{2n+1})\geq 1\quad\text{and}\quad\dot{\beta}(\varpi_{2n})\geq 1.\tag{2.28}$$

From equations (2.27) and (2.28), we conclude that

$$\varpi_{n+1}\in\mathcal{T}\varpi_n\quad\text{with}\quad\dot{\alpha}(\varpi_n)\geq 1\quad\text{and}\quad\dot{\beta}(\varpi_n)\geq 1,\quad\forall n\geq 0.$$

We have $\dot{\alpha}(\varpi_0)\dot{\beta}(\varpi_1)\geq 1$. If $\varpi_0=\varpi_1$, then no proof is required. Let $\varpi_0\neq\varpi_1$ and suppose that $\varpi_1\notin\mathcal{T}\varpi_1$, as otherwise ϖ_1 would be a fixed-point of $\mathcal{T}$. Suppose κ is a real number satisfying $\kappa\in(1,s)$. Now,

$$0<\dot{\psi}(\mathcal{D}(\varpi_1,\mathcal{T}\varpi_1))<\kappa\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1)).$$

Thus, there exists $\varpi_2\in\mathcal{T}\varpi_1$ such that

$$\dot{\psi}(d(\varpi_1,\varpi_2))\leq\kappa\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1)).\tag{2.29}$$

Since $\mathcal{T}$ is a multi-valued $(\dot{\alpha},\dot{\beta})$-$(\dot{\psi},\dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping therefore taking $\varpi=\varpi_0$ and $\omega=\varpi_1$ in (2.26), we get

$$\begin{aligned}&\dot{\zeta}\left(\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1))\right)\geq\mathcal{C}_{\mathcal{G}}\\ \Rightarrow\mathcal{C}_{\mathcal{G}}&\leq\dot{\zeta}\left(\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1))\right)\\ &<\mathcal{G}\left(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1)),\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1))\right).\end{aligned}$$

Applying $(\mathcal{G}_1)$, we obtain

$$\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1))<\Lambda(\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1)).\tag{2.30}$$

From (2.29) and (2.30), we get

$$\begin{aligned}\dot{\psi}(d(\varpi_1,\varpi_2))&\leq\kappa\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1))\\&<\kappa\Lambda(\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1))\\&<\tfrac{\kappa}{s}\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1))+\kappa\dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1)),\end{aligned}\tag{2.31}$$

where

$$\begin{aligned}\mathcal{M}(\varpi_0,\varpi_1) &= \max\{d(\varpi_0,\varpi_1),\mathcal{D}(\varpi_0,\mathcal{T}\varpi_0),\mathcal{D}(\varpi_1,\mathcal{T}\varpi_1),\tfrac{\mathcal{D}(\varpi_0,\mathcal{T}\varpi_1)+\mathcal{D}(\varpi_1,\mathcal{T}\varpi_0)}{2s}\}\\ &\leq \max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),\tfrac{\mathcal{D}(\varpi_0,\mathcal{T}\varpi_1)}{2s}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(\varpi_0,\varpi_1) &= \min\{\mathcal{D}(\varpi_0,\mathcal{T}\varpi_0),\mathcal{D}(\varpi_1,\mathcal{T}\varpi_1),\mathcal{D}(\varpi_0,\mathcal{T}\varpi_1),\mathcal{D}(\varpi_1,\mathcal{T}\varpi_0)\}\\ &\leq \min\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_2),d(\varpi_1,\varpi_1)\}=0.\end{aligned}$$

Since

$$\begin{aligned}\tfrac{\mathcal{D}(\varpi_0,\mathcal{T}\varpi_1)}{2s} \leq \tfrac{d(\varpi_0,\varpi_2)}{2s} \leq \tfrac{sd(\varpi_0,\varpi_1)+d(\varpi_1,\varpi_2)}{2s} &\leq \tfrac{d(\varpi_0,\varpi_1)+d(\varpi_1,\varpi_2)}{2}\\ &\leq \max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2)\},\end{aligned}$$

then we have

$$\mathcal{M}(\varpi_0,\varpi_1)=\max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2)\}.$$

Suppose $\mathcal{M}(\varpi_0,\varpi_1)=\max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2)\}=d(\varpi_1,\varpi_2)$, then from (2.31), we get

$$\dot{\psi}(d(\varpi_1,\varpi_2))<\frac{\kappa}{s}\dot{\psi}(d(\varpi_1,\varpi_2)),$$

a contradiction. Hence, we obtain $\mathcal{M}(\varpi_0,\varpi_1)=\max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2)\}=d(\varpi_0,\varpi_1)$, so

$$\dot{\psi}(d(\varpi_1,\varpi_2))<\frac{\kappa}{s}\dot{\psi}(d(\varpi_0,\varpi_1)).$$

As $\frac{\kappa}{s}<1$ and $\dot{\psi}$ is non-decreasing, we have

$$\dot{\psi}(\frac{s}{\kappa}d(\varpi_1,\varpi_2))\leq\frac{s}{\kappa}\dot{\psi}(d(\varpi_1,\varpi_2))<\dot{\psi}(d(\varpi_0,\varpi_1)).$$

Therefore, $d(\varpi_1,\varpi_2)\leq\frac{\kappa}{s}d(\varpi_0,\varpi_1)$.

Since $\varpi_2\in\mathcal{T}\varpi_1$ and $\varpi_1\notin\mathcal{T}\varpi_1$, so it clear that $\varpi_1\neq\varpi_2$. Take $\kappa_1=\frac{\kappa}{s}\frac{\dot{\psi}(d(\varpi_0,\varpi_1))}{\dot{\psi}(d(\varpi_1,\varpi_2))}>1$.

Now, if $\varpi_2\in\mathcal{T}\varpi_2$, then ϖ_2 is a fixed-point of $\mathcal{T}$. Assume that $\varpi_2\notin\mathcal{T}\varpi_2$, and since $\dot{\alpha}(\varpi_1)\dot{\beta}(\varpi_2)\geq 1$, then we have

$$0<\dot{\psi}(\mathcal{D}(\varpi_2,\mathcal{T}\varpi_2))<\kappa_1\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_1,\mathcal{T}\varpi_2)).$$

Hence, there exists $\varpi_3\in\mathcal{T}\varpi_2$, such that

$$\dot{\psi}(d(\varpi_2,\varpi_3))\leq\kappa_1\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_1,\mathcal{T}\varpi_2)).\tag{2.32}$$

By taking $\varpi=\varpi_1$ and $\omega=\varpi_2$ in (2.26), and using $\mathcal{T}$ is a multi-valued $(\dot{\alpha},\dot{\beta})$-$(\dot{\psi},\dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping, we get

$$\begin{aligned}&\dot{\zeta}(\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_1,\mathcal{T}\varpi_2)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2)))\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_1,\varpi_2)))\geq\mathcal{C}_{\mathcal{G}}\\ \Rightarrow\ &\mathcal{C}_{\mathcal{G}}\leq\dot{\zeta}(\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_1,\mathcal{T}\varpi_2)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2)))\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_1,\varpi_2)))\\ &\quad<\mathcal{G}(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2)))\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_1,\varpi_2)),\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_1,\mathcal{T}\varpi_2))).\end{aligned}$$

By applying $(\mathcal{G}_1)$, we obtain that

$$\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_1,\mathcal{T}\varpi_2))<\Lambda(\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2)))\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_1,\varpi_2)).\tag{2.33}$$

Using the inequalities (2.32) and (2.33), we obtain

$$\begin{aligned}\dot{\psi}(d(\varpi_2,\varpi_3)) &\leq \kappa_1\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_1,\mathcal{T}\varpi_2))\\ &< \kappa_1\Lambda(\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2)))\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2))+\kappa_1\dot{L}\dot{\phi}(\mathcal{N}(\varpi_1,\varpi_2))\\ &< \tfrac{\kappa_1}{s}\dot{\psi}(\mathcal{M}(\varpi_1,\varpi_2))+\kappa_1\dot{L}\dot{\phi}(\mathcal{N}(\varpi_1,\varpi_2)).\end{aligned}\tag{2.34}$$

Similarly as above we get $\mathcal{M}(\varpi_1,\varpi_2)\leq d(\varpi_1,\varpi_2)$ and $\mathcal{N}(\varpi_1,\varpi_2)=0$. Hence, from (2.34), we get

$$\dot{\psi}(d(\varpi_2,\varpi_3))<\frac{\kappa_1}{s}\dot{\psi}(d(\varpi_1,\varpi_2))<\frac{\kappa}{s^2}\dot{\psi}(d(\varpi_0,\varpi_1)).$$

Also, since $\frac{\kappa}{s^2} < 1$, and using the property of $\dot{\psi}$, we have

$$\dot{\psi}(\frac{s^2}{\kappa}d(\varpi_2,\varpi_3)) \leq \frac{s^2}{\kappa}\dot{\psi}(d(\varpi_2,\varpi_3)) < \dot{\psi}(d(\varpi_0,\varpi_1)),$$

it leads us $d(\varpi_2,\varpi_3) \leq \frac{\kappa}{s^2}d(\varpi_0,\varpi_1)$. Since $\varpi_3 \in \mathcal{T}\varpi_2$ and $\varpi_2 \notin \mathcal{T}\varpi_2$, so it clear that $\varpi_2 \neq \varpi_3$. Take $\kappa_2 = \frac{\kappa}{s^2}\frac{\dot{\psi}(d(\varpi_0,\varpi_1))}{\dot{\psi}(d(\varpi_2,\varpi_3))} > 1$. By proceeding this way, we generate a sequence $(\varpi_n) \subseteq \mathcal{X}$ such that for all $n \in \mathbb{N}, \varpi_n \in \mathcal{T}\varpi_{n-1}, \varpi_{n-1} \neq \varpi_n, \dot{\alpha}(\varpi_{n-1})\dot{\beta}(\varpi_n) \geq 1$, and $d(\varpi_n,\varpi_{n+1}) \leq \kappa(\frac{1}{s})^n d(\varpi_0,\varpi_1)$.
By completeness of $\mathcal{X}$ and by Proposition 1.1, since $\sum\limits_{n=1}^{\infty} d(\varpi_n,\varpi_{n+1}) < \infty$ we have (ϖ_n) is a Cauchy sequence that converges to some $\varpi^* \in \mathcal{X}$. Through the continuity of $\mathcal{T}$, we derive

$$\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*) = \mathcal{D}(\lim_{n\to\infty}\varpi_{n+1},\mathcal{T}\varpi^*) \leq \mathcal{H}(\lim_{n\to\infty}\mathcal{T}\varpi_n,\mathcal{T}\varpi^*) = 0.$$

Which gives us $\varpi^* \in \mathcal{T}\varpi^*$. This completes the proof. □

Theorem 2.6 *In Theorem 2.5, replace* $(a_{3_{iii}})$ *with* $(a^*_{3_{iii}})$*: If* $(\varpi_n) \subseteq \mathcal{X}$ *converges to* ϖ^* *as* $n \to \infty$ *and* $\dot{\beta}(\varpi_n) \geq 1$ *for all* n*, then* $\dot{\beta}(\varpi^*) \geq 1$*. Consequently,* $\mathcal{T}$ *possesses a fixed-point.*

Proof: By the condition $(a^*_{3_{iii}})$ of the theorem $\dot{\beta}(\varpi^*) \geq 1$. Since $\dot{\alpha}(\varpi_n)\dot{\beta}(\varpi^*) \geq 1$ for all $n \geq 0$. As $(\mathcal{X},d)$ is a $\mathcal{S}b\mathcal{MS}$, we have

$$\begin{aligned}\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*) &\leq sd(\varpi^*,\varpi_{n+1}) + d(\varpi_{n+1},u), u \in \mathcal{T}\varpi^* \\ &\leq sd(\varpi^*,\varpi_{n+1}) + \kappa\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*), \kappa \in (1,s).\end{aligned}$$

Taking as $n \to \infty$ and using the property of $\dot{\psi}$, we have

$$\dot{\psi}(\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)) \leq \kappa \lim_{n\to\infty}\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)). \tag{2.35}$$

By taking $\varpi = \varpi_n, \omega = \varpi^*$ in (2.26), and using $\mathcal{T}$ is a multi-valued $(\dot{\alpha},\dot{\beta})$-$(\dot{\psi},\dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping, we obtain

$$\begin{aligned}&\dot{\zeta}\left(\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*))\right) \geq \mathcal{C}_{\mathcal{G}} \\ \Rightarrow\ &\mathcal{C}_{\mathcal{G}} \leq \dot{\zeta}\left(\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*))\right) \\ &\quad < \mathcal{G}\left(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)),\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))\right).\end{aligned}$$

By utilizing $(\mathcal{G}_1)$, we derive that

$$\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)) < \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)). \tag{2.36}$$

By applying inequalities (2.35) and (2.36), we derive

$$\begin{aligned}\dot{\psi}(\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)) &\leq \kappa\lim_{n\to\infty}\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)) \\ &< \kappa\lim_{n\to\infty}\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)) + \dot{L}\kappa\lim_{n\to\infty}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)).\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}\mathcal{M}(\varpi_n,\varpi^*) &= \max\{d(\varpi_n,\varpi^*),\mathcal{D}(\varpi_n,\mathcal{T}\varpi_n),\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*),\tfrac{\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*)+\mathcal{D}(\varpi^*,\mathcal{T}\varpi_n)}{2s}\} \\ &\leq \max\{d(\varpi_n,\varpi^*),d(\varpi_n,\varpi_{n+1}),\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*),\tfrac{\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(\varpi_n,\varpi^*) &= \min\{\mathcal{D}(\varpi_n,\mathcal{T}\varpi_n),\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*),\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*),\mathcal{D}(\varpi^*,\mathcal{T}\varpi_n)\} \\ &\leq \min\{d(\varpi_n,\varpi_{n+1}),\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*),\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*),d(\varpi^*,\varpi_{n+1})\}.\end{aligned}$$

Since $(\mathcal{X}, d)$ is a $\mathcal{S}b\mathcal{MS}$, we have

$$\frac{\mathcal{D}(\varpi_n, \mathcal{T}\varpi^*) + d(\varpi^*, \varpi_{n+1})}{2s} \leq \frac{sd(\varpi_n, \varpi^*) + \mathcal{D}(\varpi^*, \mathcal{T}\varpi^*) + d(\varpi^*, \varpi_{n+1})}{2s}.$$

Therefore,

$$\lim_{n\to\infty} \frac{\mathcal{D}(\varpi_n, \mathcal{T}\varpi^*) + d(\varpi^*, \varpi_{n+1})}{2s} \leq \frac{\mathcal{D}(\varpi^*, \mathcal{T}\varpi^*)}{2s}.$$

This implies that $\lim_{n\to\infty} \mathcal{M}(\varpi_n, \varpi^*) \leq \mathcal{D}(\varpi^*, \mathcal{T}\varpi^*)$ and $\lim_{n\to\infty} \mathcal{N}(\varpi_n, \varpi^*) = 0$.
Hence, by (2.37), we get

$$\begin{aligned} &\lim_{n\to\infty} \dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n, \mathcal{T}\varpi^*)) = \dot{\psi}(\mathcal{D}(\varpi^*, \mathcal{T}\varpi^*)) \text{ and} \\ &\lim_{n\to\infty} \left[\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi^*)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi^*))\right] = \dot{\psi}(\mathcal{D}(\varpi^*, \mathcal{T}\varpi^*)). \end{aligned}$$

Employing (b) of Definition 1.4 and (2.26) together, we derive

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} \quad &\leq \limsup_{n\to\infty} \dot{\zeta}\left(\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n, \mathcal{T}\varpi^*)), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n, \varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n, \varpi^*)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_n, \varpi^*))\right) \\ &< \mathcal{C}_{\mathcal{G}}, \end{aligned}$$

which means that $\dot{\psi}(\mathcal{D}(\varpi^*, \mathcal{T}\varpi^*)) = 0$, then $\mathcal{D}(\varpi^*, \mathcal{T}\varpi^*) = 0$; and that $\varpi^* \in \mathcal{T}\varpi^*$ and hence that ϖ^* is a fixed-point of $\mathcal{T}$. □

Definition 2.4 Let $(\mathcal{X}, d)$ be a $\mathcal{S}b\mathcal{MS}$ and $\mathcal{T} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ be a map, and let $\dot{\alpha} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function. We say that $\mathcal{T}$ is a multi-valued $(\dot{\alpha}, \dot{\psi}, \dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping with respect to the $\mathcal{C}_{\mathcal{G}}$-simulation function $\dot{\zeta}$ if for $\varpi, \omega \in \mathcal{X}$, and there exist $\dot{\psi}, \dot{\phi} \in \dot{\Psi}_b$, $\Lambda \in \mathfrak{F}_b$ and $\dot{L} \geq 0$ such that

$$\dot{\zeta}\left(\dot{\alpha}(\varpi, \omega)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi, \mathcal{T}\omega)), \Lambda(\dot{\psi}(\mathcal{M}(\varpi, \omega)))\dot{\psi}(\mathcal{M}(\varpi, \omega)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi, \omega))\right) \geq \mathcal{C}_{\mathcal{G}}, \tag{2.38}$$

where $\mathcal{M}(\varpi, \omega) = \max\{d(\varpi, \omega), \mathcal{D}(\varpi, \mathcal{T}\varpi), \mathcal{D}(\omega, \mathcal{T}\omega), \frac{\mathcal{D}(\varpi, \mathcal{T}\omega) + \mathcal{D}(\omega, \mathcal{T}\varpi)}{2s}\}$, and
$\mathcal{N}(\varpi, \omega) = \min\{\mathcal{D}(\varpi, \mathcal{T}\varpi), \mathcal{D}(\omega, \mathcal{T}\omega), \mathcal{D}(\varpi, \mathcal{T}\omega), \mathcal{D}(\omega, \mathcal{T}\varpi)\}$.

Theorem 2.7 *Let (X, d) be a complete $\mathcal{S}b\mathcal{MS}$ and $\mathcal{T} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ be a multi-valued $(\dot{\alpha}, \dot{\psi}, \dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping and also the following conditions hold:*

(a_{4_i}) *$\mathcal{T}$ is a multi-valued $\dot{\alpha}$-admissible mapping;*

($a_{4_{ii}}$) *there exists $\varpi_0 \in \mathcal{X}$ and $\varpi_1 \in \mathcal{T}\varpi_0$ such that $\dot{\alpha}(\varpi_0, \varpi_1) \geq 1$;*

($a_{4_{iii}}$) *$\mathcal{T}$ is continuous.*

Then $\mathcal{T}$ has a fixed-point in $\mathcal{X}$.

Proof: According to condition ($a_{4_{ii}}$) of the theorem, there exists $\varpi_0 \in \mathcal{X}$ and $\varpi_1 \in \mathcal{T}\varpi_0$ such that $\dot{\alpha}(\varpi_0, \varpi_1) \geq 1$. If $\varpi_1 = \varpi_0$, then there is no need for further proof. Suppose $\varpi_1 \neq \varpi_0$ and $\varpi_1 \notin \mathcal{T}\varpi_1$, as otherwise ϖ_1 would be a fixed-point of $\mathcal{T}$. Let τ be a real number such that $\tau \in (1, s)$. Then

$$0 < \dot{\psi}(\mathcal{D}(\varpi_1, \mathcal{T}\varpi_1)) < \tau\dot{\alpha}(\varpi_0, \varpi_1)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0, \mathcal{T}\varpi_1)).$$

Hence, there exists $\varpi_2 \in \mathcal{T}\varpi_1$ such that

$$\dot{\psi}(d(\varpi_1, \varpi_2)) \leq \tau\dot{\alpha}(\varpi_0, \varpi_1)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0, \mathcal{T}\varpi_1)). \tag{2.39}$$

Since $\mathcal{T}$ is a multi-valued $(\dot{\alpha}, \dot{\psi}, \dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping therefore taking $\varpi = \varpi_0$ and $\omega = \varpi_1$ in (2.38), we get

$$\begin{aligned} &\dot{\zeta}\left(\dot{\alpha}(\varpi_0, \varpi_1)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0, \mathcal{T}\varpi_1)), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_0, \varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0, \varpi_1)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_0, \varpi_1))\right) \geq \mathcal{C}_{\mathcal{G}} \\ \Rightarrow \mathcal{C}_{\mathcal{G}} &\leq \dot{\zeta}\left(\dot{\alpha}(\varpi_0, \varpi_1)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0, \mathcal{T}\varpi_1)), \Lambda(\dot{\psi}(\mathcal{M}(\varpi_0, \varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0, \varpi_1)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_0, \varpi_1))\right) \\ &< \mathcal{G}\left(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_0, \varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0, \varpi_1)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_0, \varpi_1)), \dot{\alpha}(\varpi_0, \varpi_1)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0, \mathcal{T}\varpi_1))\right). \end{aligned}$$

By applying $(\mathcal{G}_1)$, we obtain that

$$\dot{\alpha}(\varpi_0,\varpi_1)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1)) < \Lambda(\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1)). \tag{2.40}$$

From (2.39) and (2.40), we get

$$\begin{aligned} \dot{\psi}(d(\varpi_1,\varpi_2)) &\leq \tau\dot{\alpha}(\varpi_0,\varpi_1)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_0,\mathcal{T}\varpi_1)) \\ &< \tau\Lambda(\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)))\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1)) \\ &< \tfrac{\tau}{s}\dot{\psi}(\mathcal{M}(\varpi_0,\varpi_1)) + \kappa\dot{L}\dot{\phi}(\mathcal{N}(\varpi_0,\varpi_1)), \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} \mathcal{M}(\varpi_0,\varpi_1) &= \max\{d(\varpi_0,\varpi_1),\mathcal{D}(\varpi_0,\mathcal{T}\varpi_0),\mathcal{D}(\varpi_1,\mathcal{T}\varpi_1),\tfrac{\mathcal{D}(\varpi_0,\mathcal{T}\varpi_1)+\mathcal{D}(\varpi_1,\mathcal{T}\varpi_0)}{2s}\} \\ &\leq \max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),\tfrac{\mathcal{D}(\varpi_0,\mathcal{T}\varpi_1)}{2s}\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}(\varpi_0,\varpi_1) &= \min\{\mathcal{D}(\varpi_0,\mathcal{T}\varpi_0),\mathcal{D}(\varpi_1,\mathcal{T}\varpi_1),\mathcal{D}(\varpi_0,\mathcal{T}\varpi_1),\mathcal{D}(\varpi_1,\mathcal{T}\varpi_0)\} \\ &\leq \min\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_2),d(\varpi_1,\varpi_1)\} = 0. \end{aligned}$$

Since

$$\begin{aligned} \tfrac{\mathcal{D}(\varpi_0,\mathcal{T}\varpi_1)}{2s} \leq \tfrac{d(\varpi_0,\varpi_2)}{2s} \leq \tfrac{sd(\varpi_0,\varpi_1)+d(\varpi_1,\varpi_2)}{2s} &\leq \tfrac{d(\varpi_0,\varpi_1)+d(\varpi_1,\varpi_2)}{2} \\ &\leq \max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2)\}, \end{aligned}$$

then we have

$$\mathcal{M}(\varpi_0,\varpi_1) = \max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2)\}.$$

Suppose $\mathcal{M}(\varpi_0,\varpi_1) = \max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2)\} = d(\varpi_1,\varpi_2)$, then from (2.41), we get

$$\dot{\psi}(d(\varpi_1,\varpi_2)) < \frac{\tau}{s}\dot{\psi}(d(\varpi_1,\varpi_2)),$$

a contradiction. Hence, we obtain $\mathcal{M}(\varpi_0,\varpi_1) = \max\{d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2)\} = d(\varpi_0,\varpi_1)$, so

$$\dot{\psi}(d(\varpi_1,\varpi_2)) < \frac{\tau}{s}\dot{\psi}(d(\varpi_0,\varpi_1)).$$

As $\frac{\tau}{s} < 1$ and $\dot{\psi}$ is non-decreasing, we have

$$\dot{\psi}(\frac{s}{\tau}d(\varpi_1,\varpi_2)) \leq \frac{s}{\tau}\dot{\psi}(d(\varpi_1,\varpi_2)) < \dot{\psi}(d(\varpi_0,\varpi_1)).$$

Therefore, $d(\varpi_1,\varpi_2) \leq \frac{\tau}{s}d(\varpi_0,\varpi_1)$.

Since $\varpi_2 \in \mathcal{T}\varpi_1$ and $\varpi_1 \notin \mathcal{T}\varpi_1$, so it clear that $\varpi_1 \neq \varpi_2$. Take $\tau_1 = \frac{\tau}{s}\frac{\dot{\psi}(d(\varpi_0,\varpi_1))}{\dot{\psi}(d(\varpi_1,\varpi_2))} > 1$.

Following the approach of Theorem 2.5, we construct a sequence $(\varpi_n) \subseteq \mathcal{X}$ satisfying the condition for all $n \in \mathbb{N}, \varpi_n \in \mathcal{T}\varpi_{n-1}, \varpi_{n-1} \neq \varpi_n, \dot{\alpha}(\varpi_{n-1},\varpi_n) \geq 1$, and $d(\varpi_n,\varpi_{n+1}) \leq \tau(\frac{1}{s})^n d(\varpi_0,\varpi_1)$.

By completeness of $\mathcal{X}$ and by Proposition 1.1, since $\sum\limits_{n=1}^{\infty} d(\varpi_n,\varpi_{n+1}) < \infty$ we have (ϖ_n) is a Cauchy sequence that converges to some $\varpi^* \in \mathcal{X}$. Through the continuity of $\mathcal{T}$, we derive

$$\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*) = \mathcal{D}(\lim_{n\to\infty}\varpi_{n+1},\mathcal{T}\varpi^*) \leq \mathcal{H}(\lim_{n\to\infty}\mathcal{T}\varpi_n,\mathcal{T}\varpi^*) = 0.$$

Which gives us $\varpi^* \in \mathcal{T}\varpi^*$. This completes the proof. □

Theorem 2.8 *In Theorem 2.7, replace* $(a_{4_{iii}})$ *with* $(a^*_{4_{iii}})$*: for all* $n \in \mathbb{N}$*, if* $(\varpi_n) \subseteq \mathcal{X}$ *converges to* ϖ^* *as* $n \to \infty$ *and* $\dot{\alpha}(\varpi_n,\varpi_{n+1}) \geq 1$*, then* $\dot{\alpha}(\varpi_n,\varpi^*) \geq 1$*. Consequently,* $\mathcal{T}$ *possesses a fixed-point.*

Proof: By the condition $(a^*_{4_{iii}})$ of the theorem $\dot{\alpha}(\varpi_n,\varpi^*) \geq 1$, for all n, and since $(\mathcal{X},d)$ is a $\mathcal{S}b\mathcal{MS}$, we have

$$\begin{aligned} \mathcal{D}(\varpi^*,\mathcal{T}\varpi^*) &\leq sd(\varpi^*,\varpi_{n+1}) + d(\varpi_{n+1},u), u \in \mathcal{T}\varpi^* \\ &\leq sd(\varpi^*,\varpi_{n+1}) + \tau\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*), \tau \in (1,s). \end{aligned}$$

Taking as $n \to \infty$ and using the property of $\dot{\psi}$, we have

$$\begin{aligned}\dot{\psi}(\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)) &\leq \tau \lim_{n\to\infty} \dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))\\ &\leq \tau \lim_{n\to\infty} \dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)).\end{aligned} \tag{2.42}$$

By taking $\varpi = \varpi_n, \omega = \varpi^*$ in (2.38), and using $\mathcal{T}$ is a multi-valued $(\dot{\alpha},\dot{\psi},\dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_\mathcal{G}}$-Geraghty type contraction mapping, we obtain

$$\begin{aligned}&\dot{\zeta}\left(\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*))\right) \geq \mathcal{C}_\mathcal{G}\\ \Rightarrow \mathcal{C}_\mathcal{G} &\leq \dot{\zeta}\left(\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*))\right)\\ &< \mathcal{G}\left(\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)),\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))\right).\end{aligned}$$

By utilizing $(\mathcal{G}_1)$, we derive that

$$\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)) < \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)). \tag{2.43}$$

By applying inequalities (2.42) and (2.43), we derive

$$\begin{aligned}\dot{\psi}(\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)) &\leq \tau \lim_{n\to\infty} \dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*))\\ &< \tau \lim_{n\to\infty} \Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\tau \lim_{n\to\infty} \dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*)).\end{aligned} \tag{2.44}$$

where

$$\begin{aligned}\mathcal{M}(\varpi_n,\varpi^*) &= \max\{d(\varpi_n,\varpi^*),\mathcal{D}(\varpi_n,\mathcal{T}\varpi_n),\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*),\tfrac{\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*)+\mathcal{D}(\varpi^*,\mathcal{T}\varpi_n)}{2s}\}\\ &\leq \max\{d(\varpi_n,\varpi^*),d(\varpi_n,\varpi_{n+1}),\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*),\tfrac{\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(\varpi_n,\varpi^*) &= \min\{\mathcal{D}(\varpi_n,\mathcal{T}\varpi_n),\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*),\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*),\mathcal{D}(\varpi^*,\mathcal{T}\varpi_n)\}\\ &\leq \min\{d(\varpi_n,\varpi_{n+1}),\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*),\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*),d(\varpi^*,\varpi_{n+1})\}.\end{aligned}$$

Since $(\mathcal{X},d)$ is a $\mathcal{S}b\mathcal{MS}$, we have

$$\frac{\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s} \leq \frac{sd(\varpi_n,\varpi^*)+\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s}.$$

Therefore,

$$\lim_{n\to\infty} \frac{\mathcal{D}(\varpi_n,\mathcal{T}\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s} \leq \frac{\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)}{2s}.$$

This implies that $\lim_{n\to\infty} \mathcal{M}(\varpi_n,\varpi^*) \leq \mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)$ and $\lim_{n\to\infty} \mathcal{N}(\varpi_n,\varpi^*) = 0$. Hence, by (2.44), we get

$$\begin{aligned}&\lim_{n\to\infty} \dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)) = \dot{\psi}(\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*))\\ &\lim_{n\to\infty} \left[\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*))\right] = \dot{\psi}(\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)).\end{aligned}$$

From (2.38) and (b) of Definition 1.4, we have

$$\begin{aligned}\mathcal{C}_\mathcal{G} &\leq \dot{\zeta}\left(\dot{\alpha}(\varpi_n,\varpi^*)\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi_n,\mathcal{T}\varpi^*)),\Lambda(\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*)))\dot{\psi}(\mathcal{M}(\varpi_n,\varpi^*))+\dot{L}\dot{\phi}(\mathcal{N}(\varpi_n,\varpi^*))\right)\\ &< \mathcal{C}_\mathcal{G},\end{aligned}$$

which means that $\dot{\psi}(\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*)) = 0$, then $\mathcal{D}(\varpi^*,\mathcal{T}\varpi^*) = 0$; and that $\varpi^* \in \mathcal{T}\varpi^*$ and hence that ϖ^* is a fixed-point of $\mathcal{T}$. □

Corollary 2.2 [[35] Theorem 2.5 and 2.6] *Let $(\mathcal{X},d)$ be a complete $\mathcal{S}b\mathcal{MS}$ and $\mathcal{T}:\mathcal{X}\to\mathcal{CB}(\mathcal{X})$ be a multi-valued mapping satisfying the following conditions:*

(i) $\dot\alpha(\varpi,\omega)\dot\psi(\mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega)) < \Lambda(\dot\psi(\mathcal{M}(\varpi,\omega)))\dot\psi(\mathcal{M}(\varpi,\omega)) + \dot{L}\dot\phi(\mathcal{N}(\varpi,\omega))$,
where $\mathcal{M}(\varpi,\omega) = \max\{d(\varpi,\omega), \mathcal{D}(\varpi,\mathcal{T}\varpi), \mathcal{D}(\omega,\mathcal{T}\omega), \frac{\mathcal{D}(\varpi,\mathcal{T}\omega)+\mathcal{D}(\omega,\mathcal{T}\varpi)}{2s}\}$, *and*
$\mathcal{N}(\varpi,\omega) = \min\{\mathcal{D}(\varpi,\mathcal{T}\varpi), \mathcal{D}(\omega,\mathcal{T}\varpi)\}$.

(ii) there exist $\omega_0 \in \alpha$ *and* $\omega_1 \in \mathcal{T}\omega_0$ *such that* $\dot\alpha(\omega_0,\omega_1) \geq 1$;

(iii) $\mathcal{T}$ *is triangular* $\dot\alpha$*-orbital admissible;*

(iv) either

(iv$_a$) $\mathcal{T}$ *is continuous, or (iv$_b$) if* $\{\omega_n\} \subset \mathcal{X}$ *satisfies* $\dot\alpha(\omega_n,\omega_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $\omega_n \to \omega$ *as* $n \to \infty$, *then* $\dot\alpha(\omega_n,\omega) \geq 1$ *for all* $n \in \mathbb{N}$.

Then $\mathcal{T}$ *has a fixed-point.*

Proof: Define $\dot\zeta_\mathcal{G}(\varpi,\omega) = \kappa\omega - \varpi, \kappa \in (0,1), \mathcal{C}_\mathcal{G} = 0$. Then Theorems 2.7 and 2.8 ensure the result. □

Corollary 2.3 *Let* $(\mathcal{X}, d)$ *be a complete* $\mathcal{S}b\mathcal{MS}$ *and* $\mathcal{T} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *be a multi-valued mapping satisfying the following conditions:*

(i) $\dot\zeta\Big(\dot\alpha(\varpi,\omega)\mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega), \Lambda((\mathcal{M}(\varpi,\omega))\mathcal{M}(\varpi,\omega) + \dot{L}\mathcal{N}(\varpi,\omega)\Big) \geq \mathcal{C}_\mathcal{G}$,
where $\mathcal{M}(\varpi,\omega) = \max\{d(\varpi,\omega), \mathcal{D}(\varpi,\mathcal{T}\varpi), \mathcal{D}(\omega,\mathcal{T}\omega), \frac{\mathcal{D}(\varpi,\mathcal{T}\omega)+\mathcal{D}(\omega,\mathcal{T}\varpi)}{2s}\}$, *and*
$\mathcal{N}(\varpi,\omega) = \min\{\mathcal{D}(\varpi,\mathcal{T}\varpi), \mathcal{D}(\omega,\mathcal{T}\omega), \mathcal{D}(\varpi,\mathcal{T}\omega), \mathcal{D}(\omega,\mathcal{T}\varpi)\}$.

(ii) there exist $\omega_0 \in \alpha$ *and* $\omega_1 \in \mathcal{T}\omega_0$ *such that* $\dot\alpha(\omega_0,\omega_1) \geq 1$;

(iii) $\mathcal{T}$ *is triangular* $\dot\alpha$*-orbital admissible;*

(iv) either

(iv$_a$) $\mathcal{T}$ *is continuous, or (iv$_b$) if* $\{\omega_n\} \subset \mathcal{X}$ *satisfies* $\dot\alpha(\omega_n,\omega_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $\omega_n \to \omega$ *as* $n \to \infty$, *then* $\dot\alpha(\omega_n,\omega) \geq 1$ *for all* $n \in \mathbb{N}$.

Then $\mathcal{T}$ *has a fixed-point.*

Proof: Define $\dot\psi(t) = \dot\phi(t) = t, \forall\, t > 0$. Then Theorems 2.7 and 2.8 ensure the result. □

The purpose of the following example is to demonstrate the validity of Theorem 2.6.

Example 2.2 Let $\mathcal{X} = [0., \infty)$ equipped with the metric $d(x,y) = \max\{|x-y|, 2|x-y|-1\}$. Then clearly $(\mathcal{X}, d)$ is a complete $\mathcal{S}b\mathcal{MS}$ with coefficient $s = 2$. Let $\mathcal{T} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ is defined as follows:

$$\mathcal{T}(\varpi) = \begin{cases} (\frac{\varpi}{20}), & 0 \leq \varpi \leq 1, \\ \left[\varpi + \frac{1}{\varpi} - \frac{1}{m}, m+1\right], & m-1 \leq \varpi \leq m, m \geq 2. \end{cases}$$

Let $\dot\alpha, \dot\beta : \mathcal{X} \to [0,\infty)$ be respectively defined as follows:

$$\dot\alpha(\varpi) = \begin{cases} e^\varpi, & 0 \leq \varpi \leq 1, \\ \frac{1}{8}, & \varpi > 1, \end{cases} \quad \text{and } \dot\beta(\varpi) = \begin{cases} \varpi + 1, & 0 \leq \varpi \leq 1, \\ \frac{1}{4}, & \varpi > 1. \end{cases}$$

Let $\dot\psi, \dot\phi : [0,\infty) \to [0,\infty)$ be respectively defined as follows:

$$\dot\psi(\omega) = \begin{cases} \frac{\omega}{2}, & 0 \leq \omega \leq 1, \\ \frac{1}{2}, & \omega > 1, \end{cases} \quad \text{and } \dot\phi(\omega) = \begin{cases} \frac{\omega}{3}, & 0 \leq \omega \leq 1, \\ \frac{1}{3}, & \omega > 1. \end{cases}$$

Clearly $\dot\psi, \dot\phi \in \dot\Psi_b$. The functions defined as $\dot\zeta(\varpi,\omega) = \frac{8}{9}\omega - \varpi$, $\mathcal{G}(\omega,\varpi) = \omega - \varpi$, and $\Lambda(l) = \frac{1}{1+l}$ for all $l > 0, \varpi, \omega \in [0,\infty)$ are specified mappings. It is evident that Λ, $\mathcal{G}$, and $\dot\zeta$ correspond to a Geraghty function, a $\mathcal{C}$-function, and a $\mathcal{C}_\mathcal{G}$-simulation function, respectively.
(i) Assume that $\varpi \in \mathcal{X}$ and $\dot\alpha(\varpi) \geq 1$. Then, $\varpi \in [0,1]$ and $\mathcal{T}\varpi \subseteq [0,1]$. Consequently, $\dot\beta(\omega) \geq 1$ for every $\omega \in \mathcal{T}\varpi$. Likewise, if $\omega \in \mathcal{X}$ and $\dot\beta(\omega) \geq 1$, it can be demonstrated that $\dot\alpha(\varpi) \geq 1$ for all $\varpi \in \mathcal{T}\omega$. Thus, $\mathcal{T}$ is a cyclic $(\dot\alpha, \dot\beta)$-admissible mapping.
(ii) $\dot\alpha(\varpi) \geq 1$ and $\dot\beta(\varpi) \geq 1$ for every $\varpi \in [0,1]$.
(iii) Assume that (ϖ_n) is a sequence in $\mathcal{X}$ converging to ϖ^* as n approaches ∞, with $\dot\beta(\varpi_n) \geq 1$ for all

n. Consequently, (ϖ_n) is a sequence in $[0,1]$, and ϖ^* also lies in $[0,1]$. It follows that $\dot{\beta}(\varpi^*) \geq 1$.
(iv) Suppose $\varpi, \omega \in \mathcal{X}$ such that $\dot{\alpha}(\varpi)\dot{\beta}(\omega) \geq 1$. This condition implies that both ϖ and ω belong to $[0,1]$. Hence, it is sufficient to check the inequality (2.20) only for $\varpi, \omega \in [0,1]$. Without loss of generality, we assume $\varpi, \omega \in [0,1]$ with $\varpi \geq \omega$. Then

$$\begin{array}{l}\mathcal{H}(\mathcal{T}\varpi, \mathcal{T}\omega) = \frac{\varpi-\omega}{20}, d(\varpi,\omega) = \varpi - \omega, \mathcal{D}(\varpi, \mathcal{T}\varpi) = \varpi - \frac{\varpi}{20},\\ \mathcal{D}(\omega, \mathcal{T}\omega) = \omega - \frac{\omega}{20}, \mathcal{D}(\varpi, \mathcal{T}\omega) = \varpi - \frac{\omega}{20}, \mathcal{D}(\omega, \mathcal{T}\varpi) = |\omega - \frac{\varpi}{20}| \quad \text{and let } \dot{L} \geq 0.\end{array}$$

We consider

$$\begin{aligned}\dot{\psi}(\mathcal{H}(\mathcal{T}\varpi, \mathcal{T}\omega)) = \tfrac{\varpi-\omega}{40} \quad &\leq \tfrac{8}{9}\left[\left(\frac{\frac{\varpi-\omega}{2}}{1+\frac{\varpi-\omega}{2}}\right)\frac{\varpi-\omega}{2} + \dot{L}\,\frac{\min\{\varpi-\frac{\varpi}{20}, \omega-\frac{\omega}{20}, \varpi-\frac{\omega}{20}, |\omega-\frac{\varpi}{20}|\}}{3}\right]\\ &\leq \tfrac{8}{9}\left[\left(\frac{\frac{\mathcal{M}(\varpi,\omega)}{2}}{1+\frac{\mathcal{M}(\varpi,\omega)}{2}}\right)\frac{\mathcal{M}(\varpi,\omega)}{2} + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega))\right]\\ &= \tfrac{8}{9}\left[\left(\Lambda(\dot{\psi}(\mathcal{M}(\varpi,\omega)))\right)\dot{\psi}(\mathcal{M}(\varpi,\omega)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega))\right],\end{aligned}$$

where $\mathcal{M}(\varpi,\omega) = \max\{d(\varpi,\omega), \mathcal{D}(\varpi,\mathcal{T}\varpi), \mathcal{D}(\omega,\mathcal{T}\omega), \frac{\mathcal{D}(\varpi,\mathcal{T}\omega)+\mathcal{D}(\omega,\mathcal{T}\varpi)}{2s}\}$.
So,

$$\begin{aligned}&\dot{\zeta}\left(\mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega)), \Lambda(\dot{\psi}(\mathcal{M}(\varpi,\omega)))\dot{\psi}(\mathcal{M}(\varpi,\omega)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega))\right)\\ &= \tfrac{8}{9}\left[\Lambda(\dot{\psi}(\mathcal{M}(\varpi,\omega)))\dot{\psi}(\mathcal{M}(\varpi,\omega)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega))\right] - \mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega))\\ &= \tfrac{8}{9}\left[\left(\frac{\frac{\mathcal{M}(\varpi,\omega)}{2}}{1+\frac{\mathcal{M}(\varpi,\omega)}{2}}\right)\frac{\mathcal{M}(\varpi,\omega)}{2} + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega))\right] - \mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega))\end{aligned} \tag{2.45}$$

and

$$\begin{aligned}&\mathcal{G}(\Lambda(\dot{\psi}(\mathcal{M}(\varpi,\omega)))\dot{\psi}(\mathcal{M}(\varpi,\omega)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega)), \mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega)))\\ &= \Lambda(\dot{\psi}(\mathcal{M}(\varpi,\omega)))\dot{\psi}(\mathcal{M}(\varpi,\omega)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega)) - \mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega))\\ &= \left(\frac{\frac{\mathcal{M}(\varpi,\omega)}{2}}{1+\frac{\mathcal{M}(\varpi,\omega)}{2}}\right)\frac{\mathcal{M}(\varpi,\omega)}{2} + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega)) - \mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega))\end{aligned} \tag{2.46}$$

Based on equations (2.45) and (2.46), we deduce

$$\begin{aligned}0 \quad &< \dot{\zeta}(\mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega)), \Lambda(\dot{\psi}(\mathcal{M}(\varpi,\omega)))\dot{\psi}(\mathcal{M}(\varpi,\omega)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega)))\\ &< \mathcal{G}(\Lambda(\dot{\psi}(\mathcal{M}(\varpi,\omega)))\dot{\psi}(\mathcal{M}(\varpi,\omega)) + \dot{L}\dot{\phi}(\mathcal{N}(\varpi,\omega)), \mathcal{H}(\mathcal{T}\varpi,\mathcal{T}\omega)))\end{aligned} \tag{2.47}$$

By applying inequality (2.47) and Definition 2.3, it is evident that the mapping $\mathcal{T}$ constitutes a multi-valued $(\dot{\alpha}, \dot{\beta})$-$(\dot{\psi}, \dot{\phi})$-$\mathcal{Z}_{\mathcal{C}_{\mathcal{G}}}$-Geraghty type contraction mapping with $\mathcal{C}_{\mathcal{G}} = 0$. Hence, the conditions of Theorem 2.6 are satisfied, confirming that $\mathcal{T}$ has fixed-points in $\mathcal{X}$.

3. Applications

3.1. Application to integral equation

Suppose $\tilde{\Omega} = \tilde{C}[a_l, b_u]$ is a set of real valued continuous functions on $[a_l, b_u]$. We define $d : \tilde{\Omega} \times \tilde{\Omega} \to \mathbb{R}^+$ by $d(\zeta, \xi) = \max\limits_{\varsigma \in [a_l, b_u]} \{|\zeta(\tau) - \xi(\tau)|, 2|\zeta(\tau) - \xi(\tau)| - 1\}$, for all $\zeta, \xi \in \tilde{\Omega}$. Then $(\tilde{\Omega}, d)$ is a complete $\mathcal{S}b\mathcal{MS}$ with $s = 2$. In this section, we present a solution to nonlinear integral equations of the Fredholm type defined by

$$\zeta(\varsigma) = \aleph(\varsigma) + \mu \int\limits_{a_l}^{b_u} \tilde{\mathcal{D}}(\varsigma, \tau, \zeta(\varsigma)) d\tau \tag{3.1}$$

where $\zeta \in \tilde{C}[a_l, b_u], \mu \in \mathbb{R}, \varsigma, \tau \in [a_l, b_u], \tilde{\mathcal{D}} : [a_l, b_u] \times [a_l, b_u] \times \mathbb{R} \to \mathbb{R}$ and $\aleph : [a_l, b_u] \to \mathbb{R}$ are continuous. Let $\Im : \tilde{\Omega} \to \tilde{\Omega}$ be a mapping defined as

$$\Im(\zeta(\varsigma)) = \aleph(\varsigma) + \mu \int\limits_{a_l}^{b_u} \tilde{\mathcal{D}}(\varsigma, \tau, \zeta(\varsigma)) d\tau \tag{3.2}$$

Considering the following:

$(\Im_1)$ $\gamma : [a_l, b_u] \times [a_l, b_u] \to \mathbb{R}^+$ is continuous with $\int\limits_{a_l}^{b_u} \gamma(\varsigma,\tau)d\tau \leq \frac{1}{b_u - a_l}$ and $|\mu| \leq 1$;

$(\Im_2)$ there exists $\zeta_0 \in \tilde{\Omega}$ such that $\eta_1(\ \zeta_0) \geq 0$ and $\eta_2(\zeta_0) \geq 0$;

$(\Im_3)$ if $\{\zeta_n\} \subseteq \tilde{\Omega}$ such that $\lim\limits_{n\to\infty} \zeta_n = \zeta$ and $\eta_2(\zeta_n) \geq 0$ for all n, then $\eta_2(\zeta) \geq 0$;

$(\Im_4)$ $\eta_1(\zeta) \geq 0$ for some $\zeta \in \tilde{\Omega} \Rightarrow \eta_2(\Im\zeta) \geq 0$ and
$\eta_2(\zeta) \geq 0$ for some $\zeta \in \tilde{\Omega} \Rightarrow \eta_1(\Im\zeta) \geq 0$;

$(\Im_5)$ $\eta_1(u) \geq 0$ and $\eta_2(v) \geq 0$ whenever $\Im u = u$ and $\Im v = v$;

$(\Im_6)$ if $\eta_1(\Im\zeta) \geq 0$, $\eta_2(\Im\xi) \geq 0$, for all $\zeta, \xi \in \tilde{\Omega}$ such that for $\varsigma, \tau \in [a_l, b_u]$, the following holds:
$|\tilde{\mathcal{D}}(\varsigma,\tau,\zeta(\varsigma)) - \tilde{\mathcal{D}}(\varsigma,\tau,\xi(\varsigma))| \leq \gamma(\varsigma,\tau)\frac{c(b_u-a_l)}{2k}\left[\frac{\mathcal{M}(\zeta,\xi)}{1+\mathcal{M}(\zeta,\xi)} + \dot{L}\mathcal{N}(\zeta,\xi)\right]$
where $\mathcal{M}(\zeta,\xi) = \max\{d(\zeta,\xi), d(\zeta,\Im\zeta), d(\xi,\Im\xi), \frac{d(\zeta,\Im\xi)+d(\xi,\Im\zeta)}{2s}\}$ and
$\mathcal{N}(\zeta,\xi) = \min\{d(\zeta,\Im\zeta), d(\xi,\Im\xi), d(\zeta,\Im\xi), d(\xi,\Im\zeta)\}$.

Theorem 3.1 *Let $\Im : \tilde{\Omega} \to \tilde{\Omega}$ defined by (3.2) for which the conditions $(\Im_1) - (\Im_6)$ hold. Then (3.1) has a unique solution in $\tilde{\Omega}$.*

Proof: We define $\dot{\alpha}, \dot{\beta} : \tilde{\Omega} \to [0,\infty)$ as
$\dot{\alpha}(\zeta) = \begin{cases} 1, & \eta_1(\zeta) > 0 \text{ where } \zeta \in \tilde{\Omega}, \\ 0, & \text{otherwise,} \end{cases}$ and $\dot{\beta}(\zeta) = \begin{cases} 1, & \eta_2(\zeta) > 0 \text{ where } \zeta \in \tilde{\Omega}, \\ 0, & \text{otherwise.} \end{cases}$
From the condition $(\Im_6), \eta_1(\Im\zeta) \geq 0$ and $\eta_2(\Im\xi) \geq 0$, for all $\zeta, \xi \in \tilde{\Omega}$ so that $\dot{\alpha}(\Im\zeta)\dot{\beta}(\Im\xi) \geq 1$. Therefore $\Im$ is a cyclic $(\dot{\alpha},\dot{\beta})$-admissible map. $\zeta, \xi \in \tilde{\Omega}$ and from $(\Im_1)$ and $(\Im_6)$, for all ς. So we have

$$\begin{aligned} |\Im\zeta(\varsigma) - \Im\xi(\varsigma)| &= \left|\mu\left(\int\limits_{a_l}^{b_u}\tilde{\mathcal{D}}(\varsigma,\tau,\zeta(\varsigma))d\tau - \int\limits_{a_l}^{b_u}\tilde{\mathcal{D}}(\varsigma,\tau,\xi(\varsigma))\right)d\tau\right| \\ &\leq |\mu|\int\limits_{a_l}^{b_u}\left|\tilde{\mathcal{D}}(\varsigma,\tau,\zeta(\varsigma)) - \tilde{\mathcal{D}}(\varsigma,\tau,\xi(\varsigma))\right|d\tau \\ &\leq \int\limits_{a_l}^{b_u}\gamma(\varsigma,\tau)\frac{c(b_u-a_l)}{2k}\left[\frac{\mathcal{M}(\zeta,\xi)}{(1+\mathcal{M}(\zeta,\xi))} + \dot{L}\mathcal{N}(\zeta,\xi)\right]d\tau \\ &= \frac{c}{2k}\left[\frac{\mathcal{M}(\zeta,\xi)}{(1+\mathcal{M}(\zeta,\xi))} + \dot{L}\mathcal{N}(\zeta,\xi)\right]. \end{aligned}$$

Now we have two cases

(I_i) If $d(\Im\zeta,\Im\xi) = |\Im\zeta - \Im\xi|$, then

$$d(\Im\zeta,\Im\xi) \leq \frac{c}{2k}\left[\frac{\mathcal{M}(\zeta,\xi)}{(1+\mathcal{M}(\zeta,\xi))} + \dot{L}\mathcal{N}(\zeta,\xi)\right].$$

(I_{ii}) If $d(\Im\zeta,\Im\xi) = 2|\Im\zeta - \Im\xi| - 1$, then

$$d(\Im\zeta,\Im\xi) \leq \frac{c}{k}\left[\frac{\mathcal{M}(\zeta,\xi)}{(1+\mathcal{M}(\zeta,\xi))} + \dot{L}\mathcal{N}(\zeta,\xi)\right] - 1.$$

From the above two cases we derive that

$$d(\Im\zeta,\Im\xi) \leq \frac{c}{k}\left[\frac{\mathcal{M}(\zeta,\xi)}{(1+\mathcal{M}(\zeta,\xi))} + \dot{L}\mathcal{N}(\zeta,\xi)\right].$$

The hypotheses of Theorem 2.1 is satisfied by taking $\dot{\zeta}(\varpi,\omega) = \frac{c}{k}\omega - \varpi, \mathcal{G}(\omega,\varpi) = \omega - \varpi, \Lambda(t) = \frac{1}{1+t}$ and $\dot{\psi}(t) = \dot{\phi}(t) = t$, where $c \in (1,s)$ and $k > s$, Hence $\Im$ defined in (3.2) has a solution. □

3.2. Application to dynamic programming

We discuss the following existence of bounded solution for functional equations that arises in dynamic programming for more see [8,9,10]. Let $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ are two Banach spaces; $\tilde{\mathcal{D}} \subseteq \tilde{\Theta}_1$ is the decision space; $\tilde{\mathcal{S}} \subseteq \tilde{\Theta}_2$ is the state space; $\mho(\tilde{\mathcal{S}})$, the set of all bounded real valued functions on $\tilde{\mathcal{S}}$ with strong b-metric is defined by:

$$d(p_x, p_y) = \max_{t \in \tilde{\mathcal{S}}}\{|p_x(t) - p_y(t)|, 2|p_x(t) - p_y(t)| - 1\}, \text{ for all } p_x, p_y \in \mho(\tilde{\mathcal{S}}).$$

Now we consider the following functional equations:

$$f(v_s) = \sup_{v_d \in \tilde{\mathcal{D}}}\{\iota(v_s, v_d) + \mathcal{C}(v_s, v_d, f(\omega(v_s, v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}, \tag{3.3}$$

where v_d is a decision vector, v_s is a state vector where as ω denotes the transformation of the process, and $f(v_s)$ indicates the optimal return function.

Let $\Im : \mho(\tilde{\mathcal{S}}) \to \mho(\tilde{\mathcal{S}})$ be a mapping defined by:

$$\Im f(v_s) = \sup_{v_d \in \tilde{\mathcal{D}}}\{\iota(v_s, v_d) + \mathcal{C}(v_s, v_d, f(\omega(v_s, v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}, \tag{3.4}$$

where $(v_d, v_s, f) \in \tilde{\mathcal{D}} \times \tilde{\mathcal{S}} \times \mho(\tilde{\mathcal{S}})$.

Let $\xi_1, \xi_2 : \mho(\tilde{\mathcal{S}}) \to \mathbb{R}$. Assume the following:

(DP_1) there exists $f_0 \in \mho(\tilde{\mathcal{S}})$ such that $\xi_1(f_0) \geq 0$ and $\xi_2(f_0) \geq 0$,

(DP_2) if $\{f_n\}$ is a sequence in $\mho(\tilde{\mathcal{S}})$ such that $\lim_{n\to\infty} f_n = f$ and $\xi_2(f_n) \geq 0$ for all n then $\xi_2(f) \geq 0$,

(DP_3) if $\xi_1(f) \geq 0$, $\xi_2(g) \geq 0$, for all $f, g \in \mho(\tilde{\mathcal{S}})$ and there exist $a \in (1, s), b > s$, and$\dot{L} \geq 0$ such that

$$|\mathcal{C}(v_s, v_d, f(\omega(v_s, v_d))) - \mathcal{C}(v_s, v_d, g(\omega(v_s, v_d)))| < \frac{a}{2b}\left[\frac{\mathcal{M}(f,g)}{1+\mathcal{M}(f,g)} + \dot{L}\mathcal{N}(f,g)\right],$$

where $\mathcal{M}(f,g) = \max\{d(f,g), \mathcal{D}(f,\Im f), \mathcal{D}(g,\Im g), \frac{\mathcal{D}(f,\Im g)+\mathcal{D}(g,\Im f)}{2s}\}$ and
$\mathcal{N}(f,g) = \max\{\mathcal{D}(f,\Im f), \mathcal{D}(g,\Im g), \mathcal{D}(f,\Im g), \mathcal{D}(g,\Im f)\}$,

(DP_4) $\xi_1(f) \geq 0$ for some $f \in \mho(\tilde{\mathcal{S}}) \Rightarrow \xi_2(\Im f) \geq 0$ and
$\xi_2(f) \geq 0$ for some $f \in \mho(\tilde{\mathcal{S}}) \Rightarrow \xi_1(\Im f) \geq 0$,

(DP_5) $\omega, \mathcal{C}$ are bounded.

Theorem 3.2 *Suppose $\Im : \mho(\tilde{\mathcal{S}}) \to \mho(\tilde{\mathcal{S}})$ are defined by (3.4) for which the conditions $(DP_1) - (DP_5)$ hold. Then (3.3) has a bounded solution in $\mho(\tilde{\mathcal{S}})$.*

Proof: Take $\epsilon > 0$. Let $v_s \in \tilde{\mathcal{S}}, v_d \in \tilde{\mathcal{D}}, f, g \in \mho(\tilde{\mathcal{S}})$. Now we demonstrate $\Im$ is a multi-valued cyclic $(\dot{\alpha}, \dot{\beta})$-admissible map.
We define $\dot{\alpha}, \dot{\beta} : \mho(\tilde{\mathcal{S}}) \to [0, \infty)$ as
$\dot{\alpha}(f) = \begin{cases} 1, & \xi_1(f) > 0 \text{ where } f \in \mho(\tilde{\mathcal{S}}), \\ 0, & \text{otherwise,} \end{cases}$ and $\dot{\beta}(f) = \begin{cases} 1, & \xi_2(f) > 0 \text{ where } f \in \mho(\tilde{\mathcal{S}}), \\ 0, & \text{otherwise.} \end{cases}$
From the condition $(DP_4), \xi_1(f) \geq 0$ and $\xi_2(g) \geq 0$, for all $f, g \in \mho(\tilde{\mathcal{S}})$ so that $\dot{\alpha}(\Im f)\dot{\beta}(\Im g) \geq 1$. By using (3.4), we can find $v_d \in \tilde{\mathcal{D}}$ and $(v_s, f, g) \in \tilde{\mathcal{S}} \times \mho(\tilde{\mathcal{S}}) \times \mho(\tilde{\mathcal{S}})$ such that

$$\Im f(v_s) < \mathcal{C}(v_s, v_d, f(\omega(v_s, v_d))) + \iota(v_s, v_d) + \epsilon \tag{3.5}$$

$$\Im g(v_s) < \mathcal{C}(v_s, v_d, g(\omega(v_s, v_d))) + \iota(v_s, v_d) + \epsilon, \tag{3.6}$$

$$\Im f(v_s) \geq \mathcal{C}(v_s, v_d, f(\omega(v_s, v_d))) + \iota(v_s, v_d), \tag{3.7}$$

$$\Im g(v_s) \geq \mathcal{C}(v_s, v_d, g(\omega(v_s, v_d))) + \iota(v_s, v_d). \tag{3.8}$$

From (3.5) and (3.8), we get

$$\begin{aligned}\Im f(v_s) - \Im g(v_s) &< \mathcal{C}(v_s, v_d, f(\omega(v_s, v_d))) - \mathcal{C}(v_s, v_d, g(\omega(v_s, v_d))) + \epsilon \\ &\leq |\mathcal{C}(v_s, v_d, f(\omega(v_s, v_d))) - \mathcal{C}(v_s, v_d, g(\omega(v_s, v_d)))| + \epsilon \\ &< \tfrac{a}{2b}\left[\tfrac{\mathcal{M}(f,g)}{1+\mathcal{M}(f,g)} + \dot{L}\mathcal{N}(f,g)\right] + \epsilon.\end{aligned} \tag{3.9}$$

Also by using (3.6) and (3.7), we have

$$\begin{aligned}\Im g(v_s) - \Im f(v_s) &< \mathcal{C}(v_s, v_d, g(\omega(v_s, v_d))) - \mathcal{C}(v_s, v_d, f(\omega(v_s, v_d))) + \epsilon \\ &\leq |\mathcal{C}(v_s, v_d, f(\omega(v_s, v_d))) - \mathcal{C}(v_s, v_d, g(\omega(v_s, v_d)))| + \epsilon \\ &< \tfrac{a}{2b}\left[\tfrac{\mathcal{M}(f,g)}{1+\mathcal{M}(f,g)} + \dot{L}\mathcal{N}(f,g)\right] + \epsilon.\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$|\Im f(v_s) - \Im g(v_s)| < \frac{a}{2b}[\Lambda(\mathcal{M}(f,g))\mathcal{M}(f,g) + \dot{L}\mathcal{N}(f,g)] + \epsilon.$$

Since $\epsilon > 0$ is taken arbitrary, we have

$$\mathcal{H}(\Im(f), \Im(g)) < \frac{a}{b}[\Lambda(\mathcal{M}(f,g))\mathcal{M}(f,g) + \dot{L}\mathcal{N}(f,g)].$$

Therefore all the hypotheses of Theorem 2.5 are satisfied by taking $\dot{\zeta}(\varpi,\omega) = \frac{a}{b}\omega - \varpi, \mathcal{G}(\omega,\varpi) = \omega - \varpi, \Lambda(t) = \frac{1}{1+t}$ and $\dot{\psi}(t) = \dot{\phi}(t) = t$, where $a \in (1, s)$ and $b > s$, hence fixed-point for $\Im$ exists in $\mho(\tilde{\mathcal{S}})$, implies that the functional equation that is defined in (3.3) has a bounded solution. □

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