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Elliptic Sombor Stress Energy of Graphs

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ABSTRACT: In this paper, we present the elliptic Sombor stress matrix $ESSM(G)$ for a connected graph G, which is linked to the elliptic Sombor stress index. We analyze the matrix's structural properties, derive bounds for its eigenvalues, and define the elliptic Sombor stress energy $E_{ESSM}(G)$ as the total sum of the absolute values of its eigenvalues. Additionally, we explore its applicability in chemistry by comparing $E_{ESSM}(G)$ with the π-electron energy of benzene derivatives.

Keywords: Graph, stress of a vertex, energy, cangul stress eigenvalues.

Contents

1. Introduction

In this article, we will be focusing on finite, unweighted, simple, and undirected graphs. Let $G = (V, E)$ denote a graph. The degree of a vertex v in G is denoted by $d(v)$. The distance between two vertices u and v in G, denoted $d(u, v)$, is the number of edges in the shortest path (or geodesic) connecting them. A geodesic path P is said to pass through a vertex v if v is an internal vertex of P, meaning v lies on P but is not an endpoint of P. For standard terminology and notion in graph theory, we follow the text-book of Harary [10].

Gutman [8] defined the energy of a graph G as the sum of the absolute values of its eigenvalues, denoted by $\mathcal{E}(G)$. Eigenvalues play a crucial role in understanding graphs as they are closely linked to major graph invariants and properties. Graph energy, a specific matrix norm, has garnered interest from both pure and applied mathematicians. Spectral graph theory examines the eigenvalues and energies of matrices associated with graphs, offering essential insights into their structural and dynamic properties. This measure reflects the collective influence of a graph's eigenvalues and has diverse applications, from chemical graph theory to network analysis. Various graph energies related to topological indices have been introduced and studied, emphasizing their importance in comprehending complex systems. Numerous matrices can be related to a graph, and their spectrums provide certain helpful information about the graph [2,4,6,7,9,13,14,16,17,23,27,28,29,30,41,47].

In 1953, Alfonso Shimbel [42] introduced the notion of vertex stress for graphs as a centrality measure. Stress of a vertex v in a graph G is the number of shortest paths (geodesics) passing through v. This concept has many applications including the study of biological and social networks. Many stress related concepts in graphs and topological indices have been defined and studied by several authors [1,3,5,12,15,18-21,25,26,31-34,38-40,43-46]. A graph G is k-stress regular [5] if $str(v) = k$ for all $v \in V(G)$.

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Othman et al. [24] introduced the concept of elliptic Sombor stress index. The elliptic Sombor stress index $ESI(G)$ of a graph G is defined as

$$ESI(G) = \sum_{uv \in E(G)} (str(u) + str(v))\sqrt{str(u)^2 + str(v)^2}.$$

In this paper, we introduce the elliptic Sombor stress matrix for a graph G and define the corresponding elliptic Sombor stress energy $E_{ESSM}(G)$ via its eigenvalues. This new framework extends graph energy by incorporating stress-related measures, offering a fresh view on graph invariants. We establish bounds for $E_{ESSM}(G)$ relative to other invariants and examine its connection to the π-electron energy in benzenoid hydrocarbons. Our study aims to deepen insights into graph energy for molecular and structural analysis.

2. Elliptic Sombor Stress Matrix and Energy

The elliptic Sombor stress matrix of a graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$ is defined as $ESSM(G) = (x_{ij})$, where

$$x_{ij} = \begin{cases} (str(v_i) + str(v_j))\sqrt{str(v_i)^2 + str(v_j)^2}, & \text{if } v_iv_j \in E(G); \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

The elliptic Sombor stress characteristic polynomial of a graph G is defined as

$$P_{ESSM}(G) = |\lambda I - ESSM(G)|, \tag{2.2}$$

where I is an $n \times n$ identity matrix.
Since the matrix $ESSM(G)$ is real and symmetric, all roots of the equation $P_{ESSM}(G) = 0$ are real. These eigenvalues can be ordered in descending sequence as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, with λ_1 representing the largest eigenvalue and λ_n the smallest eigenvalue.
The elliptic Sombor stress energy $E_{ESSM}(G)$ of a graph G is defined by

$$E_{ESSM}(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{2.3}$$

3. Preliminary Results

In this section, we will document the necessary results to support our main findings in section 4.

Lemma 3.1 *Let c_i and d_i, for $1 \leq i \leq n$, be non-negative real numbers. Then*

$$\sum_{i=1}^{n} c_i^2 \sum_{i=1}^{n} d_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} c_id_i\right)^2,$$

where $M_1 = \max_{1\leq i\leq n}\{c_i\}$; $M_2 = \max_{1\leq i\leq n}\{d_i\}$; $m_1 = \min_{1\leq i\leq n}\{c_i\}$ and $m_2 = \min_{1\leq i\leq n}\{d_i\}$.

Lemma 3.2 *Let c_i and d_i, for $1 \leq i \leq n$ be positive real numbers. Then*

$$\sum_{i=1}^{n} c_i^2 \sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} c_id_i\right)^2 \leq \frac{n^2}{4}(M_1M_2 - m_1m_2)^2,$$

where $M_1 = \max_{1\leq i\leq n}\{c_i\}$; $M_2 = \max_{1\leq i\leq n}\{d_i\}$; $m_1 = \min_{1\leq i\leq n}\{c_i\}$ and $m_2 = \min_{1\leq i\leq n}\{d_i\}$.

Lemma 3.3 *(BPR Inequality) Let c_i and d_i, for$1 \leq i \leq n$ be non-negative real numbers. Then*

$$\left| n\sum_{i=1}^{n} c_id_i - \sum_{i=1}^{n} c_i \sum_{i=1}^{n} d_i \right| \leq \alpha(n)(A - a)(B - b),$$

where a, b, A and B are real constants, that for each $i, 1 \le i \le n, a \le c_i \le A$ and $b \le d_i \le B$. Further, $\alpha(n) = n \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil\right)$.

Lemma 3.4 *(Diaz–Metcalf Inequality) If c_i and $d_i, 1 \le i \le n$, are nonnegative real numbers. Then*

$$\sum_{i=1}^{n} d_i^2 + rR \sum_{i=1}^{n} c_i^2 \le (r + R) \left(\sum_{i=1}^{n} c_i d_i \right),$$

where r and R are real constants, so that for each $i, 1 \le i \le n$, holds $rc_i \le d_i \le Rc_i$.

Lemma 3.5 *(The Cauchy-Schwarz inequality) If $c = (c_1, c_2, \ldots, c_n)$ and $d = (d_1, d_2, \ldots, d_n)$ are real n-vectors, then*

$$\left(\sum_{i=1}^{n} c_i d_i \right)^2 \le \left(\sum_{i=1}^{n} c_i^2 \right) \left(\sum_{i=1}^{n} d_i^2 \right).$$

4. Bounds for the Elliptic Sombor Stress Eigenvalues and Energy

Lemma 4.1 *Let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ be the eigenvalues of the elliptic Sombor stress matrix $ESSM(G)$. Then*

[(i)]

1. $\displaystyle\sum_{i=1}^{n} \lambda_i = 0$
2. $\displaystyle\sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1 \le i < j \le n} \left[(str(v_i) + str(v_j)) \sqrt{str(v_i)^2 + str(v_j)^2} \right]^2 = 2\mathbb{ES}$

   where $\mathbb{ES} = \sum_{1 \le i < j \le n} \left[(str(v_i) + str(v_j)) \sqrt{str(v_i)^2 + str(v_j)^2} \right]^2$

Proof: i) The first equality is a direct consequence of $ESSM(G)_{ii} = 0$ for all $1, 2, \ldots, n$.
ii) We have

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= \text{trace}[ESSM(G)]^2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left[(str(v_i) + str(v_j)) \sqrt{str(v_i)^2 + str(v_j)^2} \right]^2 \\
&= 2 \sum_{v_i v_j \in E(G)} \left[(str(v_i) + str(v_j)) \sqrt{str(v_i)^2 + str(v_j)^2} \right]^2 \\
&= 2\mathbb{ES},
\end{aligned}$$

where $\mathbb{ES} = \sum_{1 \le i < j \le n} \left[(str(v_i) + str(v_j)) \sqrt{str(v_i)^2 + str(v_j)^2} \right]^2$.

□

Lemma 4.2 *If a, b, c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\lambda + a) I_{n \times n} - aJ_{n \times n} & -cJ_{n \times m} \\ -dJ_{m \times n} & (\lambda + b) I_{m \times m} - bJ_{m \times m} \end{vmatrix}$$

$$= (\lambda + a)^{n-1} (\lambda + b)^{m-1} \left[(\lambda - (n-1) a) (\lambda - (m-1) b) - mncd \right].$$

Theorem 4.1 *If $K_{m,n}$ is a complete bipartite graph, then the elliptic Sombor stress characteristic polynomial is given by*

$$P_{ESSM}(K_{m,n}) = \lambda^{m+n-2}\left(\lambda^2 - \frac{(n(n-1)+m(m-1))^2 \cdot (n^2(n-1)^2+m^2(m-1)^2) \cdot mn}{16}\right).$$

Proof: In a complete bipartite graph $K_{m,n}$, the vertex set $V(K_{m,n})$ can be partitioned into two disjoint sets $A = \{u_1, u_2, ..., u_m\}$ and $B = \{v_1, v_2, ..., v_n\}$. The stress of any vertex v in $K_{m,n}$ is given by

$$str(v) = \begin{cases} \frac{n(n-1)}{2} & if\ v \in A \\ \frac{m(m-1)}{2} & if\ v \in B \end{cases}$$

Hence,
$ESSM\,(K_{m,n})$=

$$\begin{bmatrix} 0_{m\times m} & \left[\frac{n(n-1)}{2}+\frac{m(m-1)}{2}\right]\sqrt{\frac{n^2(n-1)^2}{4}+\frac{m^2(m-1)^2}{4}}J_{m\times n} \\ \left[\frac{n(n-1)}{2}+\frac{m(m-1)}{2}\right]\sqrt{\frac{n^2(n-1)^2}{4}+\frac{m^2(m-1)^2}{4}}J_{n\times m} & 0_{n\times n} \end{bmatrix}.$$

$$P_{ESSM}(K_{m,n}) = |\lambda I - ESSM(K_{m,n})|\,.$$

Thus we have
$P_{ESSM}(K_{m,n}) =$

$$\begin{vmatrix} \lambda I_m & -\left[\frac{n(n-1)}{2}+\frac{m(m-1)}{2}\right]\sqrt{\frac{n^2(n-1)^2}{4}+\frac{m^2(m-1)^2}{4}}J_{m\times n} \\ -\left[\frac{n(n-1)}{2}+\frac{m(m-1)}{2}\right]\sqrt{\frac{n^2(n-1)^2}{4}+\frac{m^2(m-1)^2}{4}}J_{n\times m} & \lambda I_n \end{vmatrix},$$

where I_r is the identity matrix of order $r \times r$, $0_{m\times m}$ is the zero matrix of order $m \times m$, and $J_{m\times n}$ is the $m \times n$ matrix with all entries equal to 1.
Thus, by applying Lemma 4.2, we obtain the desired result.

□

Theorem 4.2 *If $K_{1\times n-1}$ is a star graph, then the elliptic Sombor stress characteristic polynomial is given by*

$$P_{ESSM}\,(K_{1\times n-1}) = \lambda^{n-2}\left(\lambda^2 - \frac{(n-1)^5(n-2)^4}{16}\right).$$

Proof: The star graph $K_{1\times n-1}$ has two types of vertices: Internal vertex has stress $\frac{(n-1)(n-2)}{2}$ and remaining vertex have stress 0. Hence,

$$ESSM\,(K_{1\times n-1}) = \begin{bmatrix} (0)_{1\times 1} & \frac{(n-1)(n-2)}{2}\sqrt{\frac{(n-1)^2(n-2)^2}{4}}J_{1\times(n-1)} \\ \frac{(n-1)(n-2)}{2}\sqrt{\frac{(n-1)^2(n-2)^2}{4}}J_{(n-1)\times 1} & (0)_{(n-1)\times(n-1)} \end{bmatrix}.$$

$$P_{ESSM}(K_{1\times n-1}) = |\lambda I - ESSM(K_{1\times n-1})|$$
$$= \begin{vmatrix} \lambda I_1 & -\frac{(n-1)(n-2)}{2}\sqrt{\frac{(n-1)^2(n-2)^2}{4}}J_{1\times(n-1)} \\ -\frac{(n-1)(n-2)}{2}\sqrt{\frac{(n-1)^2(n-2)^2}{4}}J_{(n-1)\times 1} & \lambda I_{(n-1)} \end{vmatrix},$$

where I_r is the identity matrix of order $r \times r$, $0_{m\times m}$ is the zero matrix of order $m \times m$, and $J_{m\times n}$ is the $m \times n$ matrix with all entries equal to 1.
Thus, by applying Lemma 4.2, we obtain the desired result. □

Theorem 4.3 *Let G be any graph with n-vertices. Then*

$$\lambda_1 \leq \sqrt{\frac{(2\mathbb{ES})(n-1)}{n}}.$$

Proof:
Setting $c_i = 1, d_i = \lambda_i$, for $i = 2, 3, \ldots, n$ in Lemma 3.5, we have

$$\left(\sum_{i=2}^{n} \lambda_i\right)^2 \leq (n-1)\sum_{i=2}^{n} \lambda_i^2. \tag{4.1}$$

From Lemma 4.1, we find that

$$\sum_{i=2}^{n} \lambda_i = -\lambda_1 \text{ and } \sum_{i=2}^{n} \lambda_i^2 = -\lambda_1^2 + 2\mathbb{ES}.$$

Employing the above in (4.1), we obtain

$$(-\lambda_1)^2 \leq (n-1)\left(2\mathbb{ES} - \lambda_1^2\right)$$
$$\lambda_1 \leq \sqrt{\frac{(2\mathbb{ES})(n-1)}{n}}.$$

□

Theorem 4.4 *Let G be any graph with n-vertices. Then*

$$E_{ESSM}(G) \leq \sqrt{(2\mathbb{ES})n}.$$

Proof: Choosing $c_i = 1, d_i = |\lambda_i|$, for $i = 2, 3, \ldots, n$ in Lemma 3.5, we get

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq n\sum_{i=1}^{n} \lambda_i^2$$

$$\Longrightarrow (E_{ESSM}(G))^2 \leq n(2\mathbb{ES})$$

$$\Longrightarrow E_{ESSM}(G) \leq \sqrt{n(2\mathbb{ES})}.$$

□

Theorem 4.5 *If G is a graph with n vertices and $E_{ESSM}(G)$ be the elliptic Sombor stress energy of G, then*

$$\sqrt{2\mathbb{ES}} \leq E_{ESSM}(G).$$

Proof: By the definition of $E_{ESSM}(G)$, we have

$$[E_{ESSM}(G)]^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \geq \sum_{i=1}^{n} |\lambda_i|^2 = 2\mathbb{ES}.$$

$$\Longrightarrow \quad \sqrt{2\mathbb{ES}} \leq E_{ESSM}(G).$$

□

Theorem 4.6 *Let G be any graph with n-vertices and Φ be the absolute value of the determinant of the elliptic Sombor stress matrix $ESSM(G)$. Then*

$$\sqrt{(2\mathbb{ES}) + n(n-1)\Phi^{2/n}} \leq E_{ESSM}(G).$$

Proof: By the definition of elliptic Sombor stress energy, we find that

$$(E_{ESSM}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} |s_{\lambda_i}|^2 + 2\sum_{i<j} |\lambda_i||\lambda_j|$$
$$= (2\mathbb{ES}) + \sum_{i \neq j} |\lambda_i||\lambda_j|.$$

Since for non-negative numbers, the Arithmetic mean is greater than Geometric mean, we have

$$\begin{aligned}\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| &\geq \left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \prod_{i=1}^{n} |\lambda_i|^{2/n} \\ &= \Phi^{2/n}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{i \neq j} |\lambda_i|\,|\lambda_j| \geq n(n-1)\Phi^{\frac{2}{n}} \\ \implies\ & [E_{ESSM}(G)]^2 \geq 2\mathbb{ES} + n(n-1)\Phi^{2/n} \\ \implies\ & E_{ESSM}(G) \geq \sqrt{2\mathbb{ES} + n(n-1)\Phi^{2/n}}.\end{aligned}$$

Equality in AM-GM inequality is attained if and only if all $\lambda_i; i = 1, 2, \ldots, n$ are equal. □

Lemma 4.3 *Let $c_1, c_2, \ldots, c_n$ be non-negative numbers. Then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n} c_i - \left(\prod_{i=1}^{n} c_i\right)^{1/n}\right] \leq n\sum_{i=1}^{n} c_i - \left(\sum_{i=1}^{n} \sqrt{c_i}\right)^2 \leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} c_i - \left(\prod_{i=1}^{n} c_i\right)^{1/n}\right].$$

Theorem 4.7 *Let G be a connected graph with n vertices. Then*

$$\sqrt{(2\mathbb{ES}) + n(n-1)\Phi^{2/n}} \leq$$
$$E_{ESSM}(G) \leq \sqrt{(2\mathbb{ES})(n-1) + n\Phi^{2/n}}.$$

Proof: Let $c_i = |\lambda_i|^2, i = 1, 2, \ldots, n$ and

$$V = n\left[\frac{1}{n}\sum_{i=1}^{n} |\lambda_i|^2 - \left(\prod_{i=1}^{n} |\lambda_i|^2\right)^{1/n}\right]$$

$$= n\left[\frac{(2\mathbb{ES})}{n} - \left(\prod_{i=1}^{n}|\lambda_i|\right)^{2/n}\right]$$
$$= n\left[\frac{(2\mathbb{ES})}{n} - \Phi^{2/n}\right]$$
$$= (2\mathbb{ES}) - n\Phi^{2/n}.$$

By Lemma4.3, we obtain

$$V \leq n\sum_{i=1}^{n}|\lambda_i|^2 - \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \leq (n-1)V.$$

Upon simplification of the above equation, we find that

$$\sqrt{(2\mathbb{ES}) + n(n-1)\Phi^{2/n}} \leq$$

$$E_{ESSM}(G) \leq \sqrt{(2\mathbb{ES})(n-1) + n\Phi^{2/n}}.$$

□

Theorem 4.8 *Let G be a graph of order n. Then*

$$E_{ESSM}(G) \geq \sqrt{(2\mathbb{ES})n - \frac{n^2}{4}\left(\lambda_1 - \lambda_{\min}\right)^2},$$

where $\lambda_1 = c_{s\max} = \max_{1\leq i\leq n}\{|\lambda_i|\}$ *and* $\lambda_{\min} = \min_{1\leq i\leq n}\{|\lambda_i|\}$.

Proof: Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $ESSM(G)$. We choose $c_i = 1$ and $d_i = |\lambda_i|$, which by Lemma 3.2 implies

$$\sum_{i=1}^{n}1^2\sum_{i=1}^{n}|\lambda_i|^2 - \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \leq \frac{n^2}{4}\left(\lambda_1 - \lambda_{\min}\right)^2$$
$$\text{i.e.,}\quad (2\mathbb{ES})n - \left(E_{ESSM}(G)\right)^2 \leq \frac{n^2}{4}\left(\lambda_1 - \lambda_{\min}\right)^2$$
$$\implies\quad E_{ESSM}(G) \geq \sqrt{(2\mathbb{ES})n - \frac{n^2}{4}\left(\lambda_1 - \lambda_{\min}\right)^2}.$$

□

Theorem 4.9 *Suppose zero is not an eigenvalue of $ESSM(G)$, then*

$$E_{ESSM}(G) \geq \frac{2\sqrt{\lambda_1\lambda_{\min}}\sqrt{(2\mathbb{ES})n}}{\lambda_1 + \lambda_{\min}},$$

where $\lambda_1 = \lambda_{\max} = \max_{1\leq i\leq n}\{|\lambda_i|\}$ *and* $\lambda_{\min} = \min_{1\leq i\leq n}\{|\lambda_i|\}$.

Proof: Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $ESSM(G)$.
Setting $c_i = |\lambda_i|$ and $d_i = 1$ in Lemma3.1, we have

$$\sum_{i=1}^{n}|\lambda_i|^2\sum_{i=1}^{n}1^2 \leq \frac{1}{4}\left(\sqrt{\frac{\lambda_1}{\lambda_{\min}}} + \sqrt{\frac{\lambda_{\min}}{\lambda_1}}\right)^2\left(\sum_{i=1}^{n}|\lambda_i|\right)^2$$

$$\text{i.e.,}\quad (2\mathbb{ES})n \leq \frac{1}{4}\left(\frac{(\lambda_1+\lambda_{\min})^2}{\lambda_1\lambda_{\min}}\right)(E_{ESSM}(G))^2$$

$$\Longrightarrow\quad E_{ESSM}(G) \geq \frac{2\sqrt{\lambda_1\lambda_{\min}}\sqrt{(2\mathbb{ES})n}}{\lambda_1+\lambda_{\min}}.$$

□

Theorem 4.10 *Let G be a graph of order n and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the non zero eigenvalues of $ESSM(G)$. Then*

$$E_{ESSM}(G) \geq \frac{(2\mathbb{ES}) + n\lambda_1\lambda_{\min}}{\lambda_1+\lambda_{\min}},$$

where $\lambda_1 = \lambda_{\max} = \max_{1\leq i\leq n}\{|\lambda_i|\}$ and $\lambda_{\min} = \min_{1\leq i\leq n}\{|\lambda_i|\}$.

Proof: Assigning $d_i = |\lambda_i|, c_i = 1, R = |\lambda_1|$ and $r = |\lambda_{\min}|$ in Lemma 3.4, we get

$$\sum_{i=1}^{n}|\lambda_i|^2 + \lambda_1\lambda_{\min}\sum_{i=1}^{n}1^2 \leq (\lambda_1+\lambda_{\min})\sum_{i=1}^{n}|\lambda_i|$$

$$(2\mathbb{ES}) + n\lambda_1\lambda_{\min} \leq (\lambda_1+\lambda_{\min})\,E_{ESSM}(G)$$

After simplifying and using the definition of $E_{ESSM}(G)$, we obtain

$$E_{ESSM}(G) \geq \frac{(2\mathbb{ES}) + n\lambda_1\lambda_{\min}}{\lambda_1+\lambda_{\min}}.$$

□

Theorem 4.11 *Let G be a graph of order n and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of $ESSM(G)$. Then*

$$E_{ESSM}(G) \geq \sqrt{(2\mathbb{ES})n - \alpha(n)(\lambda_1-\lambda_{\min})^2},$$

where $\lambda_1 = \lambda_{\max} = \max_{1\leq i\leq n}\{|\lambda_i|\}$ and $\lambda_{\min} = \min_{1\leq i\leq n}\{|\lambda_i|\}$ and $\alpha(n) = n\left\lceil\frac{n}{2}\right\rceil\left(1-\frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$.

Proof: Setting $c_i = |\lambda_i| = d_i, A \leq |\lambda_i| \leq B$ and $a \leq |\lambda_n| \leq b$ in Lemma3.3, we get

$$\left|n\sum_{i=1}^{n}|\lambda_i|^2 - \left(\sum_{i=1}^{n}|\lambda_i|\right)^2\right| \leq \alpha(n)(\lambda_1-\lambda_{\min})^2$$

$$\left|(2\mathbb{ES})n - (E_{ESSM}(G))^2\right| \leq \alpha(n)(\lambda_1-\lambda_{\min})^2$$

$$E_{ESSM}(G) \geq \sqrt{(2\mathbb{ES})n - \alpha(n)(\lambda_1-\lambda_{\min})^2}.$$

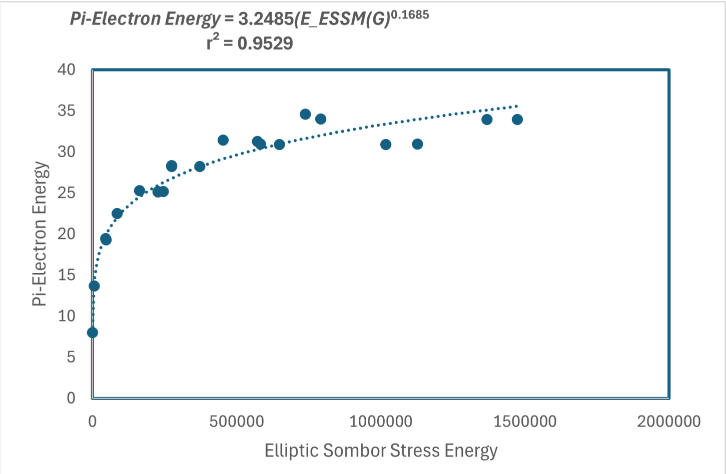
□

5. Computational Analysis of Elliptic Sombor Stress Energy and π-Electron Energy

In this section, we conduct a comprehensive computational analysis of the elliptic Sombor stress energy $E_{ESSM}(G)$ and the π-electron energy associated with various benzene derivatives. We focus on power regression models to effectively capture the nonlinear trends commonly observed in real-world data. These flexible modeling techniques help researchers identify the best fit for their datasets. The power regression model, expressed as $y = ax^b$, facilitates the analysis of nonlinear relationships and yields a correlation coefficient of 0.976, indicating a strong relationship between the variables.

Table 1: Elliptic Sombor Stress Energy and π-Electron Energy of Derivatives of Benzene

Derivatives of benzene	$E_{ESSM}(G)$	π-electron energy
Benzene	203.646	8
Naphthalene	6109.891	13.683
Phenanthrene	46164.609	19.448
Anthracene	48054.086	19.314
Chrysene	244934.868	25.192
Benzo[a]anthracene	227123.223	25.101
Triphenylene	163548.378	25.275
Tetracene	227951.17	25.188
Benzo[a]pyrene	372508.817	28.222
Benzo[e]pyrene	274860.795	28.336
Perylene	273946.443	28.245
Anthanthrene	572455.455	31.253
Benzo[ghi]perylene	453145.409	31.425
Dibenz[a,c]anthracene	583042.878	30.942
Dibenz[a,h]anthracene	1017096.906	30.881
Dibenz[a,j]anthracene	648422.785	30.880
Picene	1127197.025	30.943
Coronene	738063.019	34.572
Dibenzo[a,h]pyrene	1368396.954	33.928
Dibenzo[a,i]pyrene	1473548.066	33.954
Dibenzo[a,l]pyrene	790893.653	34.031
Pyrene	84843.678	22.506



6. Conclusion

In conclusion, the analysis of elliptic Sombor stress energy and π-electron energy using power regression models reveals significant nonlinear relationships. The strong correlation coefficient of 0.976 underscores the effectiveness of this modeling approach. These findings enhance our understanding of the properties of benzene derivatives and their interactions.

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