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Exact Similarity Solutions and Classification of Symmetries for (C-B) Equation

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Abstract: In this paper, we show that the symmetry algebra admitted by the Carleman-Boltzmann equation is solvable, not semi simple and not nilpotent. Furthermore, by applying the Ovsiannikov's approach, we construct one, two and three dimensional optimal systems. Based on the structurally important informations containing in the obtained optimal systems, we construct numerous reduction equations and we get some exact solutions.

Keywords: Invariant solutions, Lie algebra classification, symmetry reduction, optimal system.

Contents

1. Introduction

The Lie symmetry analysis, is a powerful mathematical tool for the determination of exact solutions for nonlinear ordinary and fractional differential equations [1,2,3,4,5]. In fact, Lie point symmetries help us to simplify differential equations by means of similarity transformations, which reduce the number of independent variables or the order of the studied differential equation. These reduction process is based on the existence of functions that are invariant under a specific group of point transformations. When these invariants are used as new variables, then the differential equation can be reduced. The obtained solutions that are found with the application of those invariant functions are called similarity solutions [1,2,3]. Here, by applying the Lie symmetry analysis we construct one, two and three dimensional optimal systems of Lie subalgebras corresponding to the Carleman-Boltzmann (C-B) equation [6,7,8].

Recall first that, the model describing the collision dynamics of a one dimensional gas particules's with specific initial and final velocities leads to the standard Boltzmann equation after a suitable change of variables [8,9,10]. The local existence and uniqueness was the subject of different works in the literature [11] and was studied extensively by different methods to construct exact and numerical solutions [12,13,14,8]. A classes of solutions were discovered and found in the literature by different authors based on the specializing of forms for particular choices of a similarity variables that transform the equation into ordinary differential equations, for example the singularity theory for semilinear waves [8,15].

The work presented here is devoted to the study of the structure of the symmetry algebra admitted by the (C-B) equation. The invariance properties of the equation were the subject of various works to construct some invariant solutions [10,11,16,17]. In our work, we will focus mainly on the characteristics of the structure of the symmetry algebra especially solvability, semi simplicity and nilpotency. In addition, great importance is devoted to the problem of classification of one two and three-dimensional subalgebras

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of the symmetry algebra. This allowed the determination of optimal systems, a class of inequivalent reduced equations and the construction of some exact solutions.

This paper is organized as follows: we carry out the structure properties of the symmetry algebra admitted by the (C-B) equation and determine invaraint functions with some details in Section 2. In Section 3, we construct one, two and three optimal systems. In Section 4, we establish symmetry reductions and we get some invaraint solutions.

2. Carleman-Boltzmann Equation and Invariance Analysis

2.1. The structure of the Lie symmetry algebra for C-B equation

The Carleman-Boltzmann Equation (C-B) is given by the following expression:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} = 0. \tag{1}$$

It is well known that this equation is invariant under the fourth dimensional Lie algebra $\mathcal{G}$ spanned by the following operators [8,10]:

$$Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = \frac{\partial}{\partial t}, \quad Z_3 = \frac{\partial}{\partial u}, \quad Z_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{2}$$

In order to obtain some useful properties related to the Lie algebra $\mathcal{G}$ admitted by the equation (1), we have to establish the commutator table gevin by:

Table 1: Commutator Table

$[Z_i, Z_j]$	Z_1	Z_2	Z_3	Z_4
Z_1	0	0	0	Z_1
Z_2	0	0	0	Z_2
Z_3	0	0	0	0
Z_4	$-Z_1$	$-Z_2$	0	0

From the above commutator table, we get some structure informations related to solvability, nilpotency and semi-simplicity illustrated by the following result:

Theorem 1. *The Lie algebra $\mathcal{G}$ is solvable. However, it is not nilpotent and not semi-simple.*

Proof. Let $D(\mathcal{G}) = \mathcal{G}'$ denotes the first derivation of the Lie algebra $\mathcal{G}$, i.e.,

$$D(\mathcal{G}) = [\mathcal{G}, \mathcal{G}].$$

According to commutator relations (table 1), we obtain that

$$D(\mathcal{G}) = [\mathcal{G}, \mathcal{G}] = Spann(Z_1, Z_2) \neq \mathcal{G},$$

then, $\mathcal{G}$ is not semi-simple.
If we denote by $\mathcal{G}^{(2)}$, the second derivation of $\mathcal{G}$, i.e.,

$$\mathcal{G}^{(2)} = [\mathcal{G}', \mathcal{G}'],$$

so from table 1, we obtain that

$$\mathcal{G}^{(2)} = [\mathcal{G}', \mathcal{G}'] = \{0\}.$$

Consequently, $\mathcal{G}$ is solvable. Is not difficult to see that $Z_3 \in Z(\mathcal{G})$, where $Z(\mathcal{G})$ is the center of the Lie algebra $\mathcal{G}$ then, $Z(\mathcal{G}) \neq \{0\}$, hence, $\mathcal{G}$ is not nilpotent. □

2.2. Adjoint representation and invariant functions

We first recall the definition of the adjoint representation, which plays a crucial role in problems related to subalgebras classification.

Definition 1. *Let G be a Lie group and $\mathcal{G}$ its associated Lie algebra. The adjoint representation is denoted by Ad and it is defined by:*

$$\forall g \in G,\ \forall w \in \mathcal{G},\ Ad_g(w) = g^{-1}wg.$$

Theorem 2. *Let G be a Lie group and $\mathcal{G}$ its associated Lie algebra. We have:*

$$Ad(exp(\epsilon v))(w) = \sum_{k=0}^{+\infty} \frac{\epsilon^k}{k!}(ad(v))^k(w),$$

with ad is given by :

$$\begin{array}{rcccl} ad(v) & : & \mathcal{G} & \rightarrow & \mathcal{G} \\ & & w & \mapsto & ad(v)(w) = [w, v]\,. \end{array}$$

According to the previous theorem, we can construct the table of the adjoint representation given by:

Table 2: Adjoint representation table

$Ad(exp(\epsilon\star)\star)$	Z_1	Z_2	Z_3	Z_4
Z_1	Z_1	Z_2	Z_3	$Z_4 - \epsilon Z_1$
Z_2	Z_1	Z_2	Z_3	$Z_4 - \epsilon Z_2$
Z_3	Z_1	Z_2	Z_3	Z_4
Z_4	$e^\epsilon Z_1$	$e^\epsilon Z_2$	Z_3	Z_4

Now that the definition of the adjoint representation is given, we can provide the definition of an invariant function.

Definition 2. *Let $\mathcal{G}$ a Lie algebra. A real function ϕ on $\mathcal{G}$ is called an invariant if $\phi\left(Ad_g(Z)\right) = \phi(Z)$ for all $Z \in \mathcal{G}$ and all $g \in G$.*
Two vectors Z and T are equivalent under the adjoint action, if $\phi(Z) = \phi(T)$, for any invariant function ϕ.

Remark 1.

1. *If we let $Z = \sum_{i=1}^{4} a_i Z_i$, then, the invariant ϕ can be regarded as a function of $a_1, a_2, \ldots, a_4$.*
2. *The well-known invariant function is the invariant related to the Killing form.*

In general, it is not easy to determine invariant functions, but we have the following result:

Theorem 3. *A real function $\phi(a_1, \ldots, a_4)$ is an invariant of the Lie algebra $\mathcal{G}$ if and only if ϕ is a solution of the system :*

$$\left\{ \begin{array}{l} a_4 \dfrac{\partial \phi}{\partial a_1} = 0, \\ a_4 \dfrac{\partial \phi}{\partial a_2} = 0, \\ a_1 \dfrac{\partial \phi}{\partial a_1} + a_2 \dfrac{\partial \phi}{\partial a_2} = 0. \end{array} \right. \tag{2.1}$$

Proof. Taking any subgroup $g = e^{\epsilon Z}\left(Z = \sum_{j=1}^{4} b_j Z_j\right)$ to act on $Y = \sum_{i=1}^{4} a_i Z_i$, we have

$$\begin{aligned} Ad_{\exp(\epsilon Z)}(Y) =& e^{-\epsilon Z} Y e^{\epsilon Z} \\ =& Y - \epsilon[Z, Y] + \frac{1}{2!}\epsilon^2[Z,[Z,Y]] - \cdots \\ =& \left(a_1 Z_1 + \cdots + a_4 Z_4\right) - \epsilon\left[b_1 Z_1 + \cdots + b_4 Z_4, a_1 Z_1 + \cdots + a_4 Z_4\right] \\ & + O\left(\epsilon^2\right) \\ =& \left(a_1 Z_1 + \cdots + a_4 Z_4\right) - \epsilon\left(\theta_1 Z_1 + \cdots + \theta_4 Z_4\right) + O\left(\epsilon^2\right), \end{aligned}$$

where $\theta_i \equiv \theta_i\left(a_1, \ldots, a_4, b_1, \ldots, b_4\right)$ can be easily obtained from the commutator table.

Hence, the function ϕ is invariant if and only if:

$$\theta_1 \frac{\partial \phi}{\partial a_1} + \theta_2 \frac{\partial \phi}{\partial a_2} + \theta_3 \frac{\partial \phi}{\partial a_3} + \theta_4 \frac{\partial \phi}{\partial a_4} = 0, \quad \text{for any } b_i. \tag{4}$$

Where $\theta_1 = b_1 a_4 - b_4 a_1$, $\theta_2 = b_2 a_4 - b_4 a_2$, $\theta_3 = 0$, $\theta_4 = 0$.

Equating the coefficients of all b_i in the Eq.(4), we get the following system of the first order P.D.Es.

$$\begin{cases} a_4 \dfrac{\partial \phi}{\partial a_1} = 0, \\ a_4 \dfrac{\partial \phi}{\partial a_2} = 0, \\ a_1 \dfrac{\partial \phi}{\partial a_1} + a_2 \dfrac{\partial \phi}{\partial a_2} = 0. \end{cases} \tag{5}$$

□

Remark 2. *If $a_4 \neq 0$ we have: $\phi(a_1, a_2, a_3, a_4) = F(a_3, a_4)$, with F an arbitrary function.*

Now, we will give the definition of the famous invariant function related to the Killing form, which plays a crucial role in determining the one-dimensional optimal system.

Proposition 1. *The function defined on $\mathcal{G}$ by: $C(Z) = K(Z, Z)$ is an invariant function, where K is the Killing form on $\mathcal{G}$ given by:*

$$\begin{array}{cccc} K & : & \mathcal{G} \times \mathcal{G} & \rightarrow & \mathbb{R} \\ & & (Z, Y) & \mapsto & Tr(ad(Z)ad(Y)). \end{array}$$

In our case, we have the following invariant function:

Proposition 2. $C(Z) = K(Z, Z) = 2a_4^2$.

Proof. We have $K(Z, Z) = Tr(ad(Z)ad(Z))$, and

$$ad(Z) = \begin{pmatrix} a_4 & 0 & 0 & -a_1 \\ 0 & a_4 & 0 & -a_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then, $C(Z) = K(Z, Z) = 2a_4^2$.

□

Before stating the proposition that provides us with other invariant functions, we will need the following table:

According to the bove table, we get the following result

Table 3: Table for construction of invariant functions.

$Ad(exp(\epsilon Z_i)Z)$	coef Z_1	coef Z_2	coef Z_3	coef Z_4
$Ad(exp(\epsilon Z_1)Z)$	$a_1 - \epsilon a_4$	a_2	a_3	a_4
$Ad(exp(\epsilon Z_2)Z)$	a_1	$a_2 - \epsilon a_4$	a_3	a_4
$Ad(exp(\epsilon Z_3)Z)$	a_1	a_2	a_3	a_4
$Ad(exp(\epsilon Z_4)Z)$	$a_1 e^{\epsilon}$	$a_2 e^{\epsilon}$	a_3	a_4

Proposition 3. *The following expressions are invariant functions:*

$$L(a_1, a_2, a_3, a_4) = a_4,$$

$$A(a_1, a_2, a_3, a_4) = \begin{cases} sgn(a_1) \ , \ si \ a_4 = 0, \\ 0 \ \ otherwise. \end{cases}$$

$$B(a_1, a_2, a_3, a_4) = \begin{cases} 1 \ si \ a_2^2 + a_3^2 + a_4^2 \neq 0, \\ 0 \ \ otherwise. \end{cases} \quad .$$

Proof.

- For L it is clearly observed from table 3 that L is an invariant function.
- For A: the coefficient of Z_1, i.e a_1 remains unchanged under the action of $Ad(\exp(\epsilon Z_i)$ $i = 2$ and 3. Therefore, we investigate both the adjoint actions under $Ad(\exp(\epsilon Z_1)$ and $Ad(\exp(\epsilon Z_4)$ and the invariance condion $A(Y) = A(Ad(Y))$. Then, the $sgn(a_1)$ maps to $sgn(a_1 e^{\epsilon})$, which gives positive, negative or zero, depending on the sign of a_1.
- For B: The coefficients of Z_1, Z_3 and Z_4 remain unchanged under the action of $Ad(\exp(\epsilon Z_i)$ $i = 2,\ 3$. Hence, it is enough to check the invariance of B under the action of $Ad(\exp(\epsilon Z_i)$ $i = 2, 4$.
  Let $\tilde{a_i}$, $i = 1, \ldots, 4$ be the new transformed coefficients after the adjoint action. With the action of $Ad(\exp(\epsilon Z_2)$ and $Ad(\exp(\epsilon Z_4)$, we obtain

  $$\tilde{a_2}^2 + \tilde{a_3}^2 + \tilde{a_4}^2 = (a_2 - \epsilon a_4)^2 + a_3^2 + a_4^2, \tag{2.2}$$

  $$\tilde{a_2}^2 + \tilde{a_3}^2 + \tilde{a_4}^2 = (a_2 e^{\epsilon})^2 + a_3^2 + a_4^2 \tag{2.3}$$

  From equations (2.2) and (2.3), we get

  $$\tilde{a_2} = \tilde{a_3} = \tilde{a_4} = 0 \ \text{ iff } \ a_2 = a_3 = a_4.$$

□

2.3. Adjoint matrix of the C-B equation

In order to construct the global adjoint matrix, we will first apply the adjoint action of Z_1 to $Z = \sum_{i=1}^{4} a_i Z_i$ and with the help of table 3, we have

$$\begin{aligned} Ad_{\exp(\epsilon_1 Z_1)} Z &= Ad_{\exp(\epsilon_1 Z_1)} (a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4) \\ &= (a_1 - a_4 \epsilon_1) Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4 \\ &= (a_1, a_2, a_3, a_4) \cdot A_1 \cdot (Z_1, Z_2, Z_3, Z_4)^{\mathrm{T}}, \end{aligned}$$

where,

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon_1 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, A_2, A_3 and A_4 are found to be

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\epsilon_2 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 \\ 0 & e^{\epsilon_4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$Z = a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4,$$

and

$$g = \exp(\epsilon_1 Z_1). \exp(\epsilon_2 Z_2). \exp(\epsilon_3 Z_3). \exp(\epsilon_4 Z_4) \in G.$$

We have

$$Ad(g) = Ad(exp(\epsilon_1 Z_1)) o Ad(exp(\epsilon_2 Z_2)) o Ad(exp(\epsilon_3 Z_3)) o Ad(exp(\epsilon_4 Z_4)).$$

Consequently, the general adjoint matrix A is constructed by

$$A = A_1 A_2 A_3 A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 \\ 0 & e^{\epsilon_4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon_1 e^{\epsilon_4} & -\epsilon_2 e^{\epsilon_4} & 0 & 1 \end{pmatrix}.$$

3. Optimal Systems of Symmetry Algebra Admitted by C-B Equation

According to the idea developed by Ovsiannikov [1], the classification problem is done up to an automorphism. The author showed that the automorphisms of Lie groups transformation in the case of finite dimension are the inners ones.

3.1. A one-dimensional optimal system

Proposition 4. *The optimal system of dimension* 1 *is given by:*

$$M_1 =< Z_3 >, \qquad M_2 =< aZ_3 + Z_4 >, \qquad M_3 =< Z_1 + aZ_3 >,$$

$$M_4 =< Z_1 + Z_2 + aZ_3 >, \qquad M_5 =< -Z_1 + Z_2 + aZ_3 >,$$

with $a \in \mathbb{R}$.

Proof. As every element Z of $\mathcal{G}$ can be written in the form $Z = \sum_{i=1}^{4} a_i Z_i$. Then, we will simplify the form of Z as much as possible. To do this, we search a simple form $\tilde{Z} = \sum_{i=1}^{4} \tilde{a}_i Z_i$ of Z. Using the global adjoint matrix, we get

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = (a_1, a_2, a_3, a_4)\, A \quad \Leftrightarrow \quad \begin{cases} \tilde{a}_1 = (a_1 - \epsilon_1 a_4)\, e^{\epsilon_4}, \\ \tilde{a}_2 = (a_2 - \epsilon_2 a_4)\, e^{\epsilon_4}, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = a_4. \end{cases} \tag{3.1}$$

According to the invariant function given by the Killing form, we are led to discuss the following cases:
Case 1: $a_4 = 1$
Select a representative element $\tilde{Z}$. Substituting $\tilde{a}_1 = \tilde{a}_2 = 0, \tilde{a}_4 = 1$, and $a_4 = 1$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = (a_1 - \epsilon_1 a_4)\, e^{\epsilon_4}, \\ \tilde{a}_2 = (a_2 - \epsilon_2 a_4)\, e^{\epsilon_4}, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = a_4. \end{cases} \quad \Leftrightarrow \quad \begin{cases} a_1 - \epsilon_1 a_4 = 0, \\ a_2 - \epsilon_2 a_4 = 0, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = 1. \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \epsilon_1 = \frac{a_1}{a_4}, \\ \epsilon_2 = \frac{a_2}{a_4}, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = 1. \end{cases}$$

That is to say, that all expressions $a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + Z_4$ are equivalent to $a_3 Z_3 + Z_4$.

Case 2: $a_4 = 0$ **and** $a_1 \neq 0$**.**
Substituting $a_4 = 0$ into equation (3.1). Since the invariant function related to the Killing form is zero, so according to the system (2.1), we use the obtained invariant function, given by

$$\phi\,(a_1, a_2, a_3) = \frac{a_2}{a_1}, \qquad a_1 \neq 0.$$

Case 2.1: $\frac{a_2}{a_1} = 1$**.**
According to equation (3.1), with $a_1 = a_2 = 1$, for $\epsilon_4 = 0$ we have

$$\begin{cases} \tilde{a}_1 = 1, \\ \tilde{a}_2 = 1, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = 0. \end{cases}$$

Then,

$$\tilde{Z} = Z_1 + Z_2 + aZ_3.$$

Case 2.2: $\frac{a_2}{a_1} = -1$**.**
According to equation (3.1), with $a_1 = -1$ and $a_2 = 1$, for $\epsilon_4 = 0$ we have

$$\begin{cases} \tilde{a}_1 = -1, \\ \tilde{a}_2 = 1, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = 0. \end{cases}$$

Then, the simple form in this subcase is

$$\tilde{Z} = -Z_1 + Z_2 + aZ_3.$$

Case 2.3: $\frac{a_2}{a_1} = 0$**.**
Case 2.3.1: A new invariant is given by $\phi\,(a_1, a_3) = c$, with $c \in \mathbb{R}$. Hence, the simple form is given by

$$\tilde{Z} = Z_1 + a_3 Z_3.$$

Case 2.3.1 $a_1 = 0$**:** In this last case , we have $\tilde{Z} = Z_3$. □

Remark 3. *We conclude this subsection with a table summarizing the evaluation of different obtained invariants C, L, A and B on the one-dimensional subalgebras M_i, $i = 1, \ldots, 5$.*

Table 4: Evaluation of invariants.

M_i	C	L	A	B
M_1	0	0	1	0
M_2	0	0	1	1
M_3	0	0	-1	1
M_4	2	1	0	1
M_5	0	0	0	0

3.2. A two-dimensional optimal system

In order to construct the two-dimensional optimal subalgebra, let $< Z, Y >$ be a two-dimensional subalgebra such that:

$[Z, Y] = \alpha Z + \beta Y$, for $Z = M_i$, $i = 1, \dots, 5$ and $Y = \sum_{i=1}^{4} b_i Z_i$. The key idea is to select Y from the normaliser $Nor_{\mathcal{G}}(Z)$ of Z in $\mathcal{G}$ given by:

$$Nor_{\mathcal{G}}(Z) = \{Y \in \mathcal{G}; [Z, Y] \in Z\}, \quad i.e\, [Z, Y] = \alpha Z,$$

where α is an arbitrary constant.

Proposition 5. *The optimal system of dimension* 2 *is given by:*

$$\begin{aligned} N_1 &= < Z_3, aZ_1 + bZ_2 + Z_4 >, \qquad N_2 =< Z_3, Z_4 >, \\ N_3 &= < Z_1 + aZ_3, Z_2 >, \qquad N_4 =< Z_1 + Z_2 + aZ_3; bZ_1 + Z_2 >, \\ N_5 &= < -Z_1 + Z_2 + aZ_3; bZ_1 + Z_2 >, \end{aligned} \tag{3.2}$$

where $a, b \in \mathbb{R}$. *All* N_i, $i = 1, \dots, 5$ *are abelian sub-algebras.*

Proof. In this proof, we shall assume that $L = Span_{\mathbb{R}}\{Z, Y\}$ is a 2-dimensional Lie sub-algebra of $\mathcal{G}$.

Let $Z = Z_3$, $Y = \sum_{i=1}^{4} b_i Z_i$, and $[Z, Y] = \alpha Z + \beta Y$. Then, we have

$Y = \sum_{i=1}^{4} b_i Z_i$ and $\alpha = \beta = 0$. By a suitable change of base of L, we can assume that, $Y = a_1 Z_1 + a_2 Z_2 + Z_4$ is reduced to the case of N_1 and we can not simplify more this sub-algebra.

Let $Z = aZ_3 + Z_4$, $Y = \sum_{i=1}^{4} b_i Z_i$ and $[Z, Y] = \alpha Z + \beta Y$, we have $[Z, Y] = -b_1 Z_1 - b_2 Z_2$. Then, we obtain $\alpha = \beta = 0$, and Y becomes $Y = b_3 Z_3 + b_4 Z_4$. By a suitable change of base of L, we can assume that, $Z = Z_3$ and $Y = Z_4$ is reduced the case of N_2. The simplification is achieved.

Let $Z = Z_1 + aZ_3$, $Y = \sum_{i=1}^{4} b_i Z_i$ and $[Z, Y] = \alpha Z + \beta Y$. We have $[Z, Y] = b_4 Z_1$. Then, we get $\alpha = \beta = 0$, $Y = b_1 Z_1 + b_2 Z_2 + b_3 Z_3$. By a suitable change of base of L, we can assume that, $Y = Z_2$ is reduced to the case of N_3 and we can not further simplify this sub-algebra.

Let $Z = Z_1 + Z_2 + aZ_3$, $Y = \sum_{i=1}^{4} b_i Z_i$ and $[Z, Y] = \alpha Z + \beta Y$. We have $[Z, Y] = b_4 Z_1 + b_4 Z_2$. Then, we obtain $\alpha = \beta = 0$, $Y = b_1 Z_1 + b_2 Z_2 + b_3 Z_3$. By a suitable change of base of L, we can assume that, $Y = cZ_1 + Z_2$, is reduced to the case of N_4 and the simplification is achieved.

Let $Z = -Z_1 + Z_2 + aZ_3$, $Y = \sum_{i=1}^{4} b_i Z_i$ and $[Z, Y] = \alpha Z + \beta Y$. We have $[Z, Y] = -b_4 Z_1 + b_4 Z_2$. Then, it yields $\alpha = \beta = 0$, $Y = b_1 Z_1 + b_2 Z_2 + b_3 Z_3$. By a suitable change of base of L, we can assume that, $Y = cZ_1 + Z_2$ is reduced the case of N_5 and it can not be more simplified.

□

3.3. A three-dimensional optimal system

This three-dimensional optimal system is based on the expansion of the obtained two-dimensional optimal system. To do this, we consider any two-dimensional subalgebras from (3.2), let us consider the first subalgebra $N_i =< Y_r, Y_s >$, $i = 1, \ldots, 5$ and find a vector field $Y = a_1 Z_1 + \cdots + a_4 Z_4$ such that the triple $\{Y_r, Y_s, Y\}$ of generaters form a basis of a three-dimensional subalgebra. For that, it is necessary and sufficient that the vector field Y satisfies the equations.

$$[Y_r, Y] = \alpha_1 Y + \beta_1 Y_r + \gamma_1 Y_s, \quad [Y_s, Y] = \alpha_2 Y + \beta_2 Y_r + \gamma_2 Y_s. \tag{3.3}$$

According to the above equations, we obtain

$$\sum_{1 \leq j,k \leq 4} C^i_{jk} \mu^j_r a_k = \alpha_1 a_i + \beta_1 \mu^i_r + \gamma_1 \mu^i_s, \tag{3.4}$$

$$\sum_{1 \leq j,k \leq 4} C^i_{jk} \mu^j_s a_k = \alpha_2 a_i + \beta_2 \mu^i_r + \gamma_2 \mu^i_s, \tag{3.5}$$

where $Y_l = \sum_{i=1}^{4} \mu^i_l Z_i$, $l \in \{r, s\}$ and $[Z_j, Z_k] = \sum_{i=1}^{4} C^i_{jk} Z_i$.

The solution of system (3.4-3.5) is linearly independent of the vectors $\{Y_r, Y_s\}$ and generates a three-dimensional subalgebra. This process is also applied to the other two pairs of vector fields in (3.2).

By applying this method to all the pairs of vector fields in (3.2), we conclude that $Y = \lambda_1 Y_r + \lambda_2 Y_s$. Through an appropriate change of base, we can assume that $Y = 0$, which implies that $\mathcal{G}$ is not a three-dimensional subalgebra.
Then, we have the following result:

Proposition 6. *The Lie algebra of the symmetries $\mathcal{G}$ admitted by the (C-B) equation has no three-dimensional optimal system.*

4. Similarity and Some Invariant Solutions

This section is devoted to constructing some exact solutions of the studied Carleman-Boltzmann equation. To begin with, we apply the reduction method associated to some obtained generators.

- Reduction with $M_3 = Z_1 + aZ_3$, $a \neq 0$:
As

$$Z_1 + aZ_3 = \frac{\partial}{\partial x} + a\frac{\partial}{\partial u},$$

then, its corresponding characteristic equation is given by

$$\frac{d}{0} = \frac{dx}{1} = \frac{du}{a}. \tag{4.1}$$

The associated invariants are: $z =$ and $r = u - ax$.
Taking into account the last invariant, we assume a similarity solution of the form: $u = ax + f(z)$ and we substitute it into (1) to determine the form of the function $f(z)$. Hence, $f(z)$ has to be a solution of the following reduced differential equation

$$f'' - 2af' = 0. \tag{4.2}$$

The solution of the above equation is obtained to be of the form

$$f(z) = k_a e^{2az} + c \quad , \quad \text{with} \quad c, k_a \in \mathbb{R}.$$

Consequently, the corresponding similarity solution of Carleman-Boltzmann equation is given by

$$u(t,x) = k_a e^{2at} + ax + c.$$

- Reduction with Z_4:
The vector field Z_4 is given by

$$Z_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

In this case, the characteristic system is

$$\frac{dt}{t} = \frac{dx}{x} = \frac{du}{0}. \tag{12}$$

The corresponding invariants are $z = \dfrac{t}{x}$ and $r = u$.
Hence, a similarity solution will be of the form $u = f(z)$. Substituting into (1), we obtain the reduced differential equation

$$f'' - 2zf' - z^2 f'' + 2z(f')^2 = 0, . \tag{4.3}$$

which is a Bernoulli type equation and its solution is given by

$$f(z) = \beta + \frac{1}{2c\alpha_c}\ln\left(\frac{z+\alpha_c}{z-\alpha_c}\right), \quad c, \beta \in \mathbb{R}, \quad \alpha_c = \sqrt{\frac{1+c}{c}}.$$

Consequently,

$$u(t,x) = \beta + \frac{1}{2c\alpha_c}\ln\left(\frac{t+\alpha_c x}{t-\alpha_c x}\right), \quad c, \beta \in \mathbb{R}, \quad \alpha_c = \sqrt{\frac{1+c}{c}}.$$

Here, in the following table, we list the obtained reduced forms of the Carleman-Boltzmann equation.

Table 5: Reduced equations

j	Similarity Reduced Equations
1	$\left(1-z^2\right) f'' - 2zf' + 2z\left(f'\right)^2 = 0$
2	$\left(1-z^2\right) f'' + 2\left(a-z\right) f' + 2z\left(f'\right)^2 = \dfrac{a}{z^2}$
3	$f'' - 2af' = 0$
4	$\left(f'-a\right) f' = 0$
5	$\left(f'+a\right) f' = 0$

Having determined the infinitesimals, the invariants z_j, r_j and similarity solutions u_j are listed in the following table.

Table 6: Lie Invariants and Similarity Solutions

j	$\mathbf{X_j}$	z_j	r_j	u_j
1	$\mathbf{Z_4}$	$\frac{t}{x}$	u	$\beta + \frac{1}{2c\alpha_c} \ln\left(\frac{t+\alpha_c x}{t-\alpha_c x}\right), \quad c, \beta \in \mathbb{R}, \quad \alpha_c = \sqrt{\frac{1+c}{c}}$
2	$a\mathbf{Z_3} + \mathbf{Z_4}$	$\frac{t}{x}$	$u - a\ln(t)$	$a\ln(t) + f(z)$
3	$\mathbf{Z_1} + a\mathbf{Z_3}$		$u - ax$	$k_a e^{2at} + ax + c, \quad c \in \mathbb{R}$
4	$\mathbf{Z_1} + \mathbf{Z_2} + a\mathbf{Z_3}$	$t - x$	$u - ax$	$u(t,x) = ax + c \quad , \quad u(t,x) = at + c, \quad c \in \mathbb{R}$
5	$-\mathbf{Z_1}+\mathbf{Z_2}+a\mathbf{Z_3}$	$t + x$	$u - at$	$u(t,x) = at + c \quad , \quad u(t,x) = -ax + c, \quad c \in \mathbb{R}$

Conclusion

In this work, we showed that the symmetry algebra of the (C-B) equation is solvable, not semi simple and not nilpotent. Furthermore, we used some one dimensional optimal subalgebras to reduce the studied equation into a list of inequivalent ordinary differential equations and we determined some exact solutions of the studied equation. Finally, we notice that the used method can be extended to study the structure of Lie symmetry algebras admitted by other linear and non linear ordinary and fractional differential equations.

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