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The Bi-Fox-Wright Function Srivastava Triple Hypergeometric Function ${}^{\Psi}H^{m,n}_{A;p,q}(.)$ and Applications of the Multiple Laplace Transform

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ABSTRACT: The aim of this work is to introduce bi-Fox-Wright function Srivastava triple hypergeometric function that includes two Fox-Wright functions as its kernel, demonstrating how it simplifies to certain established results in the literature. The discussion includes properties such as integral representations, differential formulas, and recurrence relations. In addition, connections are established between the double Mellin transform and the Laplace transform. Finally, solution to the double Caputo fractional differential equation containing the bi-Fox-Wright function Srivastava triple hypergeometric function using the Laplace integral transform is formulated.

Keywords: Extended beta function, hypergeometric function, Fox-Wright function, Mellin transform, double Laplace transform.

Contents

1. Introduction and Preliminary

Special functions are obtained as a solution to the differential equation; for example, the hypergeometric differential equation is an important differential equation that arises in science and engineering, given in [1] by

$$z(1-z)\frac{d^2u}{dz^2}+[c-(a+b+1)]\frac{du}{dz}-abu=0. \tag{1.1}$$

One of the most significant solutions of (1.1) is called the Gauss hypergeometric function named after Johann Carl Friedrich Gauss (1777-1855) given in [5] as

$${}_2F_1(a,b;c;z)=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{z^r}{r!}, \tag{1.2}$$

where $|z|<1$, $a,b\in\mathbb{C}$, $c\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathbb{C}$ is the set of complex numbers, $\mathbb{Z}_0^-$ denotes the sets of non-positive integers and $(a)_r$ is the Pochhammer's symbol (also known as the rising factorial) defined for a complex

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number $(a \neq 0)$ as

$$(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2)(a+r-1), & \text{if } r \in \mathbb{N}, a \in \mathbb{C}, \\ 1, & \text{if } r = 0, a \in \mathbb{C}\backslash\{0\}, \end{cases} \tag{1.3}$$

where $\mathbb{N}$ is the set of natural numbers, $\Gamma(.)$ is the classical gamma function [19]. The following properties of the Pochhammer symbol can be found in [6,17]:

$$(a)_{r+k} = (a)_r(a+r)_k = (a)_k(a+k)_r, \tag{1.4}$$

$$(a+1)_{r+k} = (a)_{r+k}\left(1 + \frac{r}{a} + \frac{k}{a}\right), \tag{1.5}$$

$$\sum_{r=0}^{\infty}(a)_r\frac{z^r}{r!} = (1-z)^{-a}, |z| < 1. \tag{1.6}$$

In 1880 Appell considered the following product of two Gauss functions [18]:

$${}_2F_1(a,b;c;y)\,{}_2F_1(a',b';c';z) = \sum_{j,r=0}^{\infty}\frac{(a)_j(b')_j(a')_r(b)_r}{(c)_j(c')_r}\frac{y^j}{j!}\frac{z^r}{r!}, \tag{1.7}$$

this study identifies five distinct possibilities for obtaining new functions, including four Appell functions of two variables. It leads to the discovery of 14 complete Saran hypergeometric functions of three variables and of second order, together with ten types of the triple hypergeometric function from the Lauricella's set and 20 different triple hypergeometric functions of Exton. Srivastava observed the existence of three additional complete triple hypergeometric functions of the second order that were not included in Lauricella's set, which are defined as follows [20]:

$$H_A(a,b,c;d,e;x,y,z) = \sum_{j,k,r=0}^{\infty}\frac{(a)_{j+r}(b)_{j+k}(c)_{k+r}}{(d)_j(e)_{k+r}}\frac{x^j}{j!}\frac{y^k}{k!}\frac{z^r}{r!}, \tag{1.8}$$

$$H_B(a,b,c;d,e,f;x,y,z) = \sum_{j,k,r=0}^{\infty}\frac{(a)_{j+r}(b)_{j+k}(c)_{j+r}}{(d)_j(e)_k(f)_r}\frac{x^j}{j!}\frac{y^k}{k!}\frac{z^r}{r!}, \tag{1.9}$$

$$H_C(a,b,c;d;x,y,z) = \sum_{j,k,r=0}^{\infty}\frac{(a)_{j+r}(b)_{j+k}(c)_{k+r}}{(d)_{j+k+r}}\frac{x^j}{j!}\frac{y^k}{k!}\frac{z^r}{r!}. \tag{1.10}$$

Next, we give the definition of some closely related functions, integral transforms, and fractional derivative operators as follows:

Definition 1.1 *The beta function is defined in [12] as*

$$B(x,y) = \begin{cases} \int_0^1 t^{x-1}(1-t)^{y-1}dt, & \left(\min\{Re(x), Re(y)\} > 0, a \in \mathbb{C}\right), \\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+t)}, & \left(x, y \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \end{cases} \tag{1.11}$$

Definition 1.2 *The Fox-Wright function is defined by the series formula in [15]*

$${}_h\Psi_g(z) = {}_h\Psi_g\left[\begin{array}{c|c} (r_k, R_k)_{1,h} & \\ & z \\ (d_\iota, D_\iota)_{1,g} & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{k=1}^{h}\Gamma(r_k + nR_k)}{\prod_{\iota}^{g}\Gamma(d_\iota + nD_\iota)}\frac{z^n}{n!}, \tag{1.12}$$

where $z, r_k, d_\iota \in \mathbb{C}$, $R_k, D_\iota \in \mathbb{R}$, $(k = 1,2,3,\ldots,\epsilon; \imath = 1,2,3,\ldots,\varepsilon)$, $\mathbb{R}$ *is the set of real numbers.*

Definition 1.3 *[2] The integral representation of the Gauss hypergeometric function in (1.2) is*

$${}_2F_1(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b1}(1-tz)^{-a}dt, \tag{1.13}$$

where $b, c \in \mathbb{C}$, $Re(c) > Re(b) > 0$, $|1-z)| < \pi$.

Definition 1.4 *The Laplace transform given in [11] as*

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} f(t)dt, \tag{1.14}$$

for all $s \in \mathbb{C}$, $t \in \mathbb{R}^+$ and $\mathbb{R}^+$ is the set of positive real numbers. The inverse Laplace transform is

$$f(t) = \mathcal{L}^{-1}\{f(t)\}(s) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)ds, \tag{1.15}$$

where denotes i is the imaginary unit and $\gamma > 0$.

Definition 1.5 *(The double Laplace transform) The double Laplace transform [8] is*

$$\mathcal{L}_t\mathcal{L}_\tau\{f(t,\tau)\}(s,\ell) = F(s,\ell) = \int_0^\infty\int_0^\infty e^{-st-\ell\tau} f(t,\tau)dtd\tau, \tag{1.16}$$

for all $s, \ell \in \mathbb{C}$, $t, \tau \in \mathbb{R}^+$. The inverse double Laplace transform is

$$f(t,\tau) = \mathcal{L}_t^{-1}\mathcal{L}_\tau^{-1}\{f(t,\tau)\}(s,\ell) = \frac{1}{(2\pi i)^2}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st+\ell\tau}F(s,\ell)dsd\ell, \tag{1.17}$$

where $Re(s) > \gamma$ and $Re(\ell) > \epsilon$.

Definition 1.6 *The Mellin transform [11] is*

$$\mathcal{M}\{f(t)\}(s) = \int_0^\infty t^{s-1} f(t)dt, \tag{1.18}$$

for all $s \in \mathbb{C}$, $t \in \mathbb{R}^+$. The inverse Mellin transform is given by

$$\mathcal{M}^{-1}\{f(s)\}(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} t^{-s} f(s)ds, \tag{1.19}$$

for all $\sigma = Re(s)$, $t \in \mathbb{R}^+$.

Definition 1.7 *[13] (The double Mellin transform) Let $(y,z) \in \mathbb{R}^2_+ = (0,\infty)\times(0,\infty)$. The function $f(y,z)$ is Mellin transformable if the function $f(y,z)y^a z^b$ is absolute Lebesgue integrable in $\mathbb{R}^2_+$ for some real numbers a and b. Then the double Mellin transform of $f(y,z)$ is defined as the complex function*

$$\mathcal{M}\{f(y,z)\}(s,\ell) = \int_0^\infty\int_0^\infty y^{s-1}z^{\ell-1}f(y,z)dydz, \tag{1.20}$$

where s and ℓ are complex numbers such that $Re(s) = a$ and $Re(\ell) = b$.
Let the function $f(y,z)$ be continuous on $\mathbb{R}^2_+$ with the Mellin transform $f(s,\ell)$. Then the inverse of double Mellin transform exists and satisfies

$$f(y,z) = \frac{1}{(2\pi i)^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\delta-i\infty}^{\delta+i\infty} y^{-s}z^{-\ell}f(s,\ell)dsd\ell. \tag{1.21}$$

Definition 1.8 *[14] The Caputo fractional differential operator is*

$${}^{C}D_z^{\alpha}\{f(t)\} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^z (z-t)^{-\alpha-1}\frac{d^m}{dz^m}\{f(t)\}dt, & (0 \le m-1 < Re(\alpha) < m, m \in \mathbb{N}), \\ \frac{d^m}{dz^m}\{f(t)\}, & (\alpha = m, m \in \mathbb{N}). \end{cases} \tag{1.22}$$

Definition 1.9 *[3] The double Caputo fractional derivative operator is*

$$ {}_{q}^{C}D_{0+}^{\zeta}\,{}_{p}^{C}D_{0+}^{\xi}\{y(p,q)\} = \frac{1}{\Gamma(m-\zeta)\Gamma(n-\xi)} \int_0^q \int_0^p (p-t)^{m-\zeta-1}(q-\tau)^{n-\xi-1} \frac{\partial^{m+n}}{\partial t^m \partial \tau^n}\{f(t,\tau)\}dtd\tau, \quad (1.23) $$

where $m-1 \leq Re(\zeta) < m$, $n-a \leq Re(\xi) < n$, $m,n \in \mathbb{N}$.
The double Laplace transform of the double Caputo fractional derivative operator is

$$ \begin{aligned} &\mathcal{L}_q\mathcal{L}_p\{{}_{q}^{C}D_{0+}^{\zeta}\,{}_{p}^{C}D_{0+}^{\xi}\{y(p,q)\}\}(s,\ell) = s^{\xi}\ell^{\zeta}\left[\mathcal{L}_q\mathcal{L}_p\{y(p,q)\}(s,\ell) - \sum_{k=0}^{m-1} s^{-1-k}\mathcal{L}_q\left\{\frac{\partial^k y(0,q)}{\partial p^k}\right\}(\ell)\right. \\ &\left. -\sum_{l=0}^{n-1} \ell^{-1-l}\mathcal{L}_p\left\{\frac{\partial^l y(p,0)}{\partial q^l}\right\}(s) + \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} s^{-1-k}\ell^{-1-l}\mathcal{L}_q\left\{\frac{\partial^{k+l} y(0,q)}{\partial p^k \partial q^l}\right\}\right]. \end{aligned} \quad (1.24) $$

Taking $m = n = 2$ in (1.24), then the following result is obtained:

$$ \begin{aligned} &\mathcal{L}_q\mathcal{L}_p\{{}_{q}^{C}D_{0+}^{\zeta}\,{}_{p}^{C}D_{0+}^{\xi}\{y(p,q)\}\}(s,\ell) = s^{\xi}\ell^{\zeta}\left[\mathcal{L}_q\mathcal{L}_p\{y(p,q)\}(s,\ell) - s^{-1}\mathcal{L}_q\{y(0,q)\}(s)\right. \\ &- s^{-2}\mathcal{L}_q\left\{\frac{\partial y(0,q)}{\partial p}\right\}(s) - \ell^{-1}\mathcal{L}_p\{y(p,0)\}(\ell) - \ell^{-2}\mathcal{L}_p\left\{\frac{\partial y(p,0)}{\partial q}\right\}(\ell) + s^{-1}\ell^{-1}y(0,0) \\ &\left. + s^{-1}\ell^{-2}\frac{\partial y(0,0)}{\partial p} + s^{-2}\ell^{-1}\frac{\partial y(0,0)}{\partial q} + s^{-2}\ell^{-2}\frac{\partial y(0,0)}{\partial p \partial q}\right]. \end{aligned} \quad (1.25) $$

Definition 1.10 *[9] The bi-Fox-Wright function beta function is*

$$ \begin{aligned} {}^{\Psi}B_{p,q}^{m,n}(x,y) &= {}^{\Psi}B_{p,q}^{m,n}\left[\begin{array}{c|c|c} (\tau_i,T_i)_{1,h} & (r_k,R_k)_{1,f} & \\ & & x,y \\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right] \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_{h}\Psi_{g}\left(-\frac{p}{t^m}\right){}_{f}\Psi_{u}\left(-\frac{q}{(1-t)^n}\right)dt, \end{aligned} \quad (1.26) $$

where $\min\{Re(m), Re(n)\} > 0$, $\min\{Re(p), Re(q)\} > 0$, $\min\{Re(x), Re(y)\} > 0$.

Lemma 1.1 *[10] The double Laplace transform of the bi-Fox-Wright function beta function is*

$$ \mathcal{L}\left\{{}^{\Psi}B_{p,q}^{m,n}(x,y)\right\}(p,q) = \frac{1}{s\ell}\,{}^{\Psi}B_{\frac{1}{s},\frac{1}{\ell}}^{m,n}\left[\begin{array}{c|c|c} (\tau_i,T_i)_{1,h},(1,1) & (r_k,R_k)_{1,f},(1,1) & \\ & & x,y \\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right], \quad (1.27) $$

where s and ℓ are the Laplace parameters and $s,\ell > 0$.

Lemma 1.2 *The double Mellin transform of the bi-Fox-Wright function beta function is*

$$ \mathcal{M}\left\{{}^{\Psi}B_{p,q}^{m,n}(x,y)\right\} = {}^{\Psi}\Gamma(s)\,{}^{\Psi}\Gamma(\ell)B(x+ms, y+n\ell), \quad (1.28) $$

Where s,ℓ are the Mellin transform parameters with $s,\ell > 0$, $Re(x) > -ms$, $Re(y) > -n\ell$ and ${}^{\Psi}\Gamma(.)$ is the extended gamma function presented in [4].

Lemma 1.3 *[10]*

$$ \frac{\partial^\lambda}{\partial p^\lambda}\,{}^{\Psi}B_{p,q}^{m,n}(x,y) = (-1)^{\lambda}\,{}^{\Psi}B_{p,q}^{m,n}\left[\begin{array}{c|c|c} (\tau_i+\lambda T_i,T_i)_{1,h} & (r_k,R_k)_{1,f} & \\ & & x-\lambda m,y \\ (\ell_j+\lambda L_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right], \quad (1.29) $$

where $Re(x) > \lambda m$.

Lemma 1.4 *[10]*

$$\frac{\partial^{\eta}}{\partial q^{\eta}}{}^{\Psi}B^{m,n}_{p,q}(x,y)=(-1)^{\eta}{}^{\Psi}B^{m,n}_{p,q}\left[\begin{array}{c|c|c}(\tau_i,T_i)_{1,h} & (r_k+\eta R_k,R_k)_{1,f} & \\ & & x,y-\eta n\\ (\ell_j,L_j)_{1,g} & (d_\iota+\eta D_\iota,D_\iota)_{1,u} & \end{array}\right], \tag{1.30}$$

where $Re(y)>\eta n$.

Lemma 1.5 *[10]*

$$\frac{\partial^{\eta+\lambda}{}^{\Psi}B^{m,n}_{p,q}(x,y)}{\partial p^{\lambda}\partial q^{\eta}}=(-1)^{\eta+\lambda}{}^{\Psi}B^{m,n}_{p,q}\left[\begin{array}{c|c|c}(\tau_i+\lambda T_i,T_i)_{1,h} & (r_k+\eta R_k,R_k)_{1,f} & \\ & & x-\lambda m,y-\eta n\\ (\ell_j+\lambda L_j,L_j)_{1,g} & (d_\iota+\eta D_\iota,D_\iota)_{1,u} & \end{array}\right], \tag{1.31}$$

where $Re(x)>\lambda m$, $Re(y)>\eta n$.

In this work, bi-Fox-Wright function Srivastava triple hypergeometric function will be introduced and properties such as integral representations, differential formulas, recurrence relation, integral transforms and solution to double Caputo fractional differential equation containing bi-Fox-Wright function Srivastava triple hypergeometric function using double Laplace will be demonstrated.

The subsequent sections of this manuscript are organized as follows: Section 2 includes definitions and special cases of the bi-Fox-Wright function Srivastava triple hypergeometric function. Section 3 discusses integral representations properties. Section4 details the differential formulas. Section 5 illustrates recurrence relation. Section 6 presents integral transforms. Section 7 examines the application to the double Caputo fractional differential equation. Lastly, Section 8 contains the conclusion.

2. Definition and Special Cases of ${}^{\Psi}H^{m,n}_{A;p,q}(.)$

We are ready to present the definition and special cases in the following.

Definition 2.1 *The extended Srivastava triple hypergeometric ${}^{\Psi}H^{m,n}_{A;p,q}(.)$ is defined by*

$$\begin{aligned}{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)&={}^{\Psi}H^{m,n}_{A;p,q}\left[\begin{array}{c|c|c}(\tau_i,T_i)_{1,h} & (r_k,R_k)_{1,f} & \\ & & a,b,c;d,e;x,y,z\\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right]\\ &=\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!},\end{aligned} \tag{2.1}$$

where $\min\{Re(m),Re(n)\}>0$, $\min\{Re(p),Re(q)\}>0$, $\min\{Re(x),Re(y)\}>0$.

If the parameters are substituted appropriately, the following special cases can be obtained:

i. Srivastava and Karlsson [20]

$$H_A(a,b,c;d,e;x,y,z)={}^{\Psi}H^{1,1}_{A;0,0}\left[\begin{array}{c|c|c}(1,0)_{1,1} & (1,0)_{1,1} & \\ & & a,b,c;d,e;x,y,z\\ (1,0)_{1,1} & (1,0)_{1,1} & \end{array}\right].$$

ii. Cetinkaya et al., [7]

$$H_{A;p}(a,b,c;d,e;x,y,z)={}^{\Psi}H^{1,1}_{A;p,p}\left[\begin{array}{c|c|c}(1,0)_{1,1} & (1,0)_{1,1} & \\ & & a,b,c;d,e;x,y,z\\ (1,0)_{1,1} & (1,0)_{1,1} & \end{array}\right].$$

iii. Parmar and Pogany [16]

$$H_{A;p,q}(a,b,c;d,e;x,y,z)={}^{\Psi}H^{1,1}_{A;p,q}\left[\begin{array}{c|c|c}(1,0)_{1,1} & (1,0)_{1,1} & \\ & & a,b,c;d,e;x,y,z\\ (1,0)_{1,1} & (1,0)_{1,1} & \end{array}\right].$$

3. Integral Representation of ${}^{\Psi}H_{A;p,q}^{m,n}(.)$

This section addresses various integral representations of the bi-Fox-Wright function Srivastava triple hypergeometric function ${}^{\Psi}H_{A;p,q}^{m,n}(.)$. We commence the section with the subsequent theorem.

Theorem 3.1

$$\begin{aligned}{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z) =& \frac{1}{B(c,e-c)}\int_0^1 t^{c-1}(1-t)^{e-c-1}(1-ty)^{-b}(1-tz)^{-a}\\ &\times {}_2F_1\left(a,b;d;\frac{x}{(1-ty)(1-tz)}\right){}_h\Psi_g\left(-\frac{p}{t^m}\right){}_f\Psi_u\left(-\frac{q}{(1-t)^n}\right)dt. \end{aligned} \tag{3.1}$$

Proof: Applying the bi-Fox-Wright function beta function in (1.26) to (2.1), leads to

$$\begin{aligned}{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z) = &\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r B(c,e-c)}\left\{\int_0^1 t^{c-1}(1-t)^{e-c-1}\right.\\ &\left.\times\, {}_h\Psi_g\left(-\frac{p}{t^m}\right){}_f\Psi_u\left(-\frac{q}{(1-t)^n}\right)dt\right\}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}. \end{aligned} \tag{3.2}$$

Changing the order of summation and integration in Eq. (3.2) and later using the property of Pochhammer symbol in (1.4), gives

$$\begin{aligned}{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z) =& \frac{1}{B(c,e-c)}\int_0^1 t^{c-1}(1-t)^{e-c-1}{}_h\Psi_g\left(-\frac{p}{t^m}\right){}_f\Psi_u\left(-\frac{q}{(1-t)^n}\right)\\ &\times\left\{\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(d)_r}\frac{x^r}{r!}\sum_{k}^{\infty}(b+r)_k\frac{(ty)^k}{k!}\sum_{l}^{\infty}(a+r)_l\frac{(tz)^l}{l!}\right\}. \end{aligned} \tag{3.3}$$

Using the generalized binomial theorem in Eq. (1.6), the Gauss hypergeometric function in (1.2) to (3.3), the required result in (3.1) is obtained. □

Theorem 3.2

$$\begin{aligned}{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z) =& \frac{1}{B(c,e-c)B(b,d-b)}\int_0^1\int_0^1 t^{c-1}w^{b-1}(1-t)^{e-c-1}(1-w)^{d-b-1}(1-ty)^{a-b}\\ &\times[(1-ty)(1-tz)-xy]^{-a}\,{}_h\Psi_g\left(-\frac{p}{t^m}\right){}_f\Psi_u\left(-\frac{q}{(1-t)^n}\right)dt. \end{aligned} \tag{3.4}$$

Proof: Putting the integral representation of the Gauss hypergeometric function in Eq. (1.13) into (3.1), the required result in (3.4) is obtained. □

Little simplification of Eq. (3.4), the following result is obtained:

Corollary 3.1

$$\begin{aligned}{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z) = &\frac{1}{B(c,e-c)B(b,d-b)}\int_0^1\int_0^1 t^{c-1}v^{b-1}(1-t)^{e-c-1}(1-v)^{d-b-1}(1-ty)^{-b}\\ \times(1-xv-tz)^{-a}&\left(1-\frac{xytv}{(1-ty)(1-vx-tz)}\right)^{a}{}_h\Psi_g\left(-\frac{p}{t^m}\right){}_f\Psi_u\left(-\frac{q}{(1-t)^n}\right)dt. \end{aligned} \tag{3.5}$$

Theorem 3.3

$$\begin{aligned}{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z) = &\frac{2}{B(c,e-c)}\int_0^{\frac{\pi}{2}}\sin^{2c-1}\theta\cos^{e-c-1}\theta(1-y\sin^2\theta)^{-b}(1-z\sin^2\theta)^{-a}\\ \times\,{}_2F_1\left(a,b;d;\frac{x}{(1-y\sin^2\theta)(1-z\sin^2\theta)}\right)&{}_h\Psi_g\left(-\frac{p}{\sin^m\theta}\right){}_f\Psi_u\left(-\frac{q}{\cos^n\theta}\right)d\theta. \end{aligned} \tag{3.6}$$

Proof: Substituting $t=\sin^2\theta$ then $1-t=\cos^2\theta$ and $dt=\sin\theta\cos\theta$ also this implies at $t=0$, $\theta=0$ and at $t=1$, $\theta=\frac{\pi}{2}$ in Eq. (3.1), we obtained the desired result in (3.6). ☐

Theorem 3.4

$$
\begin{aligned}
{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z) &= \frac{1}{B(c,e-c)}\int_0^\infty \frac{t^{c-1}}{(1-t)^{a+b+e}}(1+v-vy)^{-b}(1+v-vz)^{-a}\\
\times\, {}_2F_1\left(a,b;d;\frac{x(1+v)^2}{(1+v-vy)(1+v-vz)}\right) &{}_h\Psi_g\left(-p\left(1+\frac{1}{v}\right)^m\right) {}_f\Psi_u\left(-q(1+v)^n\right)dv. \qquad (3.7)
\end{aligned}
$$

Proof: Letting $v=\frac{v}{1+v}$ then $1-t=\frac{1}{1+v}$ and $dt=\frac{dv}{(1+v)^2}$ also this implies at $t=0$, $v=0$ and at $t=1$, $v=1$ into Eq. (3.1), we obtained the desired result in (3.7). ☐

Theorem 3.5

$$
\begin{aligned}
&{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\\
&= \frac{(\phi-\varphi)^{1+a+b-e}}{B(c,e-c)}\int_\varphi^\phi (v-\varphi)^{c-1}(\phi-v)^{e-c-1}\left[\phi-\varphi(v-\varphi)y\right]^{-b}\left[\phi-\varphi(v-\varphi)z\right]^{-a}\\
&\times\, {}_2F_1\left(a,b;d;\frac{x(\phi-\varphi)^2}{\left[\phi-\varphi(v-\varphi)y\right]\left[\phi-\varphi(v-\varphi)z\right]}\right) {}_h\Psi_g\left(-\frac{p(\phi-\varphi)^m}{(v-\varphi)^m}\right) {}_f\Psi_u\left(-\frac{q(\phi-\varphi)^m}{(\phi-v)^n}\right)dv. \qquad (3.8)
\end{aligned}
$$

Proof: Substituting $t=\frac{v-\varphi}{\phi-\varphi}$, then $1-t=\frac{\phi-v}{\phi-\varphi}$, $dt=\frac{dv}{\phi-\varphi}$ this also implies at $t=0$, $v=\varphi$ and at $t=1$, $v=\phi$ into Eq. (3.1), we obtained the desired result in (3.8). ☐

Theorem 3.6

$$
\begin{aligned}
{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z) &= \frac{2^{1+a+b-e}}{B(c,e-c)}\int_{-1}^1 (v+1)^{c-1}(1-v)^{e-c-1}\left[\phi-\varphi(v-\varphi)y\right]^{-b}\left[\phi+(v+1)z\right]^{-a}\\
\times\, {}_2F_1\left(a,b;d;\frac{2^2x}{\left[1+(v+1)y\right]\left[1+(v+1)z\right]}\right) &{}_h\Psi_g\left(-\frac{2^m p}{(v+1)^m}\right) {}_f\Psi_u\left(-\frac{2^m q}{(1-v)^n}\right)dv. \qquad (3.9)
\end{aligned}
$$

Proof: Putting $t=\frac{v+1}{2}$, then $1-t=\frac{1-v}{2}$ and $dt=\frac{dv}{2}$ this also implies at $t=0$, $v=-1$ and at $t=1$, $v=1$ into Eq. (3.1), we obtained the desired result in (3.9). ☐

Theorem 3.7

$$
\begin{aligned}
{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z) = \frac{(1+\kappa)^c}{B(c,e-c)}\int_0^1 &\frac{v^{c-1}(1-v)^{e-c-1}}{(1+\kappa v)^{e-b-a}}\left[1+\kappa v-(1+\kappa)vy\right]^{-b}\\
&\times\left[1+\kappa v-(1+\kappa)vz\right]^{-a} {}_2F_1\left(a,b;d;\frac{x(1+\kappa v)^2}{\left[1+\kappa v-(1+\kappa)vy\right]\left[1+\kappa v-(1+\kappa)vz\right]}\right)\\
&\times {}_h\Psi_g\left(-\frac{p(1+\kappa v)^m}{([1+\kappa]v)^m}\right) {}_f\Psi_u\left(-\frac{q(1+\kappa v)^n}{(1-v)^n}\right)dt. \qquad (3.10)
\end{aligned}
$$

Proof: Setting $t=\frac{(1+\kappa)v}{(1+\kappa v)}$ then $1-t=\frac{(1-v)}{(1+\kappa v)}$ and $dt=\frac{(1+\kappa)}{(1+\kappa v)^2}dv$ also implies at $t=0$, $v=0$ and at $t=1$, $v=1$ into Eq. (3.1), we obtained the desired result in (3.10). ☐

4. Differential Formulas of ${}^{\Psi}H_{A;p,q}^{m,n}(.)$

We start the Section with following theorem.

Theorem 4.1

$$\frac{\partial^{\lambda}}{\partial p^{\lambda}}\,{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z)=\frac{(-1)^{\lambda}B(c-\lambda m,e-c)}{B(c,e-c)}$$
$$\times\,{}^{\Psi}H_{A;p,q}^{m,n}\left[\begin{array}{c|c|c}(\tau_i+\lambda T_i,T_i)_{1,h} & (r_k,R_k)_{1,f} & \\ & & a,b,c-\lambda m;d,e-\lambda m;x,y,z\\ (\ell_j+\lambda L_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right]. \tag{4.1}$$

Proof: Differentiating Eq. (2.1) λ-time with respect to p using, gives

$$\frac{\partial^{\lambda}}{\partial p^{\lambda}}\,{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z)=\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\frac{\partial^{\lambda}}{\partial p^{\lambda}}\left\{{}^{\Psi}B_{p,q}^{m,n}(c+k+l,e-c)\right\}. \tag{4.2}$$

Applying Eq. (1.29) to (4.2), gives

$$\frac{\partial^{\lambda}}{\partial p^{\lambda}}\,{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z)$$
$$=(-1)^{\lambda}\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B_{p,q}^{m,n}\left[\begin{array}{c|c|c}(\tau_i+\lambda T_i,T_i)_{1,h} & (r_k,R_k)_{1,f} & \\ & & c-\lambda m+k+l,e-c\\ (\ell_j+\lambda L_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right]}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}$$
$$=\frac{(-1)^{\lambda}B(c-\lambda m,e-c)}{B(c,e-c)}\,{}^{\Psi}H_{A;p,q}^{m,n}\left[\begin{array}{c|c|c}(\tau_i+\lambda T_i,T_i)_{1,h} & (r_k,R_k)_{1,f} & \\ & & a,b,c-\lambda m;d,e-\lambda m;x,y,z\\ (\ell_j+\lambda L_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right].$$

□

Applying Eq. (1.30) to (2.1), the following differential formula is obtained:

Corollary 4.1

$$\frac{\partial^{\eta}}{\partial q^{\eta}}\,{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z)=\frac{(-1)^{\lambda}B(c-,e-c-\zeta n)}{B(c,e-c)}$$
$$\times\,{}^{\Psi}H_{A;p,q}^{m,n}\left[\begin{array}{c|c|c}(\tau_i,T_i)_{1,h} & (r_k+\eta R_k,R_k)_{1,f} & \\ & & a,b,c;d,e-\eta n,x,y,z\\ (\ell_j,L_j)_{1,g} & (d_\iota+\eta D_\iota,D_\iota)_{1,u} & \end{array}\right]. \tag{4.3}$$

Putting Eq. (1.31) to (2.1), gives

Corollary 4.2

$$\frac{\partial^{\eta+\lambda\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z)}{\partial p^{\lambda}\partial q^{\eta}}=\frac{(-1)^{\eta+\lambda}B(c-\lambda m,e-c-\zeta n)}{B(c,e-c)}$$
$$\times\,{}^{\Psi}H_{A;p,q}^{m,n}\left[\begin{array}{c|c|c}(\tau_i+\lambda T_i,T_i)_{1,h} & (r_k+\eta R_k,R_k)_{1,f} & \\ & & a,b,c-\lambda m;d,e--\lambda m\eta n;x,y,z\\ (\ell_j+\lambda L_j,L_j)_{1,g} & (d_\iota+\eta D_\iota,D_\iota)_{1,u} & \end{array}\right]. \tag{4.4}$$

Theorem 4.2

$$\frac{\partial^{\lambda+\zeta+\omega}}{\partial x^{\lambda}\partial y^{\zeta}\partial z^{\omega}}\left\{{}^{\Psi}H_{A;p,q}^{m,n}(a,b,c;d,e;x,y,z)\right\}$$
$$=\frac{(a)_{\lambda+\omega}(b)_{\lambda+\zeta}(c)_{\zeta+\omega}}{(d)_{\lambda}(e)_{\zeta+\omega}}\,{}^{\Psi}H_{A;p,q}^{m,n}(a+\lambda+\omega,b+\lambda+\zeta,c+\zeta+\omega;d+\lambda,e+\zeta+\omega;x,y,z), \tag{4.5}$$

where $\lambda,\zeta,\omega\in\mathbb{N}$.

Proof: Differentiating Eq. (2.1) partially with respect to x, gives

$$\frac{\partial}{\partial x}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}=\sum_{r=1}^{\infty}\sum_{k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^{r-1}}{(r-1)!}\frac{y^k}{k!}\frac{z^l}{l!}. \tag{4.6}$$

Setting $r\to r+1$ in Eq. (4.6) and using the property of Pochhammer symbol in (1.4), yields

$$\begin{aligned}&\frac{\partial}{\partial x}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}\\&=\frac{(a)(b)}{(d)}\sum_{r,k,l=0}^{\infty}\frac{(a+1)_{r+l}(b+1)_{r+k}}{(d+1)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}.\end{aligned} \tag{4.7}$$

Differentiating Eq. (4.7) $(\lambda-1)$-times partially with respect to x, gives

$$\begin{aligned}&\frac{\partial}{\partial x}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}\\&=\frac{(a)_\lambda(b)_\lambda}{(d)_\lambda}\sum_{r,k,l=0}^{\infty}\frac{(a+\lambda)_{r+l}(b+\lambda)_{r+k}}{(d+\lambda)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\\&=\frac{(a)_\lambda(b)_\lambda}{(d)_\lambda}{}^{\Psi}H^{m,n}_{A;p,q}(a+\lambda,b+\lambda,c;d+\lambda,e;x,y,z).\end{aligned} \tag{4.8}$$

Similarly, differentiating Eq. (4.8) ζ-times partially with respect to y, yields

$$\begin{aligned}&\frac{\partial^{\lambda+\zeta}}{\partial x^\lambda\partial y^\zeta}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}\\&=\frac{(a)_\lambda(b)_{\lambda+\zeta}(c)_\zeta}{(d)_\lambda}\sum_{r,k,l=0}^{\infty}\frac{(a+\lambda)_{r+l}(b+\lambda+\zeta)_{r+k}}{(d+\lambda)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+\zeta+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\\&=\frac{(a)_\lambda(b)_{\lambda+\zeta}(c)_\zeta}{(d)_\lambda}{}^{\Psi}H^{m,n}_{A;p,q}(a+\lambda,b+\lambda+\zeta,c+\zeta;d+\lambda,e;x,y,z).\end{aligned} \tag{4.9}$$

Again, Differentiating Eq. (4.10) ω-times partially with respect to z, gives

$$\begin{aligned}&\frac{\partial^{\lambda+\zeta+\omega}}{\partial x^\lambda\partial y^\zeta\partial z^\omega}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}\\&=\frac{(a)_{\lambda+\omega}(b)_{\lambda+\zeta}(c)_{\zeta+\omega}}{(d)_\lambda(e)_\omega}\sum_{r,k,l=0}^{\infty}\frac{(a+\lambda)_{r+l}(b+\lambda+\zeta)_{r+k}}{(d+\lambda)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+\zeta+\omega+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\\&=\frac{(a)_{\lambda+\omega}(b)_{\lambda+\zeta}(c)_{\zeta+\omega}}{(d)_\lambda(e)_\omega}{}^{\Psi}H^{m,n}_{A;p,q}(a+\lambda+\omega,b+\lambda+\zeta,c+\zeta+\omega;d+\lambda,e+\omega;x,y,z).\end{aligned} \tag{4.10}$$

□

5. Recurrence Relation of ${}^{\Psi}H^{m,n}_{A;p,q}(.)$

We can see the bi-Fox-Wright Function Srivastava triple hypergeometric function ${}^{\Psi}H^{m,n}_{A;p,q}(.)$ satisfies following recurrence relation

Theorem 5.1

$$\begin{aligned}&{}^{\Psi}H^{m,n}_{A;p,q}(a+1,b,c;d,e;x,y,z)={}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\\&+\frac{bx}{a}{}^{\Psi}H^{m,n}_{A;p,q}(a+1,b+1,c;d+1,e;x,y,z)+\frac{z}{c}{}^{\Psi}H^{m,n}_{A;p,q}(a+1,b,c+1;d,e;x,y,z).\end{aligned} \tag{5.1}$$

Proof: The left-hand side of Eq. (5.1) can be expressed as

$$
{}^{\Psi}H^{m,n}_{A;p,q}(a+1,b,c;d,e;x,y,z)=\sum_{r,k,l=0}^{\infty}\frac{(a+1)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}. \tag{5.2}
$$

Applying the Pochhammer symbol property in Eq. (1.5) to (5.2), we get the following

$$
\begin{aligned}
&{}^{\Psi}H^{m,n}_{A;p,q}(a+1,b,c;d,e;x,y,z)\\
&=\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\left(1+\frac{r}{a}+\frac{l}{a}\right)\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}.
\end{aligned} \tag{5.3}
$$

In simplification of Eq. (5.3), it gives the following

$$
\begin{aligned}
{}^{\Psi}H^{m,n}_{A;p,q}(a+1,b,c;d,e;x,y,z)&=\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\\
&+\frac{x}{a}\sum_{r=1}^{\infty}\sum_{k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^{r-1}}{(r-1)!}\frac{y^k}{k!}\frac{z^l}{l!}\\
&+\frac{z}{a}\sum_{l=1}^{\infty}\sum_{r,k=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^{l-1}}{(l-1)!}.
\end{aligned} \tag{5.4}
$$

Putting $r\to r+1$ and $l\to l+1$ in the third and fourth summations and applying the property of the Pochhammer symbol in Eq. (1.4), gives

$$
\begin{aligned}
{}^{\Psi}H^{m,n}_{A;p,q}(a+1,b,c;d,e;x,y,z)&=\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\\
&+\frac{bx}{a}\sum_{r,k,l=0}^{\infty}\frac{(a+1)_{r+l}(b+1)_{r+k}}{(d+1)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\\
&+\frac{z}{c}\sum_{r,k,l=0}^{\infty}\frac{(a+1)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{p,q}(c+k+l+1,e-c)}{B(c+1,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}.
\end{aligned} \tag{5.5}
$$

Writing Eq. (5.5) in compact form, the result required in (5.1) is obtained. □

6. Integral Transforms of ${}^{\Psi}H^{m,n}_{A;p,q}(.)$

Theorem 6.1 *(Double Laplace transform)*

$$
\begin{aligned}
&\mathcal{L}\left\{H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}(p,q)\\
&=\frac{1}{s\ell}\,{}^{\Psi}H^{m,n}_{A;\frac{1}{s},\frac{1}{\ell}}\left[\begin{array}{c|c|c}(\tau_i,T_i)_{1,h},(1,1) & (r_k,R_k)_{1,f},(1,1) & \\ & & a,b,c;d,e;x,y,z\\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right],
\end{aligned} \tag{6.1}
$$

where s and ℓ are the Laplace parameters and $s,\ell>0$.

Proof: Applying double Laplace transform in Eq. (1.20) to (2.1) and changing the order of summation and integration, gives

$$
\mathcal{L}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}(s,\ell)=\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_rB(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\mathcal{L}\left\{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)\right\}. \tag{6.2}
$$

Applying Eq. (1.27) to (6.2) with little simplification yields

$$\begin{aligned}
&\mathcal{L}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}(s,\ell)\\
&=\frac{1}{s\ell}\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{{}^{\Psi}B^{m,n}_{A;\frac{1}{s},\frac{1}{\ell}}\left[\begin{array}{c|c|c}(\tau_i,T_i)_{1,h},(1,1) & (r_k,R_k)_{1,f},(1,1) & \\ & & c+k+l,e-c\\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right]}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\\
&=\frac{1}{s\ell}{}^{\Psi}H^{m,n}_{A;\frac{1}{s},\frac{1}{\ell}}\left[\begin{array}{c|c|c}(\tau_i,T_i)_{1,h},(1,1) & (r_k,R_k)_{1,f},(1,1) & \\ & & a,b,c;d,e;x,y,z\\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right].
\end{aligned}$$

□

Corollary 6.1

$$\begin{aligned}
{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)&=\frac{1}{(2\pi i)^2}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{\epsilon-i\infty}^{\epsilon+i\infty}e^{st+\ell\tau}\\
\times\frac{1}{s\ell}{}^{\Psi}H^{m,n}_{A;\frac{1}{s},\frac{1}{\ell}}&\left[\begin{array}{c|c|c}(\tau_i,T_i)_{1,h},(1,1) & (r_k,R_k)_{1,f},(1,1) & \\ & & a,b,c;d,e;x,y,z\\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right]ds d\ell,
\end{aligned}\tag{6.3}$$

where $\gamma,\epsilon>0$.

Next, we shall show that the double Melli transform is also true in the following theorem

Theorem 6.2 *(Double Mellin transform)*

$$\begin{aligned}
&\mathcal{M}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}(s,\ell)\\
&=\frac{{}^{\Psi}\Gamma(s){}^{\Psi}\Gamma(\ell)B(b+ms,e+n\ell-c)}{B(c,e-c)}H_A(a,b,c+ms;d,e+ms+n\ell;x,y,z).
\end{aligned}\tag{6.4}$$

Proof: Using direct substitution of Eq. (2.1) into the left-hand side of (1.28), gives

$$\begin{aligned}
&\mathcal{M}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}(s,\ell)\\
&=\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_rB(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\mathcal{M}\left\{{}^{\Psi}B^{m,n}_{p,q}(c+k+l,e-c)\right\}.
\end{aligned}\tag{6.5}$$

Applying Eq. (1.28) to (6.5) with little simplification yields

$$\begin{aligned}
&\mathcal{M}\left\{{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)\right\}(s,\ell)\\
&=\frac{1}{s\ell}\sum_{r,k,l=0}^{\infty}\frac{(a)_{r+l}(b)_{r+k}}{(d)_r}\frac{B(b+ms+k+l,e+n\ell-c)}{B(c,e-c)}\frac{x^r}{r!}\frac{y^k}{k!}\frac{z^l}{l!}\\
&=\frac{{}^{\Psi}\Gamma(s){}^{\Psi}\Gamma(\ell)B(b+ms,e+n\ell-c)}{B(c,e-c)}H_A(a,b,c+ms;d,e+ms+n\ell;x,y,z).
\end{aligned}$$

□

Corollary 6.2

$$\begin{aligned}
{}^{\Psi}H^{m,n}_{A;p,q}(a,b,c;d,e;x,y,z)&=\frac{1}{(2\pi i)^2B(c,e-c)}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\delta-i\infty}^{\delta+i\infty}{}^{\Psi}\Gamma(s){}^{\Psi}\Gamma(\ell)B(b+ms,e+n\ell-c)\\
&\times H_A(a,b,c+ms;d,e+ms+n\ell;x,y,z)p^{-s}q^{-\ell}ds d\ell.
\end{aligned}\tag{6.6}$$

7. Double Caputo Fractional Differential Equation Containing ${}^{\Psi}H^{m,n}_{A;p,q}(.)$

Problem 1 *Consider the double Caputo fractional differential equation*

$$ {}^{C}_{q}D^{\zeta}_{0+}\,{}^{C}_{p}D^{\xi}_{0+}\{y(p,q)\} = {}^{\Psi}H^{m,n}_{A;p\xi,q\zeta}(a,b,c;d,e;x,y,z), \tag{7.1}$$

with initial condition

$$\left.\begin{aligned} y(0,0) &= \frac{\partial y(0,0)}{\partial p} = \frac{\partial y(0,0)}{\partial q} = \frac{\partial^2 y(0,0)}{\partial p\partial q} = 0\\ y(p,0) &= \frac{\partial y(p,0)}{\partial q} = 0\\ y(0,q) &= \frac{\partial y(0,q)}{\partial p} = 0 \end{aligned}\right\}. \tag{7.2}$$

Solution: Taking the double Laplace transform of both sides of Eq. (7.1), gives

$$\begin{aligned} & s^{\xi}\ell^{\zeta}\Bigg[\mathcal{L}_q\mathcal{L}_p\{y(p,q)\}(s,\ell) - s^{-1}\mathcal{L}_q\{y(0,q)\}(s) - s^{-2}\mathcal{L}_q\left\{\frac{\partial y(0,q)}{\partial p}\right\}(s) - \ell^{-1}\mathcal{L}_p\{y(p,0)\}(\ell)\\ & -\ell^{-2}\mathcal{L}_p\left\{\frac{\partial y(p,0)}{\partial q}\right\}(\ell) + s^{-1}\ell^{-1}y(0,0) + s^{-1}\ell^{-2}\frac{\partial y(0,0)}{\partial p} + s^{-2}\ell^{-1}\frac{\partial y(0,0)}{\partial q} + s^{-2}\ell^{-2}\frac{\partial y(0,0)}{\partial p\partial q}\Bigg]\\ & = \frac{\zeta\xi}{s\ell}\,{}^{\Psi}H^{m,n}_{A;\frac{\zeta}{s},\frac{\xi}{\ell}}\left[\begin{array}{c|c|c} (\tau_i,T_i)_{1,h},(1,1) & (r_k,R_k)_{1,f},(1,1) & \\ & & a,b,c;d,e;x,y,z \\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right]. \end{aligned} \tag{7.3}$$

Applying the initial condition in Eq. (7.2) to (7.3), yields

$$\begin{aligned} & \mathcal{L}_q\mathcal{L}_p\{y(p,q)\}(s,\ell)\\ & = \frac{\zeta\xi}{s^{\zeta+1}\ell^{\xi+1}}\,{}^{\Psi}H^{m,n}_{A;\frac{\zeta}{s},\frac{\xi}{\ell}}\left[\begin{array}{c|c|c} (\tau_i,T_i)_{1,h},(1,1) & (r_k,R_k)_{1,f},(1,1) & \\ & & a,b,c;d,e;x,y,z \\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right]. \end{aligned} \tag{7.4}$$

Taking the inverse double Laplace transform of Eq. (7.4), yields

$$ y(p,q) = p^{\zeta}q^{\xi}\,{}^{\Psi}H^{m,n}_{A;p\zeta,q\xi}\left[\begin{array}{c|c|c} (\tau_i,T_i)_{1,h},(1,1) & (r_k,R_k)_{1,f},(1,1) & \\ & & a,b,c;d,e;x,y,z \\ (\ell_j,L_j)_{1,g} & (d_\iota,D_\iota)_{1,u} & \end{array}\right]. \tag{7.5}$$

□

8. Conclusion

The extended Srivastava triple hypergeometric function known as the bi-Fox-Wright Function Srivastava triple hypergeometric function because its kernel consists of two Fox-Wright functions was introduced and studied due to its potential application in science and engineering.
Areas for further research include, but are not limited to:

i. Applying fractional derivative and integral operators to the bi-Fox-Wright Function Srivastava triple hypergeometric function.

ii. Investigating multi-variable and multi-index extended forms of the bi-Fox-Wright Function Srivastava triple hypergeometric function.

iii. Enhancing numerical techniques for evaluating the presented bi-Fox-Wright function Srivastava triple hypergeometric function.

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