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A Two-Step Iterative Fixed-Point Method for New General Absolute Value Equations *

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ABSTRACT: In this work, we have studied a class of new general absolute value equation (NGAVE) of type: $Ax - |Bx| = b$, (A, $B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$) are given. Some weaker sufficient conditions for the unique solvability of the NGAVE are also obtained. For its numerical solution, a two-steps Picard's fixed-point iterative method is proposed. Moreover, we have proved under an appropriate assumption that the proposed algorithm is well-defined and converges globally linearly to the unique solution of NGAVE. Finally, we present a various set of numerical results to confirm the efficiency of our proposed approach.

Keywords: Absolute value equations, unique solution, Picard's fixed-point iterative method, global convergence.

Contents

1. Introduction

In this paper, we focus on the numerical solution of the new general absolute value equations (abbreviated as NGAVE) of type:

$$Ax - |Bx| = b, \tag{1.1}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given, $b \in \mathbb{R}^n$, and $|x|$ is a vector whose i-th entry is the absolute value of the i-th entry of x. If $B = I$ is the identity matrix, then the NGAVE (1.1) can be reduced to the type:

$$Ax - |x| = b. \tag{1.2}$$

The NGAVE (1.1) and AVE (1.2) are widely applicable in optimization, including linear complementarity problems, linear programming, convex quadratic programming, and binary matrix games. The latter was first introduced by Rohn [15] and later explored in a broader context by Mangasarian and Meyer (see [10]). Other studies for the AVE can be found in [1,5,12,14,18,19]. The existence and uniqueness of the solution of the AVE have been comprehensively studied. Authors concentrate on reformulating the AVE (1.2) and NGAVE (1.1) as various equivalent problems, noting that establishing the existence and uniqueness of a solution for the NGAVE (1.1) is NP-hard problem due to the presence of the nonlinear and non-differentiable term $|Bx|$ in the NGAVE (1.1). For solving the AVE (1.2), many algorithms are

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proposed, such as the Newton method and its other versions. For example, Mangasarian in [11] proposed a semi-smooth Newton's method for solving the AVE, and under suitable conditions, he showed the finite and linear convergence to a solution of the AVE. In recent years, a wide variety of procedures have been developed for solving AVE (1.2). see e.g., [4,6,7,13,17] and the references therein. Wu and Li [22] presented a special shift splitting technique for determining the solution of AVE (1.2) and performed a convergence analysis. However, other numerical approaches focus on reformulating the AVE as a horizontal linear complementarity problem (HLCP) (see Achache [3]), where they introduce an infeasible path-following interior-point method for solving the AVE by using equivalent reformulations as an HLCP. Recently, Achache and Anane [2] proposed Picard's iterative method for solving a slightingly different GAVE of the form $Ax - B|x| = b$ and showed that this method is globally linearly convergent under suitable assumptions.

In this paper, we are interested in solving the NGEVA (1.1). We study first the unique solvability for this equation and then we propose a new two-step Picard's fixed-point iterative method for solving the NGAVE (1.1). Under a new mild assumption, we show that this method is always well-defined and the generated sequence converges globally and linearly to the unique solution of the NGAVE from any starting initial point. Moreover, some specific sufficient conditions are established when the coefficient matrix A is a symmetric positive definite. Finally, numerical results are provided to illustrate the efficiency of this algorithm in solving the NGAVE. In addition, it shows that the proposed method is better than Picard's methods in practice. The outline of this paper is organized as follows. The main results of unique solvability of the NGAVE is stated in section 2. In section 3, Picard's iterative method for solving the NGAVE is presented. In section 4, a two-step Picard's fixed-point iterative method is proposed to solve the NGAVE. Moreover, under suitable conditions, the global convergence to the unique solution is proved. In section 5, some numerical results are provided to show the efficiency of the proposed algorithm. Finally, a conclusion and some remarks are drawn in the last section of the paper. At the end of this section, some notations are presented. Let $\mathbb{R}^{n\times n}$ be the set of all $n\times n$ real matrices. The scalar product and the Euclidean norm are denoted, respectively, by $x^Ty, x, y \in \mathbb{R}^n$ and $\|x\| = \sqrt{x^Tx}$. Recall that a subordinate matrix norm for $A \in \mathbb{R}^{n\times n}$ is defined as follows: $\|A\| := \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$, this definition implies:

$$\|Ax\| \leq \|A\| \|x\|, \; \|AB\| \leq \|A\| \|B\|, \; \forall A, B \in \mathbb{R}^{n\times n} \text{ and } x \in \mathbb{R}^n.$$

The $sign(x)$ denotes a vector with components equal to -1, 0, or 1 depending on whether the corresponding component is negative, zero, or positive. In addition, $D(x) := Diag(sign(x))$ will denote a diagonal matrix corresponding to $sign(x)$. The absolute value of a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and the vector of all ones are denoted by $|A| = (|a_{ij}|) \in \mathbb{R}^{n\times n}$ and $e \in \mathbb{R}^n$, respectively. $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ represent, respectively, the smallest and the largest eigenvalue of A. Finally, a matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if for all nonzero vector x, $x^TAx > 0$ and the inverse of a nonsingular matrix A, is denoted by A^{-1}.

2. On unique solvability for NGAVE

In this section, we will give some results about the unique solvability for NGAVE. For given matrices A, $B \in \mathbb{R}^{n\times n}$ and for any diagonal matrix $D \in \mathbb{R}^{n\times n}$ whose diagonal elements are ∓ 1 and 0, we define the matrix $(A - DB) \in \mathbb{R}^{n\times n}$. Then, NGAVE is uniquely solvable for any b if the matrix $(A - DB)$ is nonsingular. To achieve these results, the following lemma is crucial.

Lemma 2.1 *Given two matrices A and B with A is non singular. If*

$$\left\|A^{-1}\right\| \|B\| < 1,$$

then the matrix $(A - DB)$ is non singular for any diagonal matrix D whose elements are ∓ 1 and 0.

Proof: We prove this by contradiction. We assume that $(A - DB)$ is singular, and let a nonzero vector x with $\|x\| = 1$, then

$$(A - DB)x = 0.$$

Since the matrix A is nonsingular so $x = A^{-1}DBx$.
Further, we have

$$\begin{aligned} 1 = \quad \|x\| &= \left\|A^{-1}DBx\right\| \\ \leq \quad & \left\|A^{-1}\right\| \|D\| \|B\| \|x\| \\ \leq \quad & \left\|A^{-1}\right\| \|B\|, \end{aligned}$$

which leads to a contradiction. Hence $(A - DB)$ is nonsingular for any diagonal matrix D whose elements are ± 1 and 0. This completes the proof. □

Next theorem guarantees the unique solvability of the NGAVE (1.1).

Theorem 2.1 *If matrices A and B satisfy*

$$\|A^{-1}\|\|B\| < 1,$$

provided A is non singular, then the NGAVE (1.1) is uniquely solvable for any vector $b \in \mathbb{R}^n$.

Proof: According to $D_{(Bx)}Bx = |Bx|$ where $D_{(Bx)} = Daig\,(sign\,(Bx))$, the NGAVE (1.1) can be rewritten as the following standard linear system of equation:

$$(A - D_{(Bx)}B)x = b.$$

Based on the results in lemma 2.1, $(A - DB)$ the matrix of coefficients of the linear system is nonsingular for any diagonal matrix D whose diagonal elements are ± 1 and 0. Hence the NGAVE (1.1) is uniquely solvable for any b. This completes the proof. □

3. A one-step Picard's fixed-point method of NGAVE

In this section, we provide a one-step Picard's fixed point iteration method for computing an approximated solution of uniquely solvable NGAVEs. The principle of the latter is to use the equivalent following scheme to NGAVE:

$$x_{k+1} = A^{-1}|Bx_k| + A^{-1}b, k = 0, 1, 2, \cdots \tag{3.1}$$

to find an approximated solution. The details of Picard's iterative algorithm for solving the NGAVE (1.1) are described in Figure 1 as follows:

3.1. Algorithm

Input:
An accuracy $\varepsilon > 0$;
an initial starting point $x_0 \in \mathbb{R}^n$;
two matrices A and B and a vector b;
set $k = 0$;
while $\frac{\|Ax-|Bx|-b\|}{1+\|b\|} > \varepsilon$ **do**
begin
compute x_{k+1} from the linear system $x_{k+1} = A^{-1}(|Bx_k| + b)$;
update $k := k + 1$;
end;
end.

Figure 1. One-step Picard's algorithm for the NGAVE

The convergence of Picard's fixed point scheme is based on the Banach fixed point theorem (see [8]).

Theorem 3.1 *Let A be nonsingular matrix and if*

$$\left\|A^{-1}\right\|\|B\|<1,$$

then the sequence $\{x_k\}$ converges to the unique solution $x^\star$ of the NGAVE (1.1) for any arbitrary $x_0 \in \mathbb{R}^n$. In this case the error bound is given by

$$\|x_{k+1}-x^\star\| \leq \frac{\left\|A^{-1}\right\|\|B\|}{1-\left\|A^{-1}\right\|\|B\|}\|x_{k+1}-x_k\|,\ k=0,1,2,\cdots$$

Moreover, the sequence $\{x_k\}$ converges to the unique solution $x^\star$ as follows

$$\|x_{k+1}-x^\star\| \leq \left\|A^{-1}\right\|\|B\|\|x_k-x^\star\|,\ k=0,1,2,\cdots$$

Proof: Here, we omit the proof since it is similar to the one given in [2] □

Specifically, when $B=I\in\mathbb{R}^{n\times n}$, we can derive the following corollary.

Corollary 3.1 *Let $A\in\mathbb{R}^{n\times n}$ be nonsingular matrix. If*

$$\left\|A^{-1}\right\|<1,$$

then the Picard iteration method (3) converges linearly from any starting point to a solution x^* of the AVE (1.2)

4. Two-step Picard's fixed-point iterative method

In this section, we establish a new approach using fixed-point iteration to solve the NGAVE. Let $y=|Bx|$, then the NGAVE (1.1) is equivalent to the following system:

$$\left\{\begin{array}{l} Ax-\ \ y=b \\ -|Bx|+y=0 \end{array}\right. . \tag{4.1}$$

The latter can be expressed as follows:

$$\left(\begin{array}{cc} A & -I \\ -D_{(Bx)}B & I \end{array}\right)\left(\begin{array}{c} \mathrm{x} \\ \mathrm{y} \end{array}\right)=\left(\begin{array}{c} \mathrm{b} \\ 0 \end{array}\right), \tag{4.2}$$

where $D_{(Bx)}:=Diag\left(sign\left(Bx\right)\right)$, $x\in\mathbb{R}^n$. Note that the system (4) is nonlinear, it is generally impossible to obtain an exact solution. We will therefore be satisfied with an approximated solution. If the matrix A is nonsingular hence from (4) we can obtain the following fixed point equation:

$$\left\{\begin{array}{l} x^\star=A^{-1}\left(y^\star+b\right) \\ y^\star=\left(1-r\right)y^\star+r\ \ |Bx^\star| \end{array}\right. , \tag{4.3}$$

where $r>0$ is a suitable parameter that we shall specify later. According to the fixed-point equation, we generate a sequence $\{(x_k,y_k)\}$ converging to the solution of NGAVE (1.1). So the new fixed-point iteration is given by:

$$\left\{\begin{array}{l} x_{k+1}=A^{-1}\left(y_k+b\right) \\ y_{k+1}=\left(1-r\right)y_k+r\left|Bx_{k+1}\right|,k=0,1,\cdots \end{array}\right. . \tag{4.4}$$

Now, we can finally describe the corresponding two-step Picard's algorithm for solving the NGAVE in Figure 2 as follows:

4.1. Algorithm

Input:
An accuracy $\varepsilon > 0$;
a parameter r such that $0 < r < \frac{2}{\|A^{-1}\|\|B\|+1}$;
an initial starting point $y_0 \in \mathbb{R}^n$;
compute $x_1 = A^{-1}(y_0 + b)$, $y_1 = (1-r)\, y_0 + r\,|Bx_1|$;
set $k = 1$;
while $\frac{\|Ax-|Bx|-b\|}{1+\|b\|} > \varepsilon$ **do**
begin
compute $\left\{ \begin{array}{lcc} x_{k+1} & = & A^{-1}(y_k + b) \\ y_{k+1} & = & (1-r)\, y_k + r\,|Bx_{k+1}| \end{array} \right.$;
$k := k + 1$;
end;
end.

Figure 2 . Two-step Picard's algorithm for the NGAVE

5. Convergence analysis

We will study the global convergence of two-step Picard's algorithm for solving the NGAVE. A general convergence condition is derived firstly. Then, some specific convergence conditions are derived when the matrix A is symmetric positive definite, first we give the following lemma.

Lemma 5.1 *For all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we obtain the following results:*

$$\| |x| - |y| \| \leq \|x - y\| \ .$$

Proof: For detailed proof see [10, Lemma 5]. □

Theorem 5.1 *Let A be non singular matrix then the sequence $\{(x_k, y_k)\}$ generated by the iterative algorithm (4.4) to solve the problem (4.1) is well-defined for any starting point $y_0 \in \mathbb{R}^n$. In addition, if*

$$\left\|A^{-1}\right\| \|B\| < 1 \quad and \quad 0 < r < \frac{2}{\|A^{-1}\| \, \|B\| + 1}$$

then the sequence $\{(x_k, y_k)\}$ converges linearly to the solution $(x^\star, y^\star)$ of the nonlinear equation (4.1).

Proof: First, because A is a non singular matrix we check that the sequence $\{(x_k, y_k)\}$ is well-defined. Next, using formula (4.4) and lemma 5.1, we have, on one hand that

$$\begin{array}{rcl} \|y_{k+1} - y^\star\| & = & \|(1-r)y_k + r\,|Bx_{k+1}| - (1-r)y^\star - r\,|Bx^\star|\| \\ & = & \|(1-r)(y_k - y^\star) + r(|Bx_{k+1}| - |Bx^\star|)\| \\ & \leq & |1-r| \, \|(y_k - y^\star)\| + r\,\|Bx_{k+1} - Bx^\star\| \\ & \leq & |1-r| \, \|(y_k - y^\star)\| + r\,\|B\| \, \|x_{k+1} - x^\star\| \,. \end{array}$$

So

$$\begin{array}{rcl} \|x_{k+1} - x^\star\| & = & \left\|A^{-1}(y_k + b) - A^{-1}(y^\star + b)\right\| \\ & = & \left\|A^{-1}(y_k - y^\star)\right\| \leq \left\|A^{-1}\right\| \|(y_k - y^\star)\| \,. \end{array}$$

Therefore

$$\|y_{k+1} - y^\star\| \leq (|1-r| + r\left\|A^{-1}\right\| \|B\|) \, \|(y_k - y^\star)\| \,.$$

On the other hand

$$\begin{aligned}\|x_{k+1}-x^{\star}\| &\leq \left\|A^{-1}(y_k-y^{\star})\right\| \\ &\leq \left\|A^{-1}(y_k-(1-r)y_{k-1}+(1-r)y_{k-1}-y^{\star})\right\| \\ &\leq \left\|A^{-1}(r\left|Bx_k\right|+(1-r)y_{k-1}-y^{\star})\right\|.\end{aligned}$$

As $|Bx| = y$, we find

$$\begin{aligned}\|x_{k+1}-x^{\star}\| &\leq \left\|A^{-1}(r\left|Bx_k\right|-r\left|Bx^{\star}\right|)-\left|1-r\right|A^{-1}(y_{k-1}-y^{\star})\right\| \\ &\leq r\left\|A^{-1}\right\|\|B\|\|x_k-x^{\star}\|+\left|1-r\right|\|x_k-x^{\star}\| \\ &\leq (\left|1-r\right|+r\left\|A^{-1}\right\|\|B\|)\|x_k-x^{\star}\|.\end{aligned}$$

The sequence (x_k, y_k) is convergent if the following condition

$$|1-r|+r\left\|A^{-1}\right\|\|B\|<1,$$

holds. For that we distinguish two cases

Case1. If $0 < r \leq 1$, then

$$\begin{aligned}|1-r|+r\left\|A^{-1}\right\|\|B\|<1 &\iff 1-r+r\left\|A^{-1}\right\|\|B\|<1 \\ &\iff r(\left\|A^{-1}\right\|\|B\|-1)<0.\end{aligned}$$

Since $\left\|A^{-1}\right\|\|B\| < 1$ then,

$$r(\left\|A^{-1}\right\|\|B\|-1)<0, \forall\ 0<r\leq 1.$$

Case2. If $r > 1$, then

$$\begin{aligned}|1-r|+r\left\|A^{-1}\right\|\|B\|<1 &\iff -1+r+r\left\|A^{-1}\right\|\|B\|<1 \\ &\iff r<\frac{2}{\|A^{-1}\|\|B\|+1}.\end{aligned}$$

Finally, regrouping the two cases, this give the required result □

Since the AVE (1.2) is a special case of the NGAVE (1.1). The two-step Picard's fixed-point iterative method (7) can be directly used to solve the AVE (1.2). By simply letting $B = I$, the following corollary can be obtained.

Corollary 5.1 *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular matrix. If*

$$\left\|A^{-1}\right\|<1 \quad \textit{and} \quad 0<r<\frac{2}{\|A^{-1}\|+1},$$

then the two-step Picard's fixed-point iterative method (7) converges linearly from any starting point to a solution x^* of the AVE (1.2).

5.1. The case of symmetric positive definite matrix

In this subsection, we discuss the convergence conditions of the two-step Picard's fixed-point iterative method (7) for solving the NGAVE (1.1) and the AVE (1.2) when the matrix A is symmetric positive definite.

Theorem 5.2 *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $\lambda_{\min}$ be the smallest eigenvalue of the matrix A and $\|B\| = \tau$. If*

$$\tau<\lambda_{\min} \ \textit{and} \ \ 0<r<\frac{2}{\frac{\tau}{\lambda_{\min}}+1}, \tag{5.1}$$

then the Two-step Picard's fixed-point iterative method (7) converges linearly from any starting point to a solution x^* of the NGAVE (1.1).

Proof: According to the proof of Theorem 4, we just need to get the sufficient conditions for

$$\|A^{-1}\|\|B\| < 1 \text{ and } 0 < r < \frac{2}{\|A^{-1}\|\,\|B\| + 1}.$$

By assumptions, we have

$$\|A^{-1}\|\|B\| = \frac{\tau}{\lambda_{\min}} < 1,$$

and

$$0 < r < \frac{2}{\frac{\tau}{\lambda_{\min}} + 1} = \frac{2}{\|A^{-1}\|\,\|B\| + 1},$$

provided that $\tau < \lambda_{\min}$ (8) holds. This completes the proof. □

In particular, if $B = I \in \mathbb{R}^{n\times n}$, the following corollary can be obtained.

Corollary 5.2 *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, let $\lambda_{\min}$ be the smallest eigenvalue of the matrix A. If*

$$1 < \lambda_{\min},$$

then the Two-step Picard's fixed-point iterative method (7) converges linearly from any starting point to a solution x^* of the AVE (1.2).

6. Numerical results

In this section, we present some examples of NGAVE problems where their unique solvability is checked. Also by applying two-step Picard's iterative method we compute an approximated solution of these examples. All programs were implemented in **MATLAB** R2016a on a personal pc with 1.40 GHZ AMD E1-2500 APU Radeon (TM) HD Graphic, 8 GB memory and Windows 10 operating system.The starting point and the unique solution of the NGAVE are denoted, respectively, by y_0 and $x^\star$. In the table of numerical results we display the following notations: "Iter" and "CPU" state for the number of iterations and the elapsed times. The termination of the algorithm is as the relative residue:

$$RES := \frac{\|Ax - |Bx| - b\|}{1 + \|b\|} \leq 10^{-6}$$

Example 6.1 Consider the NGAVE problem where A, $B \in \mathbb{R}^{10\times 10}$ are given by:

$$A = \begin{bmatrix} 101 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 102 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 103 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 104 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 105 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 106 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 107 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 108 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 109 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 110 \end{bmatrix}, B = 3I.$$

Applying Theorem 2.1, we have, $\left\|A^{-1}\right\|\|B\| = 0.009 < 1$, then the problem is uniquely solvable for any b.

For $b = (0.5A - 3I)e$, and with the different starting points:

$$\begin{aligned} y_0^1 &= [1,2,3,4,5,6,7,8,9,10]^T, \\ y_0^2 &= [7,7,7,7,7,7,7,7,7,7]^T, \\ y_0^3 &= [30,30,30,30,30,30,30,30,30,30]^T. \end{aligned}$$

Then the obtained numerical results are summarized in Table 1:

Table 1: Numerical results for Example 6.1 with $r = 0.9$.

Algo→	One-step Picard's Algo		Two-steps Picard's Algo	
y_0 ↓	**Iter**	**CPU(s)**	**Iter**	**CPU(s)**
y_0^1	3	0.0102291	2	0.009932
y_0^2	3	0.013366	2	0.011541
y_0^3	5	0.015981	3	0.012425

The unique solution of this problem is given by:

$$x^\star = [0.486, 0.486, 0.486, 0.486, 0.486, 0.485, 0.485, 0.485, 0.486, 0.486]^T .$$

Example 6.2 Consider the NGAVE where $A,\ B \in \mathbb{R}^{n\times n}$ are given by:

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 & \cdots & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 2 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 3 & 3 & \cdots & 3 \\ 0 & 1 & 3 & 3 & \cdots & 3 \\ 0 & 0 & 1 & 3 & \cdots & 3 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 3 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

Applying Theorem 2.1, we have, $\left\|A^{-1}\right\| \|B\| = 0.0250 < 1$, then the problem is uniquely solvable for any b. For $b = [2, 2, ..., 2]^T$ and with the initial point $y_0 = 5e$, the obtained numerical results with different sizes of n are summarized in Table 2:

Table 2: Numerical results for Example 6.2 with $r = 2$.

Algo→	One-step Picard's Algo		Two-steps Picard's Algo	
Size n ↓	**Iter**	**CPU(s)**	**Iter**	**CPU(s)**
20	105	0.050786	20	0.013642
50	217	0.226826	50	0.148911
100	340	1.136755	101	1.080822
500	1264	74.028398	540	25.170953
1000	–	–	1040	45.170953

For example if $n = 8$, then an approximated solution of this problem is

$$x^\star = [256, 128, 64, 32, 16, 8, 4, 2]^T .$$

Example 6.3 Let the symmetric matrices A, $B \in \mathbb{R}^{n\times n}$ and the vector b be given as

$$A \;=\; = \begin{bmatrix} 4n & n & 0.5 & \cdots & 0.5 & 0.5 \\ n & 4n & n & \cdots & 0.5 & 0.5 \\ 0.5 & n & 4n & \cdots & 0.5 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 4n & n \\ 0.5 & 0.5 & 0.5 & \cdots & 4n & n \\ 0.5 & 0.5 & \cdots & 0.5 & n & 4n \end{bmatrix},$$

$$B \;=\; = \begin{bmatrix} n & \frac{1}{n} & 0.125 & \cdots & 0.125 & 0.125 \\ \frac{1}{n} & n & \frac{1}{n} & \cdots & 0.125 & 0.125 \\ 0.125 & \frac{1}{n} & n & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{n} & 0 \\ 0.125 & 0.125 & 0.125 & \cdots & n & \frac{1}{n} \\ 0.125 & 0.125 & \cdots & 0.125 & \frac{1}{n} & n \end{bmatrix},$$

$$b = [5, 6, 5, 6, ..., 6, 5]^T .$$

The initial point is defined as follows:

$$y_0 = [9, 9, ..., 9]^T .$$

The obtained numerical results with different size of n are summarized in Table 3:

Table 3: Numerical results for Example 6.3 with $r = 1.45$.

Algo→	One-step Picard's Algo		Two-steps Picard's Algo	
Size n ↓	**Iter**	**CPU(s)**	**Iter**	**CPU(s)**
50	14	0.027287	9	0.012424
120	14	0.065248	9	0.038086
500	15	1.232522	9	1.202624
1000	16	5.325272	10	4.402624
2000	16	29.25233	10	26.44262
3000	16	87.13425	10	76.26541
3500	16	99.13425	10	92.65415

Example 6.4 Consider the NGAVE Problem where A, $B \in \mathbb{R}^{n\times n}$ are given by:

$$A = Tridiag\;(-I, M, -I) = \begin{bmatrix} M & -I & 0 & \cdots & 0 & 0 \\ -I & M & -I & \cdots & 0 & 0 \\ 0 & -I & M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & M & -I \\ 0 & 0 & 0 & \cdots & -I & M \end{bmatrix} \in \mathbb{R}^{n\times n}$$

is a block-tridiagonal matrix,

$$M = Tridiag\;(-1, 6, -1) = \begin{bmatrix} 6 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 6 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 6 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 6 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 6 \end{bmatrix} \in \mathbb{R}^{m\times m},$$

and

$$B = A - 2I,$$

$$b = [24, 12, ..., 24, 12]^T.$$

The obtained numerical results with different sizes of n are summarized in Table 4.

Table 4: Numerical results for Example 6.4 with $r = 1.7$.

Algo→	**One-step Picard's Algo**		**Two-steps Picard's Algo**	
Size $(n \times m)$ ↓	**Iter**	**CPU(s)**	**Iter**	**CPU(s)**
(5×100)	42	3.664824	27	1.205434
(10×200)	41	19.10913	26	16.56523
(10×300)	40	15.87704	24	13.78974
(20×200)	41	20.87704	21	14.78974

For example if $n = 4$ and $m = 2$, The unique solution of this problem is given by:

$$x^{\star} = [18, 24, 24, 18, 18, 24, 24, 18]^T.$$

7. Conclusion

In this paper we have present a numerical study for solving a new general absolute value equations NGAVE. we applied a new two-step Picard's iterative fixed-point iteration. In particular, the sufficient conditions for the convergence of our algorithm are studied. The obtained numerical results deduced from the testing examples illustrate that the suggested algorithms are efficient and valid to solve the NGAVE problems, and has much better computing efficiency than Picard's method.

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