



Extension of the Notion of D -Symmetric Operators Using the Aluthge Transform

Elmahdi Sahtouti, Mohamed Morjane and Mohamed Ech-Chad

ABSTRACT: Let $A \in \mathcal{L}(H)$ be a bounded operator on a Hilbert space H , and let δ_A denote the inner derivation defined by $\delta_A(X) = AX - XA$. A is called D -symmetric if $\overline{\mathcal{R}(\delta_{A^*})} = \overline{\mathcal{R}(\delta_A)}$, where $\overline{\mathcal{R}(\delta_A)}$ is the norm closure of the range of δ_A . Motivated by the results of [7,10], we introduce \tilde{D} -symmetric operators for which

$$\mathcal{R}(\delta_{\tilde{A}}) \subset \overline{\mathcal{R}(\delta_A)},$$

where \tilde{A} denotes the Aluthge transform of A . We show that this class contains quasinormal operators, isometries, co-isometries, cyclic subnormal operators, and all D -symmetric operators, and we characterize \tilde{D} -symmetric operators via the range of the associated derivation in the Calkin algebra. For invertible operators, an additional characterization is obtained using suitable linear functionals, and A is D -symmetric if and only if \tilde{A} is D -symmetric.

Keywords: D -symmetric, partial isometry, Aluthge transform, subnormal, compact operator, Calkin algebra.

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1. Introduction

Let $\mathcal{L}(H)$ denote the algebra of all bounded operators on an infinite-dimensional complex Hilbert space H . The generalized derivation operator $\delta_{A,B}$ associated with the pair (A, B) , defined on $\mathcal{L}(H)$ by

$$\delta_{A,B}(X) = AX - XB,$$

was systematically studied for the first time in [12]. The properties of such operators have been extensively investigated (see, for example, [2,5,9,13,14] and the references therein). In the special case $A = B$, the operator $\delta_{A,A} = \delta_A : \mathcal{L}(H) \rightarrow \mathcal{L}(H)$, defined by $\delta_A(X) = AX - XA$, is called the inner derivation.

In the development of operator theory, Aluthge [1] introduced the operator \tilde{T} for $T \in \mathcal{L}(H)$ using the polar decomposition $T = U|T|$:

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}},$$

where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry satisfying $\ker(|T|) = \ker(U)$. The operator \tilde{T} is called the Aluthge transform of T . Recently, several studies in operator theory have focused on the relationship between operators and their Aluthge transforms (see, for example, [7,8,6] and the references therein).

An operator $A \in \mathcal{L}(H)$ is said to be D -symmetric if

$$\overline{\mathcal{R}(\delta_{A^*})} = \overline{\mathcal{R}(\delta_A)}.$$

This concept was introduced by J. H. Anderson *et al.* in [2]. It was proved therein that if A is D -symmetric, then $AT = TA$ implies $A^*T = TA^*$ for every trace-class operator $T \in \mathcal{C}_1(H)$. Operators satisfying this property are called P -symmetric. D -symmetric operators have been extensively investigated (see [3,5,13] and references therein). The notion was later extended to pairs of operators in [5]. The concept of P -symmetric operators was introduced by S. Bouali and J. Charles in [3]. M. Ech-Chad

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et al. [7] established that if A is P -symmetric, then $AT = TA$ implies $\tilde{A}T = T\tilde{A}$ for every $T \in \mathcal{C}_1(H)$, where \tilde{S} denotes the Aluthge transform of $S \in \mathcal{L}(H)$. This property is called \tilde{P} -symmetry and was later generalized to pairs of operators in [10]; see also [11].

Following the results in [7,10], we consider a generalization of D -symmetric operators. Specifically, we study operators $A \in \mathcal{L}(H)$ satisfying

$$\mathcal{R}(\delta_{\tilde{A}}) \subset \overline{\mathcal{R}(\delta_A)}.$$

Operators satisfying this property are called \tilde{D} -symmetric. In the first part, we establish a characterization and basic properties of \tilde{D} -symmetric operators using different arguments. Moreover, we show that the class of \tilde{D} -symmetric operators is strictly contained in the class of \tilde{P} -symmetric operators. In the second part, we present additional properties of this class, analogous to those known for D -symmetric operators. In these two parts, we exhibit several classes of \tilde{D} -symmetric operators that are not necessarily D -symmetric. We conclude this section with some notations.

Let $\mathcal{K}(H)$ and $\mathcal{C}_1(H)$ denote the ideals of compact and trace-class operators on H , respectively. The trace function is defined on $\mathcal{C}_1(H)$ by

$$\mathrm{Tr}(T) = \sum_n \langle Te_n, e_n \rangle,$$

where (e_n) is any complete orthonormal system in H . We denote by $\mathcal{C}(H)$ the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$, and by $[A]$ the equivalence class of $A \in \mathcal{L}(H)$ in this algebra. For a linear operator X acting on a Banach space E , we denote by $\ker(X)$, $\mathcal{R}(X)$, M^\perp , and $X|_M$ the kernel, the range, the orthogonal complement, and the restriction of X to an invariant subspace M , respectively. Additionally, $\overline{\mathcal{R}(X)}$ denotes the closure of the range of X with respect to the norm topology. Given a Banach space \mathcal{B} and a subspace $\mathcal{S} \subseteq \mathcal{B}$, let \mathcal{B}' denote the dual of \mathcal{B} . The annihilator of \mathcal{S} is defined as

$$\mathcal{S}^\circ = \{\Phi \in \mathcal{B}' : \Phi(x) = 0 \text{ for all } x \in \mathcal{S}\}.$$

2. \tilde{D} -Symmetric Operators

Theorem 2.1 ([2]) *Let $A \in \mathcal{L}(H)$. The following assertions are equivalent:*

1. A is D -symmetric, that is $\overline{\mathcal{R}(\delta_{A^*})} = \overline{\mathcal{R}(\delta_A)}$.
2. (a) $[A]$ is D -symmetric; and
(b) $AT = TA$ implies $A^*T = TA^*$ for all $T \in \mathcal{C}_1(H)$, that is, A is P -symmetric.

In what follows, we will denote by $\mathcal{D}(H)$ the class of D -symmetric operators.

Remark 2.1 *Let $A \in \mathcal{L}(H)$. Then:*

1. If A is normal, then A is D -symmetric.
2. For $f \in \mathcal{L}(H)'$ and $f^*(X) = \overline{f(X^*)}$, the operator A is D -symmetric if and only if $f^* \in \mathcal{R}(\delta_A)^\circ$ for every $f \in \mathcal{R}(\delta_A)^\circ$.
3. A is D -symmetric if and only if $\mathcal{R}(\delta_{A^*}) \subset \overline{\mathcal{R}(\delta_A)}$.

Definition 2.1 *Let $A \in \mathcal{L}(H)$. If*

$$\mathcal{R}(\delta_{\tilde{A}}) \subset \overline{\mathcal{R}(\delta_A)},$$

then A is called \tilde{D} -symmetric. The class of all \tilde{D} -symmetric operators is denoted by $\tilde{\mathcal{D}}(H)$.

Example 2.1 1. Recall that an operator $A = U|A| \in \mathcal{L}(H)$ is quasinormal if and only if $|A|U = U|A|$. In this case, $\tilde{A} = A$, and hence A is \tilde{D} -symmetric.

2. If A is normal, then A is \tilde{D} -symmetric.
3. If A is isometric, then A is quasinormal, and therefore \tilde{D} -symmetric.

Lemma 2.1 Let \mathcal{S}_1 and \mathcal{S}_2 be subspaces of $\mathcal{L}(H)$. Then

$$\mathcal{S}_1^\circ \subset \mathcal{S}_2^\circ \quad \text{if and only if} \quad \mathcal{S}_2 \subset \overline{\mathcal{S}_1}.$$

Proof: This follows directly from the bipolar theorem. \square

Lemma 2.2 ([14]) Let $A, B \in \mathcal{L}(H)$. Then

$$\mathcal{R}(\delta_{A,B})^\circ = \mathcal{R}(\delta_{A,B})^\circ \cap \mathcal{K}(H)^\circ \oplus \ker(\delta_{B,A}) \cap \mathcal{C}_1(H).$$

Theorem 2.2 Let $A \in \mathcal{L}(H)$. The following assertions are equivalent:

1. A is \tilde{D} -symmetric.
2. (a) $[A]$ is \tilde{D} -symmetric; and
(b) For all $T \in \mathcal{C}_1(H)$, $AT = TA$ implies $\tilde{A}T = T\tilde{A}$, that is, A is \tilde{P} -symmetric.

Proof: Suppose that A is \tilde{D} -symmetric, and let $[A]$ denote its class in the Calkin algebra $\mathcal{C}(H) = \mathcal{L}(H)/\mathcal{K}(H)$. Since

$$[\delta_A(X)] = \delta_{[A]}([X]) \quad \text{and} \quad [\tilde{A}] = [\tilde{A}],$$

and $\mathcal{R}(\delta_{\tilde{A}}) \subset \overline{\mathcal{R}(\delta_A)}$, it follows that $[\mathcal{R}(\delta_{\tilde{A}})] \subset \overline{[\mathcal{R}(\delta_A)]}$, that is, $\mathcal{R}(\delta_{[\tilde{A}]}) \subset \overline{\mathcal{R}(\delta_{[A]})}$. Hence, $[A]$ is \tilde{D} -symmetric. Next, let $T \in \mathcal{C}_1(H)$ satisfy $AT = TA$, and define the linear functional

$$f_T: \mathcal{L}(H) \rightarrow \mathbb{C}, \quad f_T(X) := \text{Tr}(TX).$$

Then $f_T \in \mathcal{R}(\delta_A)^\circ$. By Lemma 2.1, we have $f_T \in \mathcal{R}(\delta_{\tilde{A}})^\circ$, and Lemma 2.2 implies $\tilde{A}T = T\tilde{A}$. Conversely, let $f \in \mathcal{L}(H)' \cap \mathcal{R}(\delta_A)^\circ$. By a standard decomposition, we can write $f = f_0 + f_T$, where

$$f_0 \in \mathcal{R}(\delta_A)^\circ \cap \mathcal{K}(H)^\circ, \quad f_T(X) = \text{Tr}(TX) \text{ with } T \in \ker(\delta_A) \cap \mathcal{C}_1(H).$$

Since $\tilde{A}T = T\tilde{A}$, it follows that $f_T \in \mathcal{R}(\delta_{\tilde{A}})^\circ$. Define the functional

$$\phi: \mathcal{L}(H)/\mathcal{K}(H) \rightarrow \mathbb{C}, \quad \phi([X]) := f_0(X).$$

Since $f_0 \in \mathcal{K}(H)^\circ$, the value of ϕ is independent of the representative X , hence ϕ is well-defined. Moreover,

$$\phi \in \mathcal{R}(\delta_{[A]})^\circ \subset \mathcal{R}(\delta_{[\tilde{A}]})^\circ,$$

which implies $f_0 \in \mathcal{R}(\delta_{\tilde{A}})^\circ$. Therefore, $f \in \mathcal{R}(\delta_{\tilde{A}})^\circ$, showing that A is \tilde{D} -symmetric. \square

Definition 2.2 An operator $A \in \mathcal{L}(H)$ is said to be essentially \tilde{D} -symmetric if its class $[A]$ in the Calkin algebra $\mathcal{C}(H) = \mathcal{L}(H)/\mathcal{K}(H)$ is \tilde{D} -symmetric.

Proposition 2.1 Let $A \in \mathcal{L}(H)$ be a partial isometry such that either $I - A^*A$ or $I + A^*A$ is compact. Then A is essentially \tilde{D} -symmetric.

Proof: Since $I - A^*A$ or $I + A^*A$ is compact, it follows that $(I - A^*A)(I + A^*A)A$ is compact, and hence $A - A^*AA$ is compact. As $AA^*A = A$, we deduce that $AA^*A - A^*AA$ is compact. Consequently, the class $[A]$ is quasinormal in $\mathcal{L}(H)/\mathcal{K}(H)$, and thus $[A]$ is \tilde{D} -symmetric. \square

Example 2.2 Let $(e_k)_{k \geq 1}$ be an orthonormal basis of H , and let $S \in \mathcal{L}(H)$ denote the unilateral shift defined by $Se_k = e_{k+1}$ for all $k \geq 1$. Set $A = S^*$. Then A is a partial isometry such that $I - A^*A$ is compact (indeed, of finite rank). Hence, A is essentially \tilde{D} -symmetric.

Corollary 2.1 Let $A \in \mathcal{L}(H)$ be an essentially \tilde{D} -symmetric operator. Then A is \tilde{D} -symmetric whenever one of the following holds:

1. A is a partial isometry such that A^2 is normal;
2. A is a cyclic subnormal operator;
3. $f(A)$ is a cyclic subnormal operator, where f is a nonconstant analytic function defined on an open set containing $\sigma(A)$.

Proof: By [7], in each of the above cases, A is \tilde{P} -symmetric. Hence, by Theorem 2.2, A is \tilde{D} -symmetric. \square

Remark 2.2 S. R. Garcia proved in [8] that an operator A is nilpotent of order 2 if and only if its Aluthge transformation \tilde{A} vanishes. Therefore, A is trivially \tilde{D} -symmetric.

Corollary 2.2 Let $A \in \mathcal{C}_1(H)$ be a partial isometry. If A is nilpotent of order $n \geq 3$ then A is not \tilde{D} -symmetric.

Proof: By Proposition 1.19 in [7], A is not \tilde{P} -symmetric. Hence, by Theorem 2.2, A is not \tilde{D} -symmetric. \square

Corollary 2.3 Let $A \in \mathcal{L}(H)$ satisfy $\ker A \neq \{0\}$ and $\{0\} \neq \ker A^* \not\subset \ker(\tilde{A})^*$. Then A is not \tilde{D} -symmetric.

Proof: By Theorem 1.12 in [7], A is not \tilde{P} -symmetric. Hence, by Theorem 2.2, A is not \tilde{D} -symmetric. \square

Example 2.3 Let $(e_n)_{n \geq 1}$ be an orthonormal basis of H . Set $H_0 = \text{span}\{e_1, e_2, e_3\}$, and for $\theta \in]0, \frac{\pi}{2}[$ define

$$A_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & \sin \theta & 0 \end{pmatrix} \in \mathcal{L}(H_0).$$

A direct calculation shows that A_θ is a partial isometry. By Lemma 1.9 in [7], the Aluthge transform of A_θ is

$$\tilde{A}_\theta = A_\theta^* A_\theta^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $A = A_\theta \oplus I$ with respect to the decomposition $H = H_0 \oplus H_0^\perp$. Since A is a partial isometry, we have $\tilde{A} = A^* A^2 = \tilde{A}_\theta \oplus I$. It is easy to see that

$$Ae_3 = 0, \quad A^*(\sin \theta e_2 - \cos \theta e_3) = 0, \quad \text{but} \quad \tilde{A}^*(\sin \theta e_2 - \cos \theta e_3) \neq 0.$$

Therefore, by Corollary 2.3, A is not \tilde{D} -symmetric.

Theorem 2.3 $\mathcal{D}(H)$ is strictly contained in $\tilde{\mathcal{D}}(H)$.

Proof: Let $A \in \mathcal{L}(H)$ such that $\overline{\mathcal{R}(\delta_A)} = \overline{\mathcal{R}(\delta_{A^*})}$. If $f \in \mathcal{L}(H)'$ satisfy $f(AX) = f(XA)$ for all $X \in \mathcal{L}(H)$, then $f(A^*X) = f(XA^*)$, which implies $f(|A|^2X) = f(X|A|^2)$. Let $\{P_n(t)\}$ be a sequence of polynomials without constant term such that $P_n(t) \rightarrow t^{1/4}$ uniformly on a compact set. It follows that $f(|A|^{1/2}X) = f(X|A|^{1/2})$. By Theorem 2.1 in [6],

$$\tilde{A} = |A|^{1/2} \int_0^{+\infty} A e^{-s|A|^2} |A| ds |A|^{1/2}.$$

Hence, $f(\tilde{A}X) = f(X\tilde{A})$, and therefore A is \tilde{D} -symmetric. To show that the inclusion is proper, let $(e_n)_{n \geq 1}$ be an orthonormal basis of H , and define $S \in \mathcal{L}(H)$ by

$$Se_k = \begin{cases} 0, & \text{if } k = 1, \\ e_{k+1}, & \text{if } k \geq 2. \end{cases}$$

A direct computation shows that S is quasinormal. Consequently, S is \tilde{D} -symmetric. However, by [3, Theorem 1.6], S is not D -symmetric. This proves that $\mathcal{D}(H) \subsetneq \tilde{\mathcal{D}}(H)$. \square

Corollary 2.4 *If $A \in \mathcal{L}(H)$ is a co-isometry or a subnormal cyclic operator, then A is \tilde{D} -symmetric.*

Proof: The result follows immediately from Theorem 2.3, Theorem 2.3 in [2], and Theorem 2.5 in [3]. \square

Corollary 2.5 *Let $A \in \mathcal{L}(H)$. Suppose the following conditions hold:*

- (1) A is D -symmetric, and
- (2) $\mathcal{R}(\delta_A) \subset \overline{\mathcal{R}(\delta_{\tilde{A}})}$.

Then $\overline{\mathcal{R}(\delta_A)} = \overline{\mathcal{R}(\delta_{A^})} = \overline{\mathcal{R}(\delta_{\tilde{A}})}$, and \tilde{A} is D -symmetric.*

Proof: The result follows directly from Theorem 2.3. \square

Proposition 2.2 *If $\tilde{A} = f(A)$, where f is analytic on an open set containing $\sigma(A)$, then A is \tilde{D} -symmetric.*

Proof: Let $f(z) = \sum_{n=0}^{+\infty} \alpha_n z^n$ be analytic on an open set containing $\sigma(A)$. By the functional calculus, we have $\tilde{A} = \sum_{n=0}^{+\infty} \alpha_n A^n$. For any $X \in \mathcal{L}(H)$,

$$\tilde{A}X - X\tilde{A} = \sum_{n=1}^{+\infty} \alpha_n (A^n X - X A^n).$$

Since $A^n X - X A^n \in \mathcal{R}(\delta_A)$ for all $n \in \mathbb{N}$, it follows that $\tilde{A}X - X\tilde{A} \in \overline{\mathcal{R}(\delta_A)}$. Hence, A is \tilde{D} -symmetric. \square

Lemma 2.3 *Let $A, B \in \mathcal{L}(H)$. The following assertions are equivalent:*

1. $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ and $\ker(\delta_A|_{\mathcal{C}_1(H)}) \subset \ker(\delta_B|_{\mathcal{C}_1(H)})$,
2. For every sequence $(T_n) \subset \mathcal{C}_1(H)$,

$$\lim_{n \rightarrow +\infty} (AT_n - T_n A) = 0 \quad \Rightarrow \quad \lim_{n \rightarrow +\infty} (BT_n - T_n B) = 0.$$

Proof: Define the mapping

$$\phi : \overline{\delta_A(\mathcal{C}_1(H))} \longrightarrow \overline{\delta_B(\mathcal{C}_1(H))}, \quad \delta_A(X) = AX - XA \mapsto BX - XB = \delta_B(X).$$

Assume the first assertion holds. Since

$$\ker(\delta_A|_{\mathcal{C}_1(H)}) \subset \ker(\delta_B|_{\mathcal{C}_1(H)}),$$

the mapping ϕ is well-defined and linear. Moreover, the inclusion $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ implies, by [9], Theorem 1.5, that there exists a constant $K > 0$ such that

$$\|\delta_B(T)\| \leq K\|\delta_A(T)\|, \quad \text{for all } T \in \mathcal{C}_1(H),$$

which proves the second assertion. Conversely, assume the second assertion holds. Then ϕ is continuous at the origin and hence continuous on $\overline{\delta_A(\mathcal{C}_1(H))}$. By continuity, there exists $K > 0$ such that

$$\|\delta_B(T)\| \leq K\|\delta_A(T)\|, \quad \text{for all } T \in \mathcal{C}_1(H).$$

By [9], Theorem 1.5, this inequality is equivalent to $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$, and it follows that $\ker(\delta_A|_{\mathcal{C}_1(H)}) \subset \ker(\delta_B|_{\mathcal{C}_1(H)})$. \square

Corollary 2.6 *Let $A \in \mathcal{L}(H)$ be an operator such that for every sequence $(T_n) \subset \mathcal{C}_1(H)$,*

$$\lim_{n \rightarrow +\infty} (AT_n - T_n A) = 0 \quad \Rightarrow \quad \lim_{n \rightarrow +\infty} (\tilde{A}T_n - T_n \tilde{A}) = 0.$$

Then A is \tilde{D} -symmetric.

Proof: The result follows directly from the previous lemma. \square

Lemma 2.4 *Let $f \in \mathcal{L}(H \oplus H)'$. Then there exist unique linear functionals $f_1, f_2, f_3,$ and f_4 on $\mathcal{L}(H)$ such that, for all*

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \mathcal{L}(H \oplus H),$$

we have

$$f\left(\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\right) = f_1(X_1) + f_2(X_2) + f_3(X_3) + f_4(X_4).$$

Proof: For $f \in \mathcal{L}(H \oplus H)'$, define

$$f_1 : \mathcal{L}(H) \rightarrow \mathbb{C}, \quad X \mapsto f\left(\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\right).$$

It is straightforward that f_1 is a linear functional. \square

Theorem 2.4 *Let $A, B \in \mathcal{L}(H)$ be \tilde{D} -symmetric operators with disjoint spectra. Then $A \oplus B$ is \tilde{D} -symmetric.*

Proof: Let $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \mathcal{L}(H \oplus H)$. The derivation $\delta_{A \oplus B}(X)$ is given by

$$\delta_{A \oplus B}(X) = \begin{pmatrix} AX_1 - X_1A & AX_2 - X_2B \\ BX_3 - X_3A & BX_4 - X_4B \end{pmatrix}.$$

Let $f \in \mathcal{R}(\delta_{A \oplus B})^\circ$. By Lemma 2.4, there exist unique linear functionals f_1, f_2, f_3 , and f_4 on $\mathcal{L}(H)$ such that

$$f_1 \in \mathcal{R}(\delta_A)^\circ, \quad f_2 \in \mathcal{R}(\delta_{A,B})^\circ, \quad f_3 \in \mathcal{R}(\delta_{B,A})^\circ, \quad f_4 \in \mathcal{R}(\delta_B)^\circ.$$

Which gives us, based on the hypotheses

$$f_1 \in \mathcal{R}(\delta_{\tilde{A}})^\circ \quad \text{and} \quad f_4 \in \mathcal{R}(\delta_{\tilde{B}})^\circ.$$

Since A and B have disjoint spectra, it follows from [12] that $\delta_{B,A}$ is invertible. By Lemma 2.2, we have $f_2 \in \mathcal{R}(\delta_{A,B})^\circ \cap \mathcal{K}(H)^\circ \subset \mathcal{K}(H)^\circ$, which implies $f_2 = 0$. Similarly, $f_3 = 0$. Hence $f = f_1 + f_4$. Finally,

$$f((\tilde{A} \oplus \tilde{B})X) = f_1(\tilde{A}X_1) + f_4(\tilde{B}X_4) = f_1(X_1\tilde{A}) + f_4(X_4\tilde{B}) = f(X(\tilde{A} \oplus \tilde{B})).$$

Therefore, $A \oplus B$ is \tilde{D} -symmetric. \square

Definition 2.3 Let $A \in \mathcal{L}(H)$ be an invertible operator, and let $f \in \mathcal{L}(H)'$ be a linear functional. We define the linear functional f_A by

$$f_A(X) = f\left(|A|^{\frac{1}{2}}X|A|^{-\frac{1}{2}}\right), \quad \text{for all } X \in \mathcal{L}(H).$$

Remark 2.3 Let $A \in \mathcal{L}(H)$ be an invertible operator, and let $f \in \mathcal{L}(H)'$. Since $|A^{-1}| = |A^*|^{-1}$, it follows that

$$f_{A^{*-1}}(X) = f\left(|A|^{-\frac{1}{2}}X|A|^{\frac{1}{2}}\right), \quad \text{for all } X \in \mathcal{L}(H).$$

Moreover, we have

$$(f_{A^{*-1}})_A = (f_A)_{A^{*-1}} = f.$$

Theorem 2.5 Let $A \in \mathcal{L}(H)$ be an invertible operator. Then the following statements are equivalent:

1. A is \tilde{D} -symmetric;
2. $f \in \mathcal{R}(\delta_A)^\circ$ implies $f_A \in \mathcal{R}(\delta_A)^\circ$ for every $f \in \mathcal{L}(H)'$.

Proof: Suppose that A is \tilde{D} -symmetric. Then for every $f \in \mathcal{R}(\delta_A)^\circ$ and every $X \in \mathcal{L}(H)$, we have

$$f(\tilde{A}X) = f(X\tilde{A}),$$

where $\tilde{A} = |A|^{1/2}A|A|^{-1/2}$. Let $X \in \mathcal{L}(H)$ and define $Y = |A|^{1/2}X|A|^{-1/2} \in \mathcal{L}(H)$. Then $X = |A|^{-1/2}Y|A|^{1/2}$ and

$$\tilde{A}Y = |A|^{1/2}A|A|^{-1/2}Y = |A|^{1/2}AX|A|^{-1/2},$$

and similarly,

$$Y\tilde{A} = Y|A|^{1/2}A|A|^{-1/2} = |A|^{1/2}XA|A|^{-1/2}.$$

Thus,

$$f(|A|^{1/2}AX|A|^{-1/2}) = f(|A|^{1/2}XA|A|^{-1/2}), \quad \text{for all } X \in \mathcal{L}(H),$$

which is equivalent to

$$f_A(AX) = f_A(XA), \quad \text{for all } X \in \mathcal{L}(H).$$

Conversely, a similar argument shows that if $f_A \in \mathcal{R}(\delta_A)^\circ$ whenever $f \in \mathcal{R}(\delta_A)^\circ$, then A is \tilde{D} -symmetric. \square

Remark 2.4 Let $A \in \mathcal{L}(H)$ be an invertible operator. If $\overline{\mathcal{R}(\delta_A)} = \overline{\mathcal{R}(\delta_{\tilde{A}})}$, then

$$f \in \mathcal{R}(\delta_A)^\circ \quad \text{if and only if} \quad f_A \in \mathcal{R}(\delta_A)^\circ.$$

Lemma 2.5 *Let $A \in \mathcal{L}(H)$ be an invertible operator. The following statements are equivalent: $f \in \mathcal{R}(\delta_A)^\circ$, $f^* \in \mathcal{R}(\delta_{A^*})^\circ$, and $f_A \in \mathcal{R}(\delta_{\tilde{A}})^\circ$.*

Proof: The equivalence follows from straightforward computations using the definitions of f^* and f_A . \square

Corollary 2.7 *Let $A \in \mathcal{L}(H)$ be an invertible operator. If A is \tilde{D} -symmetric, then A^* is also \tilde{D} -symmetric.*

Proof: Let $f \in \mathcal{L}(H)'$ and assume $f \in \mathcal{R}(\delta_{A^*})^\circ$. By Lemma 2.5, we have $f^* \in \mathcal{R}(\delta_A)^\circ$. Since A is \tilde{D} -symmetric, Theorem 2.5 implies $(f^*)_A \in \mathcal{R}(\delta_A)^\circ$. Using the identity $(f_{A^{*-1}})^* = (f^*)_A$, we deduce

$$(f_{A^{*-1}})^* \in \mathcal{R}(\delta_A)^\circ.$$

Applying Lemma 2.5 again yields $f_{A^{*-1}} \in \mathcal{R}(\delta_{A^*})^\circ$. Hence,

$$f = (f_{A^{*-1}})_A \in \mathcal{R}(\delta_{\tilde{A}})^\circ.$$

\square

Lemma 2.6 *Let $A, B \in \mathcal{L}(H)$. Then*

$$\mathcal{R}(\delta_{AB}) \subseteq \overline{\mathcal{R}(\delta_A)} + \overline{\mathcal{R}(\delta_B)}.$$

Moreover,

$$\mathcal{R}(\delta_A)^\circ \cap \mathcal{R}(\delta_B)^\circ \subseteq \mathcal{R}(\delta_{AB})^\circ.$$

If A is invertible, then

$$\overline{\mathcal{R}(\delta_A)} = \overline{\mathcal{R}(\delta_{A^{-1}})}.$$

Proof: The assertions follow from elementary calculations. \square

Theorem 2.6 *Let $A \in \mathcal{L}(H)$ be invertible. Then \tilde{A} is D -symmetric if and only if A is D -symmetric.*

Proof: Suppose \tilde{A} is D -symmetric and let $f \in \mathcal{R}(\delta_A)^\circ$. By Lemma 2.5, $f_A \in \mathcal{R}(\delta_{\tilde{A}})^\circ$, and since \tilde{A} is D -symmetric, $f_A \in \mathcal{R}(\delta_{\tilde{A}^*})^\circ$. Lemma 2.5 then gives $(f_A)^* \in \mathcal{R}(\delta_{\tilde{A}})^\circ$. Using $(f_A)^* = (f^*)_{A^{*-1}}$, we have $(f^*)_{A^{*-1}} \in \mathcal{R}(\delta_{\tilde{A}})^\circ$, hence

$$f^* = (f^*)_{A^{*-1}} \in \mathcal{R}(\delta_A)^\circ.$$

Therefore, A is D -symmetric. The converse follows analogously. \square

Remark 2.5 *The invertibility assumption in Theorem 2.6 is essential, as illustrated by the following example.*

Example 2.4 *Let $(e_k)_{k \geq 1}$ be an orthonormal basis of H and let $S \in \mathcal{L}(H)$ be the unilateral shift operator defined by $Se_k = e_{k+1}$ for all $k \geq 1$. Consider $A = S^*$. Then A is D -symmetric according to Theorem 2.3 in [2] but it is not invertible. However, \tilde{A} is not D -symmetric since it is also not P -symmetric as shown in Example 2.8 in [7].*

Theorem 2.7 *Let $A \in \mathcal{L}(H)$ be an invertible operator. Then A is \tilde{D} -symmetric if and only if \tilde{A} is D -symmetric.*

Proof: Assume that A is \tilde{D} -symmetric. The mapping $f \mapsto f_A$ is a bijection on $\mathcal{L}(H)'$, so it suffices to consider functionals of the form f_A with $f_A \in \mathcal{R}(\delta_{\tilde{A}})^\circ$. By Lemma 2.5, we have $f \in \mathcal{R}(\delta_A)^\circ$. Since A is \tilde{D} -symmetric, $f \in \mathcal{R}(\delta_{\tilde{A}})^\circ$. Using $f = (f_{A^{*-1}})_A$, Lemma 2.5 gives

$$f_{A^{*-1}} \in \mathcal{R}(\delta_A)^\circ.$$

Since $f_{A^{*-1}} = (f_A)^*$, we get

$$(f_A)^* \in \mathcal{R}(\delta_A)^\circ \subset \mathcal{R}(\delta_{\tilde{A}})^\circ.$$

Applying Lemma 2.5 again yields

$$f_A \in \mathcal{R}(\delta_{\tilde{A}^*})^\circ.$$

Hence, \tilde{A} is D -symmetric. The converse follows by reversing the argument. \square

Corollary 2.8 *Let $A \in \mathcal{L}(H)$ be an invertible operator. Then A is \tilde{D} -symmetric if and only if A is D -symmetric.*

Proof: The result follows directly from Theorem 2.6 and Theorem 2.7. \square

Theorem 2.8 $\tilde{\mathcal{D}}(H)$ is strictly contained in the class of \tilde{P} -symmetric operators.

Proof: The inclusion follows directly from Theorem 2.2. We need to show that the inclusion is proper. Let $\{e_n\}_{n \geq 1}$ be an orthonormal basis of H and define an operator $S \in \mathcal{L}(H)$ by

$$S e_{2n} = 2 e_{2n+1}, \quad S e_{2n+1} = \frac{1}{2} e_{2n+2}, \quad n \geq 1.$$

According to Corollaire 3.2 in [4], the operator S is P -symmetric. Hence, by Theorem 1.14 in [7], S is \tilde{P} -symmetric. However, Lemma 2 in [13] ensures that $[S]$ is not D -symmetric. Consequently, by Theorem 2.1, the operator S itself is not D -symmetric. On the other hand, since S is invertible, the preceding corollary implies that S cannot be \tilde{D} -symmetric. \square

3. Closures of Derivation Ranges

Theorem 3.1 ([2]) *Let $A \in \mathcal{L}(H)$. The following assertions are equivalent:*

1. A is D -symmetric;
2. $\delta_{A^*}(A)\mathcal{L}(H) + \mathcal{L}(H)\delta_{A^*}(A) \subseteq \overline{\mathcal{R}(\delta_A)}$;
3. $A^*\mathcal{R}(\delta_A) + \mathcal{R}(\delta_A)A^* \subseteq \overline{\mathcal{R}(\delta_A)}$.

Theorem 3.2 *Let $A \in \mathcal{L}(H)$. The following assertions are equivalent:*

1. $\delta_{\tilde{A}}(A)\mathcal{L}(H) + \mathcal{L}(H)\delta_{\tilde{A}}(A) \subseteq \overline{\mathcal{R}(\delta_A)}$;
2. $\tilde{A} \cdot \mathcal{R}(\delta_A) + \mathcal{R}(\delta_A) \cdot \tilde{A} \subseteq \overline{\mathcal{R}(\delta_A)}$.

Proof: The result follows from the identities

$$(\tilde{A}A - A\tilde{A})X = \tilde{A}\delta_A(X) - \delta_A(\tilde{A}X), \quad X(\tilde{A}A - A\tilde{A}) = \delta_A(X)\tilde{A} - \delta_A(X\tilde{A}),$$

for all $X \in \mathcal{L}(H)$. \square

Theorem 3.3 *Let A be a \tilde{D} -symmetric operator. Then*

$$\tilde{A}\mathcal{R}(\delta_A) + \mathcal{R}(\delta_A)\tilde{A} \subseteq \overline{\mathcal{R}(\delta_A)}.$$

Proof: The result follows from the identities

$$\tilde{A} \delta_A(X) = \delta_{\tilde{A}}(AX) + \delta_A(\tilde{A}X) - A \delta_{\tilde{A}}(X),$$

and

$$\delta_A(X) \tilde{A} = \delta_{\tilde{A}}(XA) + \delta_A(X\tilde{A}) - \delta_{\tilde{A}}(X)A,$$

for all $X \in \mathcal{L}(H)$. □

Proposition 3.1 *Let $A \in \mathcal{L}(H)$ satisfy $\mathcal{R}(\delta_{|A|}) \subset \overline{\mathcal{R}(\delta_A)}$. Then A is \tilde{D} -symmetric.*

Proof: Let $f \in \mathcal{L}(H)'$ satisfy $f(AX) = f(XA)$ for all $X \in \mathcal{L}(H)$. Then

$$f(|A|X) = f(X|A|), \quad \text{for all } X \in \mathcal{L}(H).$$

By functional calculus, it follows that

$$f(|A|^{\frac{1}{2}}X) = f(X|A|^{\frac{1}{2}}), \quad \text{for all } X \in \mathcal{L}(H).$$

By Theorem 2.1 in [6], we have

$$\tilde{A} = |A|^{\frac{1}{2}} \left(\int_0^{+\infty} A e^{-s|A|^2} |A| ds \right) |A|^{\frac{1}{2}}.$$

It follows that

$$f(\tilde{A}X) = f(X\tilde{A}), \quad \text{for all } X \in \mathcal{L}(H),$$

and hence A is \tilde{D} -symmetric. □

Definition 3.1 *Let $A \in \mathcal{L}(H)$. The left and right multiplication operators L_A and R_A on $\mathcal{L}(H)$ are defined by*

$$L_A(X) = AX \quad \text{and} \quad R_A(X) = XA, \quad \text{for all } X \in \mathcal{L}(H).$$

The derivation associated with A is then $\delta_A = L_A - R_A$.

Corollary 3.1 *Let $A = U|A| \in \mathcal{L}(H)$ satisfy $\mathcal{R}(\delta_U) \subset \overline{\mathcal{R}(\delta_A)}$, where R_U has dense range. Then A is \tilde{D} -symmetric.*

Proof: Let $f \in \mathcal{L}(H)'$ satisfy $f(AX) = f(XA)$ for all $X \in \mathcal{L}(H)$. Then

$$f(U|A|X) = f(XU|A|), \quad \text{for all } X \in \mathcal{L}(H),$$

and

$$f(UX) = f(XU), \quad \text{for all } X \in \mathcal{L}(H).$$

Combining these identities, we obtain

$$f(|A|XU) = f(XU|A|), \quad \text{for all } X \in \mathcal{L}(H).$$

Since R_U has dense range, it follows that

$$f(|A|X) = f(X|A|), \quad \text{for all } X \in \mathcal{L}(H).$$

By Proposition 3.1, we conclude that A is \tilde{D} -symmetric. □

Proposition 3.2 *Let $A \in \mathcal{L}(H)$ satisfy $\overline{\mathcal{R}(\delta_{|A|})} \subset \overline{\mathcal{R}(\delta_{\tilde{A}})}$, and assume $R_{|A|}$ has dense range. Then A is \tilde{D} -symmetric if and only if*

$$\overline{\mathcal{R}(\delta_{\tilde{A}})} = \overline{\mathcal{R}(\delta_A)}.$$

Proof: It suffices to show that

$$\mathcal{R}(\delta_A) \subset \overline{\mathcal{R}(\delta_{\tilde{A}})}.$$

Let $f \in \mathcal{L}(H)'$ satisfy $f(\tilde{A}X) = f(X\tilde{A})$ for all $X \in \mathcal{L}(H)$. By hypothesis,

$$f(|A|X) = f(X|A|), \quad \text{for all } X \in \mathcal{L}(H).$$

By functional calculus,

$$f(|A|^{\frac{1}{2}}X) = f(X|A|^{\frac{1}{2}}), \quad \text{for all } X \in \mathcal{L}(H).$$

By Lemma 2.6,

$$f(\tilde{A}|A|^{\frac{1}{2}}X) = f(X\tilde{A}|A|^{\frac{1}{2}}), \quad \text{for all } X \in \mathcal{L}(H).$$

Since $\tilde{A}|A|^{\frac{1}{2}} = |A|^{\frac{1}{2}}A$, we deduce

$$f(|A|^{\frac{1}{2}}AX) = f(X|A|^{\frac{1}{2}}A), \quad \text{for all } X \in \mathcal{L}(H),$$

and hence

$$f(AX|A|^{\frac{1}{2}}) = f(X|A|^{\frac{1}{2}}A), \quad \text{for all } X \in \mathcal{L}(H).$$

Consequently, we obtain

$$f(AX|A|) = f(X|A|A), \quad \text{for all } X \in \mathcal{L}(H).$$

This shows that $f(AX) = f(XA)$ for all $X \in \mathcal{R}(R_{|A|})$. Since $R_{|A|}$ has dense range, it follows that

$$f(AX) = f(XA), \quad \text{for all } X \in \mathcal{L}(H),$$

and thus A is \tilde{D} -symmetric. □

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Elmahdi Sahtouti,
LAGA Laboratory, Department of Mathematics
Faculty of Sciences, University Ibn Tofail
P.O. Box 133, Kenitra, Morocco
E-mail address: `elmahdi.sahtouti@uit.ac.ma`

and

Mohamed Morjane,
LAGA Laboratory, Department of Mathematics
Faculty of Sciences, University Ibn Tofail
P.O. Box 133, Kenitra, Morocco
E-mail address: `mohamed.morjane@uit.ac.ma`

and

Mohamed Ech-chad,
LAGA Laboratory, Department of Mathematics
Faculty of Sciences, University Ibn Tofail
P.O. Box 133, Kenitra, Morocco
E-mail address: `m.echchad@yahoo.fr` (`mohamed.ech-chad@uit.ac.ma`)