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Secure Communication Through Function-Synchronization of Time-Delayed Uncertain Genesio-Lorenz Systems

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ABSTRACT: In this manuscript, we have developed a new technique to obtain generalised function synchronization between two non-identical chaotic systems by adaptive sliding mode control technique. The adaptive controller consists of both static state feedback as well as time-varying delayed state feedback terms. The controller is designed without using any lemmas. Further, the technique is applied to synchronize modified multi-delayed Genesio system and Lorenz system. Next, we have shown its application in achieving secure communication between non-identical Genesio system as sender and Lorenz system as the receiver system. We have modified the Chaotic Masking Scheme for achieving enhanced security. Finally, numerical simulations are performed which confirm the effectiveness of our proposed control techniques and encryption methods.

Key Words: Generalised function synchronization, parameter estimation, adaptive control, sliding mode control, time-delayed chaotic systems, chaotic masking scheme, secure communication.

Contents

1. Introduction

The control of time-delay systems is a subject of great interest and a lot of research has been going on in this direction. Since Mackey & Glass first observed chaos in delayed system [1], many researchers got interested in the dynamics and chaos-synchronization of time-delayed chaotic systems. Time-delay systems are ubiquous in nature and dissipative systems with non-linear time-delayed feedback can produce chaotic dynamics. We know that high complexity of multiple time-delayed systems helps in enhancing message security. Especially in certain communication systems, it is required to achieve chaos synchronization between multiple time-delayed transmitter and receiver systems. Thus, in a chaos synchronization problem,

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there is a chaotic system considered the master system and another identical or non-identical chaotic system considered the slave system, whose dynamics is made to synchronize with that of the master system by driving the slave system with a control input [2,3,6,4,5]. So, the problem of deriving a controller for a multi-delayed chaos synchronization system is an important area of research.

The concept of synchronization has been developed in various fields such as integer-order chaotic systems, fractional-order chaotic systems, time-delayed systems and distributed-order systems [7,8,9,10]. Synchronization of time-delayed systems has since attracted much attention. Complete synchronization, anti-synchronization, projective synchronization, hybrid synchronization, and generalized synchronization are considered in multi-time-delayed chaotic systems. In this paper, we have performed synchronization where the master and slave systems are synchronized up to a scaling factor $\alpha(t)$, which is a function of time. If $\alpha(t) = 1$, complete synchronization is obtained as a special case, if $\alpha(t) = -1$, anti-synchronization is obtained while if $\alpha(t)$ is any other non-zero scalar, we get projective synchronization as a special case. Further, if we choose different values of $\alpha(t)$ for different state variables, we obtain the case of Hybrid projective synchronzation. If these values of $\alpha(t)$ are changed to some continous function of time 't' instead of non-zero scalars, the latter is called as the case of Hybrid Projective Function Synchronization or simply, Generalized Function Synchronization.

It is worth noting that if parameters and time-delays are approximately chosen, chaotic delayed systems can exhibit unprecedented complicated behaviour. Thus, synchronization of chaotic delayed systems have been intensively investigated in last two decades. [11,12,13,14,15,16,17,18,19]. However, in most studies on synchronization of chaotic delayed systems assume that the parameters involved are known in advance but in most practical situations, values of the parameters cannot be known in advance. Thus, to tackle the uncertainities induced by the unknown parameters various strategies are employed [20,21,22,23]. In our paper, we have merged two control methods: Adaptive control method and Sliding mode control method to control chaos and give a proper estimation of uncertain parameters.

After the obtainment of Generalized Function Synchronization, we show it's application in obtaining secure communication. By virtue of the characteristics like unpredictability and sensitivity to initial conditions of the chaotic systems, secure communications have become one of the major applications of the synchronization scheme of the chaotic systems [24,25,26,27,28,29]. One of the earliest methods for obtaining chaotic secure communication is through Chaotic Masking Scheme. This scheme uses chaotic signal (which are represented by state variables defining the dynamics of the chaotic model) as a carrier to hide the secret information during transmission, and then the receiver modulator restores this secret information by removing the chaotic signal through synchronization. In this paper, we have modified the masking scheme by using more than one chaotic signals for masking the secret information. This enhances security during transmission. [30].

This article is organized as follows: In section 2, a general class of multi-time-delayed chaotic systems is introduced following which master and slave systems are introduced. In section 3, the control technique derived is explained so that Generalized Function Synchronization (GFS) via adaptive sliding mode control is achieved. In section 4, the method is applied so that GFS is achieved between modified twice-delayed Genesio and Lorenz systems. Numerical simulations establishing synchronization are performed in section 5. Further, in section 6, an application to secure communications is shown where the chaotic models are chosen as sender and receiver modulators. Section 7 verifies the proposed laws numerically and compares graphs of sent messages versus received messages. In section 8, conclusion is drawn.

2. Introduction to Delay Systems

We consider a class of discrete multi-time-delayed chaotic system with uncertain parameters of the form:

$$\dot{x}(t) \quad = \quad f(x, x_{\tau_1}, x_{\tau_2}, ..., x_{\tau_m}, t) + g(x,t)\cdot\theta + \sum_{i=1}^{m} h_i(x_{\tau_i}, t)\lambda_i \qquad (2.1)$$

where x(t)$\in \mathbb{R}^n$ denotes system's n-dimensional state vector,$f, h_i : \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear functions of its arguements. $\theta = (\theta_1, \theta_2, ..., \theta_p)^T \in \mathbb{R}^p$, g: $\mathbb{R}^n \to \mathbb{R}^{n\times p}$ is also non-linear function of its arguements. θ_i, λ_j;i=1(1)p,j=1(1)m are the uncertain parameters of the system. (1) is chosen as the master system.

$\tau_i; i = 1(1)m$ are the constant time delays of the system where,$x_{\tau_i} = x(t-\tau_i); i = 1(1)m$. The non-identical discrete multi-time-delayed chaotic slave system with uncertain parameters is of the form:

$$\dot{y}(t) = F(y, y_{\tau_1}, y_{\tau_2}, ..., y_{\tau_m}, t) + G(y,t)\cdot\Theta + \sum_{i=1}^{m} H_i(y_{\tau_i}, t)\Lambda_i + u(t) \tag{2.2}$$

where y(t)$\in \mathbb{R}^n$ denotes system's n-dimensional state vector,$F, H_i : \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear functions of its arguements. $\Theta = (\Theta_1, \Theta_2, ..., \Theta_p)^T \in \mathbb{R}^p$, G: $\mathbb{R}^n \to \mathbb{R}^{n\times p}$ is also non-linear function of its arguements. Θ_i, Λ_j;i=1(1)p,j=1(1)m are the uncertain parameters of the system. (2) is chosen as the slave system. Here $u(t) \in \mathbb{R}^n$ is the control input vector.

Remark: The parameters $\theta_i, \lambda_j, \Theta_i, \Lambda_j; i = 1(1)p, j = 1(1)m$ are unknown, these are taken as uncertainities when sliding surfaces are designed and will be estimated by an adaptive control law when reaching mode control is proposed.
Some time-delayed chaotic systems satisfying (1) are:

1. Rossler system with multiple time-delays:

$$\begin{aligned}\dot{x}_1(t) &= -x_2 - x_3 - \sum_{i=1}^{n} x_i(x_{\tau_i}, t)\lambda_i\\ \dot{x}_2(t) &= x_1 + ax_2\\ \dot{x}_3(t) &= b + x_3x_1 - cx_3\end{aligned}$$

   where a,b,c are parameters, τ_i are the time-delays and λ_i are geometric factors.

2. Perez-Malta-Coutinho equation:

$$\dot{x}(t) = \beta_o - \beta_1 x_\tau - \gamma x$$

   whereβ_o, β_1, γ are parameters, τ is the time-delay.

3. Modified multiple time-delayed Genesio system:

$$\begin{aligned}\dot{x}(t) &= y\\ \dot{y}(t) &= z\\ \dot{z}(t) &= ax + by(t-\tau_2) + cz + x^2(t-\tau_1)\end{aligned}$$

   where a,b,c are parameters, τ_1, τ_2 are the time-delays.

4. Modified multiple time-delayed Lorenz system:

$$\begin{aligned}\dot{x}(t) &= \sigma(y-x)\\ \dot{y}(t) &= \rho x - xz - y(t-\tau_2)\\ \dot{z}(t) &= xy - \beta z(t-\tau_3)\end{aligned}$$

   where σ, ρ, β are parameters, τ_2, τ_3 are the time-delays.

3. GFS Via Adaptive Sliding Mode Control Method

Let $\alpha(t) \neq 0\ \forall t$ be the scaling function. Then the generalized synchronization errors are defined as follows

$$e(t) = x(t) - \alpha(t)y(t) \tag{3.1}$$

Hence using (1), (2) and (3), the error dynamics is given by

$$\begin{aligned}\dot{e}(t) = {} & f + g(x,t)\cdot\theta + \sum_{i=1}^{m} h_i(x_{\tau_i}, t)\lambda_i - \\ & \alpha(t)\{F + G(y,t)\cdot\Theta + \sum_{i=1}^{m} H_i(y_{\tau_i}, t)\Lambda_i + u(t)\} - \dot{\alpha}(t)y(t)\end{aligned} \tag{3.2}$$

It is clear that the stability of error dynamics (4) results in generalised hybrid synchronization between the non-identical master and slave system (1) and (2). Thus, the synchronization problem is replaced by the equivalent of stabilizing the error dynamics (4) by using suitable input u(t).

The sliding mode control (SMC) theory is an efficient approach to solve the robust control problems. Its advantages are easy realization, rapid and fast response and insensitivity to variations in system parameters or external disturbances when there are uncertainities, especially those derived from uncertain parameters, adaptive control is found most suitable. Thus, combinig the robustness of the SMC with adaptability of adaptive control, we design an adaptive S.M. controller to realize generalised hybrid synchronization of non-identical multi-time-delayed chaotic systems containing uncertain parameters.

To design S.M. controller, ∃ two basic steps, first to select an appropriate switching surface and second, to establish a control law which guarantees stability of the sliding surface.

3.1. Sliding Surface Design

The sliding mode surface can be in general defined as

$$S(t) \quad = \quad Ae(t) \tag{3.3}$$

where A= diag $(a_1, a_2, ..., a_n) \in \mathbb{R}^n \times \mathbb{R}^n$. Now, the necessary condition for any state trajectory to stay on the switiching surface S(t)=0 is $\dot{S}(t) = 0$. Thus, in the sliding mode, we must have

$$S(t) \quad = \quad 0 \tag{3.4}$$

$$\dot{S}(t) \quad = \quad 0 \tag{3.5}$$

Next, we design a control law u(t) which will guarantee that the error system trajectories reach on the sliding surface S(t)=0 and stay on it for all subsequent time.

3.2. Adaptive SMC design and Parameter Adaptation laws

We assume that the constant rate reaching law is applied. thus, the law can be chosen as

$$\dot{S}(t) \quad = \quad -q\,\mathrm{sgn}\{S(t)\} \tag{3.6}$$

where q>0. Thus, using (4), (5) and (7), it follows

$$\begin{aligned} 0 \quad &= \quad \dot{S}(t) \\ &= \quad A\dot{e} \\ &= \quad A[f + g(x,t)\cdot\theta + \sum_{i=1}^{m} h_i(x_{\tau_i}, t)\lambda_i - \\ &\quad\quad \alpha(t)\{F + G(y,t)\cdot\Theta + \sum_{i=1}^{m} H_i(y_{\tau_i}, t)\Lambda_i + u(t)\} - \dot{\alpha}(t)y(t)] \end{aligned} \tag{3.7}$$

(8) and (9) are identical. In order to achieve this, we propose the following adaptive SMC laws and parameter update laws for synchronizing the multi-delay systems

1. **Adaptive Sliding Mode Control Laws**

$$\begin{aligned} \alpha(t)u(t) \quad = \quad & f + g(x,t)\cdot\hat{\theta} + \sum_{i=1}^{m} h_i(x_{\tau_i}, t)\hat{\lambda}_i - \alpha(t)F - \alpha(t)G(y,t)\cdot\hat{\Theta} - \alpha(t)\sum_{i=1}^{m} H_i(y_{\tau_i}, t)\hat{\Lambda}_i \\ & -\dot{\alpha}(t)y(t) + Ke(t) + \sum_{i=1}^{m} L_i(t)e(t-\tau_i) + qA^{-1}\,\mathrm{sgn}\{S(t)\} \end{aligned} \tag{3.8}$$

where k= diag $(k_1, k_2, ..., k_n) \in \mathbb{R}^n \times \mathbb{R}^n$ is the static state feedback matrix while L_i(t); i=1(1)m are the delayed time-varying state feedback matrices. Further, $\hat{\theta}(t), \hat{\Theta}(t)$ are the estimations of the uncertain parameter vectors θ and Θ respectively. $\hat{\lambda}_i$, $\hat{\Lambda}_i$;i=1(1)m denote estimations of the uncertain parameters λ_i, Λ_i respectively. Also, sgn$(\cdot)$ denotes the signum function, $q > 0$ is the constant gain which is so determined that sliding condition is satisfied and sliding mode motion will occur.

2. **Parameter update laws**

$$\begin{aligned}\dot{\hat{\lambda}}_i(t) &= \{Ah_i(x_{\tau_i},t)+B_ie(t-\tau_i)\}^TAe-k_{\lambda_i}\bar{\lambda}_i \;\; ; i=1(1)m\\ \dot{\hat{\Lambda}}_i(t) &= -\alpha(t)\{AH_i(y_{\tau_i},t)+C_ie(t-\tau_i)\}^TAe-k_{\Lambda_i}\bar{\Lambda}_i \;\; ; i=1(1)m\\ \dot{\hat{\theta}}(t) &= \{Ag(x,t)\}^TAe-k_\theta\bar{\theta}\\ \dot{\hat{\Theta}}(t) &= -\alpha(t)\{AG(y,t)\}^TAe-k_\Theta\bar{\Theta}\end{aligned} \tag{3.9}$$

where, k_θ= diag $(k_{\theta_1}, k_{\theta_2}, ..., k_{\theta_n}), k_\Theta$= diag $(k_{\Theta_1}, k_{\Theta_2}, ..., k_{\Theta_n}) \in \mathbb{R}^p \times \mathbb{R}^p$, $k_{\lambda_i}, k_{\Lambda_i} \in \mathbb{R}$; i=1(1)m are all control gains and $\bar{\lambda}_i = \hat{\lambda}_i - \lambda_i, \bar{\Lambda}_i = \hat{\Lambda}_i - \Lambda_i$; i=1(1)m, $\bar{\theta} = \hat{\theta} - \theta$ and $\bar{\Theta} = \hat{\Theta} - \Theta$.

3.3. Stability Analysis

It is proved in the form of the following theorem:

Theorem: If the error dynamics (4) is controlled by u(t) given by (10) where k is chosen as a positive definite matrix and $L_i(t)$ is determined by the equation

$$L_i(t) \;=\; A^{-1}\{\bar{\lambda}_iB_i+\bar{\Lambda}_iC_i\} \;\; ; i=1(1)m \tag{3.10}$$

together with parameter update laws given by (11), where k_θ, k_Θ are positive definite matrices and $k_{\lambda_i}, k_{\Lambda_i}$>0 $\forall t$, then the state trajectories will converge to sliding surface S(t)=0.

Proof: To prove this, we define the following Lyapunov-Krasovskii functional V(t) as

$$V=\frac{1}{2}S(t)^TS(t)+\frac{1}{2}\bar{\theta}^T\bar{\theta}+\frac{1}{2}\bar{\Theta}^T\bar{\Theta}+\frac{1}{2}\sum_{i=1}^{m}\bar{\lambda}_i^2+\frac{1}{2}\sum_{i=1}^{m}\bar{\Lambda}_i^2\geq 0$$

Thus,

$$\dot{V}=S(t)^T\dot{S}(t)+\bar{\theta}^T\dot{\bar{\theta}}+\bar{\Theta}^T\dot{\bar{\Theta}}+\sum_{i=1}^{m}\bar{\lambda}_i\dot{\bar{\lambda}}_i+\sum_{i=1}^{m}\bar{\Lambda}_i\dot{\bar{\Lambda}}_i$$

Now, using (9), (10), (11) and (12), it can be shown that

$$\dot{V}=-e^TA^2ke-qS(t)^T\,\text{sgn}\{S(t)\}-\bar{\theta}^Tk_\theta\bar{\theta}-\bar{\Theta}^Tk_\Theta\bar{\Theta}-\sum_{i=1}^{m}k_{\lambda_i}\bar{\lambda}_i^2-\sum_{i=1}^{m}k_{\Lambda_i}\bar{\Lambda}_i^2<0$$

Thus, the trajectories of the error dynamics are globally asymptotically driven onto the sliding surface S(t)=0 and maintained there $\forall t$>0. Hence, the proof is complete.

By Lyapunov-Krasovskii stability theorem, it follows that when the trajectories are driven onto the sliding mode S=0, the error asymptotically declines on the sliding surface which establishes the stability of the error dynamics (4). Consequently, generalised function synchronization between master and slave systems (1) and (2) is achieved.

4. GFS Between Modified Genesio and Lorenz System with Two Time-Delays: an Application

Genesio system with two time-delays is given by

$$\begin{aligned}\dot{x}_1(t) &= x_2\\ \dot{x}_2(t) &= x_3\\ \dot{x}_3(t) &= ax_1+bx_2(t-\tau_2)+cx_3+x_1^2(t-\tau_1)\end{aligned} \tag{4.1}$$

and, modified Lorenz system with two time-delays is

$$\begin{aligned}\dot{y}_1(t) &= \sigma(x_2 - x_1) + u_1\\ \dot{y}_2(t) &= \rho y_1 - y_1 y_3 - y_2(t-\tau_2) + u_2\\ \dot{y}_3(t) &= y_1 y_2 - \beta y_3(t-\tau_3) + u_3\end{aligned} \tag{4.2}$$

(13) and (14) are the master and slave system respectively. Relating (13) with (1), we find x= $\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}$, $f = \begin{pmatrix} x_2\\ x_3\\ x_1^2(t-\tau_1)\end{pmatrix}, g = \begin{pmatrix} 0 & 0\\ 0 & 0\\ x_1 & x_3\end{pmatrix}, \theta = \begin{pmatrix} a\\ c\end{pmatrix}, h_2 = \begin{pmatrix} 0\\ 0\\ x_2(t-\tau_2)\end{pmatrix}, h_1 = h_3 = \mathbf{0}_{3\times1}, \lambda_2 = b, \lambda_1 = \lambda_3 = 0$, while relating (14) with (2), we find, x= $\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix}$, $F = \begin{pmatrix} 0\\ -y_1y_3 - y_2(t-\tau_2)\\ y_1y_2\end{pmatrix}$,
$G = \begin{pmatrix} y_2 - y_1 & 0\\ 0 & y_1\\ 0 & 0\end{pmatrix}, \Theta = \begin{pmatrix} \sigma\\ \rho\end{pmatrix}, H_3 = \begin{pmatrix} 0\\ 0\\ -y_3(t-\tau_3)\end{pmatrix}, H_1 = H_2 = \mathbf{0}_{3\times1}, \Lambda_3 = \beta, \Lambda_1 = \Lambda_2 = 0.$
The error dynamical system is given by

$$\begin{aligned}\dot{e}_1(t) &= x_2 - \alpha(t)[\sigma(x_2 - x_1) + u_1] - \dot{\alpha} y_1\\ \dot{e}_2(t) &= x_3 - \alpha(t)[\rho y_1 - y_1 y_3 - y_2(t-\tau_2) + u_2] - \dot{\alpha} y_2\\ \dot{e}_3(t) &= a x_1 + b x_2(t-\tau_2) + c x_3 + x_1^2(t-\tau_1) - \alpha(t)[y_1 y_2 - \beta y_3(t-\tau_3) + u_3] - \dot{\alpha} y_3\end{aligned} \tag{4.3}$$

Let us choose A=I_3 and q=1, then S(t)= $\begin{pmatrix} e_1\\ e_2\\ e_3\end{pmatrix}$ while the reaching law is given by $\dot{S}(t) = \begin{pmatrix} \mathrm{sgn}\{e_1\}\\ \mathrm{sgn}\{e_2\}\\ \mathrm{sgn}\{e_3\}\end{pmatrix}$
Now, let us choose $k_\theta = k_\Theta = I_2, k = I_3, k_{\lambda_i} = k_{\Lambda_i} = 1; i = 1(1)3$. Further, let B= $\begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ and C= $\begin{pmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. Thus, by (12), $L_1 = \mathbf{0}_{3\times3}, L_2 = \begin{pmatrix} (\hat{b}-b) & (\hat{b}-b) & 0\\ 0 & (\hat{b}-b) & 0\\ 0 & 0 & (\hat{b}-b)\end{pmatrix}$,
$L_3 = \begin{pmatrix} (\hat{\beta}-\beta) & 0 & (\hat{\beta}-\beta)\\ 0 & (\hat{\beta}-\beta) & 0\\ 0 & 0 & (\hat{\beta}-\beta)\end{pmatrix}$.
Hence, the adaptive SMC laws are given by

$$\begin{aligned}\alpha u_1(t) &= x_2 - \alpha(t)[\hat{\sigma}(y_2 - y_1)] - \dot{\alpha} y_1 + e_1 + (\hat{b}-b)e_1(t-\tau_2) + (\hat{b}-b)e_2(t-\tau_2) +\\ &\quad (\hat{\beta}-\beta)e_1(t-\tau_3) + (\hat{\beta}-\beta)e_3(t-\tau_3)\\ \alpha u_2(t) &= x_3 - \alpha(t)[-y_1y_3 - y_2(t-\tau_2)] - \alpha(t)\hat{\rho} y_1 - \dot{\alpha} y_2 + e_2 + (\hat{b}-b)e_2(t-\tau_2) +\\ &\quad (\hat{\beta}-\beta)e_2(t-\tau_3)\\ \alpha u_3(t) &= \hat{a} x_1 + \hat{b} x_2(t-\tau_2) + \hat{c} x_3 + x_1^2(t-\tau_1) - \alpha(t)[y_1y_2 - \hat{\beta} y_3(t-\tau_3)] - \dot{\alpha} y_3 +\\ &\quad e_3(\hat{b}-b)e_3(t-\tau_2) + (\hat{\beta}-\beta)e_3(t-\tau_3)\end{aligned} \tag{4.4}$$

The parameter update laws are then,

$$\begin{aligned}
\dot{\hat{a}}(t) &= x_1e_3 - (\hat{a} - a)\\
\dot{\hat{b}}(t) &= e_1(t-\tau_2)e_1 + e_2(t-\tau_2)e_1 + e_2(t-\tau_2)e_2 + x_2(t-\tau_2)e_3 + e_3(t-\tau_2)e_3 - (\hat{b} - b)\\
\dot{\hat{c}}(t) &= x_3e_3 - (\hat{c} - c)\\
\dot{\hat{\sigma}}(t) &= -\alpha(t)(y_2 - y_1)e_1 - (\hat{\sigma} - \sigma)\\
\dot{\hat{\beta}}(t) &= e_1(t-\tau_3)e_1 + e_3(t-\tau_3)e_1 + e_2(t-\tau_3)e_2 + e_3(t-\tau_3)e_3 + \alpha(t)y_3(t-\tau_3)e_3 - (\hat{\beta} - \beta)\\
\dot{\hat{\rho}}(t) &= -\alpha(t)y_1e_2 - (\hat{\rho} - \rho) \qquad (4.5)
\end{aligned}$$

Now, for stability analysis, we consider the Lyapunov-Krasovskii functional as follows:

$$V = \frac{1}{2}S(t)^TS(t) + \frac{1}{2}(\hat{a} - a)^2 + \frac{1}{2}(\hat{b} - b)^2 + \frac{1}{2}(\hat{c} - c)^2 + \frac{1}{2}(\hat{\sigma} - \sigma)^2 + \frac{1}{2}(\hat{\beta} - \beta)^2 + \frac{1}{2}(\hat{\rho} - \rho)^2$$

Clearly, V(t) is semi positive-definite. Thus, taking time-derivative of the above and using (15), (16) and (17), it can be shown that,

$$\begin{aligned}
\dot{V} &= S(t)^T\dot{S}(t) + (\hat{a} - a)\dot{\hat{a}} + (\hat{b} - b)\dot{\hat{b}} + (\hat{c} - c)\dot{\hat{c}} + (\hat{\sigma} - \sigma)\dot{\hat{\sigma}} + (\hat{\beta} - \beta)\dot{\hat{\beta}} + (\hat{\rho} - \rho)\dot{\hat{\rho}}\\
&= -e_1^2 - e_2^2 - e_3^2 - (\hat{a} - a)^2 - (\hat{b} - b)^2 - (\hat{c} - c)^2 - (\hat{\sigma} - \sigma)^2 - (\hat{\beta} - \beta)^2 - (\hat{\rho} - \rho)^2\\
&< 0 \qquad (4.6)
\end{aligned}$$

So, stability of error dynamics (15) is established.

5. Establishment of Synchronization Using Numerical Simulations

In this section, we give numerical simulations to illustrate the effectiveness of the proposed scheme by applying the theoretical results to modified Genesio system with two time-delays and modified Lorenz system with two time-delays as discussed in section 4. We choose $\tau_1 = 3, \tau_2 = 1, \tau_3 = 2$. We note that by choosing $a = -0.1, b = -0.9, c = -1.15$ and the initial conditions as $x1[t/; t <= 0] == -0.09, x2[t/; t <= 0] == -0.1, x3[t/; t <= 0] == 0.0083$, the modified Genesio system with two time-delays shows chaotic behaviour as shown in FIG. 1(a). On the other hand, if we choose $\sigma = 0.9, \rho = 2.5, \beta = 0.1$ and the initial conditions as $y1[t/; t <= 0] == -0.9, y2[t/; t <= 0] == 0.1, y3[t/; t <= 0] == -0.01$, the modified Lorenz system with two time-delays shows chaotic behaviour. This has been shown by the 3D phase portraits in FIG. 1(b).

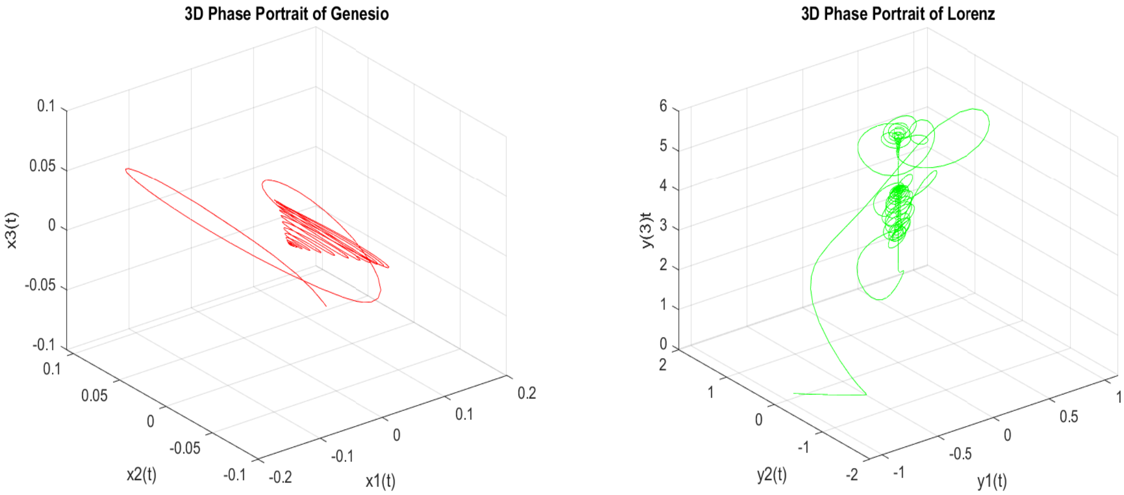
Let us now choose the initial value of the estimated parameters as $\hat{a}(0) = 2, \hat{b}(0) = 2, \hat{c}(0) = 1, \hat{\sigma}(0) = 3, \hat{\beta}(0) = -2, \hat{\rho}(0) = -3$. We shall show that by choosing different values of $\alpha(t)$, several synchronization phenomena can be obtained.

1. **Complete synchronization**
   Here $\alpha(t) = 1$. The phase portraits of the master system and the completely synchronized slave system are superimposed and shown in FIG.3(a).

2. **Anti-Synchronization**
   Here $\alpha(t) = -1$. The phase portraits of the master system and the anti-synchronized slave system are superimposed and shown in FIG. 3(b).

3. **Projective Synchronization**
   We choose as an example, $\alpha(t) = 2$. The phase portraits of the master system and the projectively synchronized slave system are superimposed and shown in FIG. 4(a).

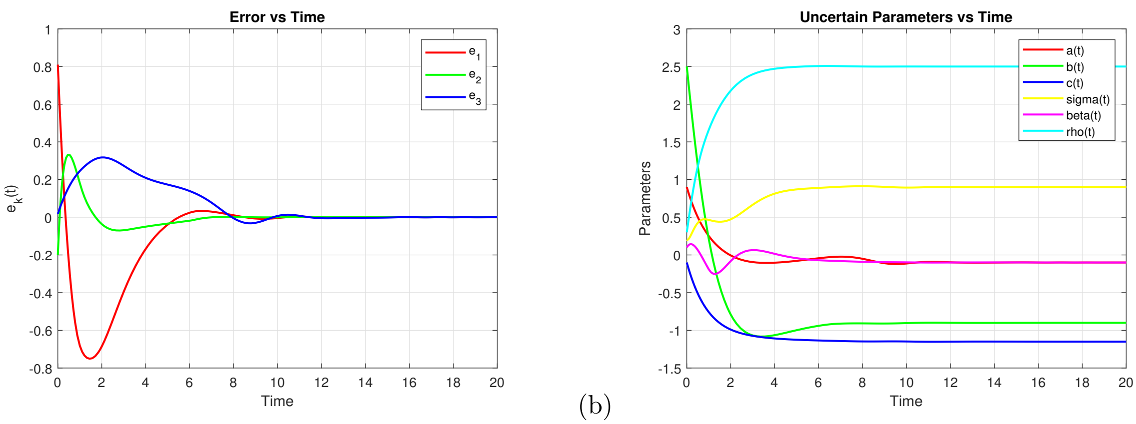
4. **Function Projective Synchronization**
   As an example, let $\alpha(t) = 0.5t + 1$. The phase portraits of the master system and the function projectively-synchronized slave system are superimposed and shown in FIG. 4(b).



(a) (b)

Figure 1: (a)Master system (b)Slave system(without controls)

The convergence of all error variables to zero is shown in FIG. 2(a). Time evolution of the estimated parameters which is shown via FIG. 2(b) which clearly indicates that $\hat{a} \to a, \hat{b} \to b, \hat{c} \to c, \hat{\sigma} \to \sigma, \hat{\beta} \to \beta, \hat{\rho} \to \rho$ as $t \to \infty$. We obtain the same error graphs and parameter estimation graphs in every synchronization phenomenon discussed above. Finally, it follows that Generalized Function Synchronization is achieved between the master system (13) and slave system (14).



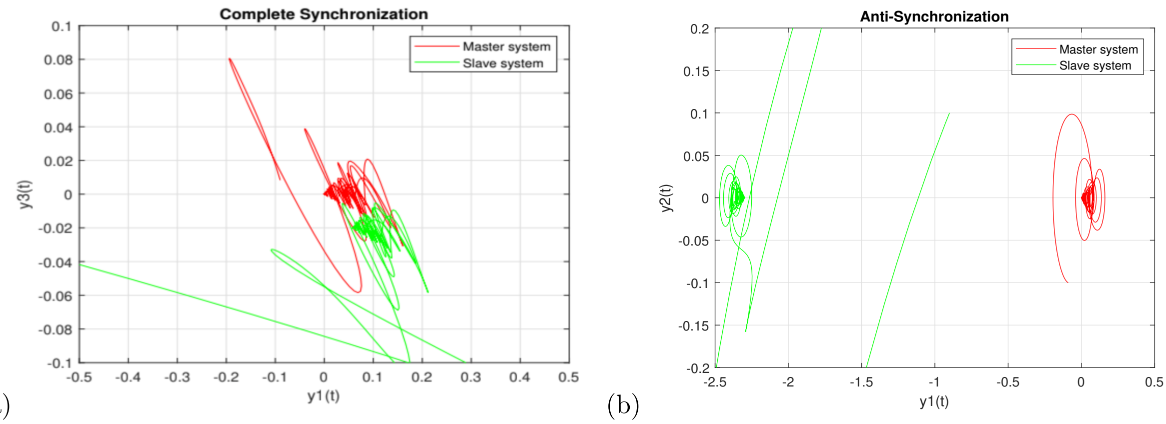
(a) (b)

Figure 2: (a) Error Dynamics (b) Parameters Estimation

6. An Application: Secure Communication of Secret Message Between Modulators Using Synchronization

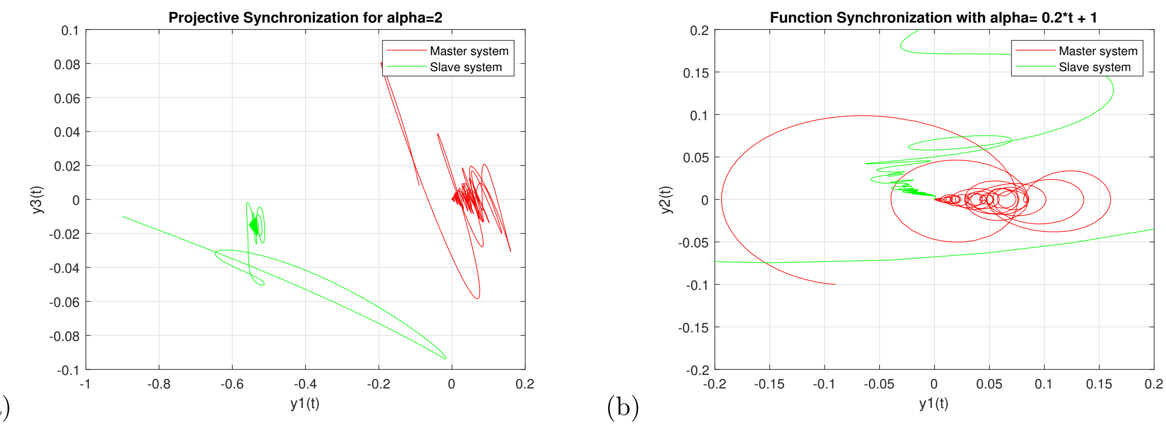
We present an application of synchronization of the two systems to achieve SECURE COMMUNICATION between them using CHAOTIC MASKING SCHEME. For achieving the above, we choose master system as the sender and the slave system as the receiver. A message signal m(t), which is a continuous function of time t, is scaled k times, and is added to the state variable $x_3(t)$. This message is then transmitted to the receiver. As the dynamics of both the sender and the receiver are synchronized, the signal is retrieved from the synchronized state variable $y_3(t)$ of the receiver model.

The above application can be explained as follows:



(a) (b)

Figure 3: (a)Complete synchronization (b)Anti-synchronization



(a) (b)

Figure 4: (a)Projective synchronization (b)Function synchronization

The message signal m(t) is scaled k times and added to $x_3(t)$ so that a new state variable s(t) is introduced as follows:

$$s(t) = km(t) + x_3(t)$$

At the sender's, modulator can be written as:

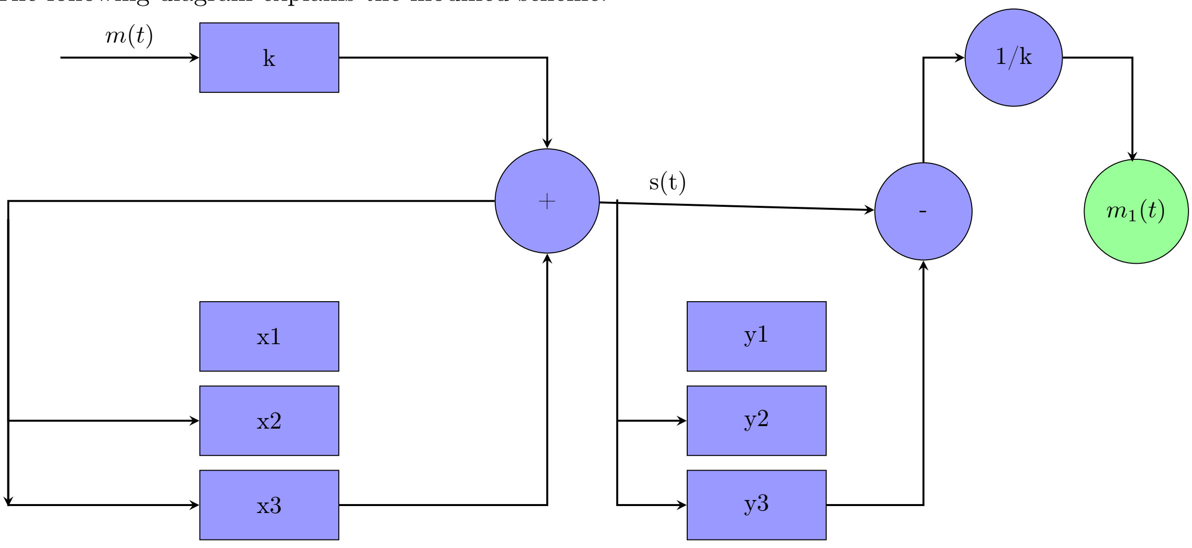
$$\begin{aligned} \dot{x}_1(t) &= x_2 \\ \dot{x}_2(t) &= s(t) \\ \dot{x}_3(t) &= ax_1 + bx_2(t-\tau_2) + cs(t) + x_1^2(t-\tau_1) \\ s(t) &= km(t) + x_3(t) \end{aligned} \tag{6.1}$$

This is the modified master system. At the receiver, the demodulator is given by:

$$\begin{aligned} \dot{y}_1(t) &= \sigma(y_1 - y_2) + u_1 \\ \dot{y}_2(t) &= \rho y_1 - y_1 y_3 - y_2(t-\tau_2) + u_2 \\ \dot{y}_3(t) &= y_1 y_2 - \beta y_3(t-\tau_3) + u_3 \\ m_1(t) &= (s(t) - y_3(t))/k \end{aligned} \tag{6.2}$$

7. Chaotic Masking Communication Scheme

The following diagram explains the modified scheme:



HERE, $m(t)$ is the sent signal, $m_1(t)$ is the retrieved signal.
The modified error dynamical system is then given by:

$$\begin{aligned}\dot{e}_1(t) &= x_2 - \alpha(t)[\sigma(y_2-y_1)+u_1] - \dot{\alpha}y_1 \\ \dot{e}_2(t) &= s(t) - \alpha(t)[\rho y_1 - y_1 s(t) - y_2(t-\tau_2)+u_2] - \dot{\alpha}y_2 \\ \dot{e}_3(t) &= ax_1 + bx_2(t-\tau_2) + cs(t) + x_1^2(t-\tau_1) - \alpha(t)[y_1y_2 - \beta s(t-\tau_3)+u_3] - \dot{\alpha}y_3\end{aligned} \tag{7.1}$$

We shall use the same controllers and synchronize the modified systems (6.1) and (6.2). Since, the two modified systems (6.1) and (6.2) have the same structure as the original master and slave systems (only $x_3(t)$ being replaced by s(t)). Hence, the modified Adaptive Control Laws are given by:

$$\begin{aligned}\alpha u_1(t) &= x_2 - \alpha(t)[\hat{\sigma}(y_2-y_1)] - \dot{\alpha}y_1 + e_1 + (\hat{b}-b)e_1(t-\tau_2) + (\hat{b}-b)e_2(t-\tau_2) + \\ &\quad (\hat{\beta}-\beta)e_1(t-\tau_3) + (\hat{\beta}-\beta)e_3(t-\tau_3) \\ \alpha u_2(t) &= s(t) - \alpha(t)[-y_1y_3 - y_2(t-\tau_2)] - \alpha(t)\hat{\rho}y_1 - \dot{\alpha}y_2 + e_2 + (\hat{b}-b)e_2(t-\tau_2) + \\ &\quad (\hat{\beta}-\beta)e_2(t-\tau_3) \\ \alpha u_3(t) &= \hat{a}x_1 + \hat{b}x_2(t-\tau_2) + \hat{c}s(t) + x_1^2(t-\tau_1) - \alpha(t)[y_1y_2 - \hat{\beta}y_3(t-\tau_3)] - \dot{\alpha}y_3 + \\ &\quad e_3(\hat{b}-b)e_3(t-\tau_2) + (\hat{\beta}-\beta)e_3(t-\tau_3)\end{aligned} \tag{7.2}$$

The modified Parameter Update Laws are,

$$\begin{aligned}\dot{\hat{a}}(t) &= x_1e_3 - (\hat{a}-a) \\ \dot{\hat{b}}(t) &= e_1(t-\tau_2)e_1 + e_2(t-\tau_2)e_1 + e_2(t-\tau_2)e_2 + x_2(t-\tau_2)e_3 + e_3(t-\tau_2)e_3 - (\hat{b}-b) \\ \dot{\hat{c}}(t) &= s(t)e_3 - (\hat{c}-c) \\ \dot{\hat{\sigma}}(t) &= -\alpha(t)(y_2-y_1)e_1 - (\hat{\sigma}-\sigma) \\ \dot{\hat{\beta}}(t) &= e_1(t-\tau_3)e_1 + e_3(t-\tau_3)e_1 + e_2(t-\tau_3)e_2 + e_3(t-\tau_3)e_3 + \alpha(t)y_3(t-\tau_3)e_3 - (\hat{\beta}-\beta) \\ \dot{\hat{\rho}}(t) &= -\alpha(t)y_1e_2 - (\hat{\rho}-\rho)\end{aligned} \tag{7.3}$$

These laws ensure the stability of the modified Error Dynamical system (7.1). Upon successful synchronization, we establish that the masked secret message m(t) has been communicated between the modulators using Numerical Simulations in the next section.

8. Establishment of Secure Communication Using Numerical Simulations

In this section, we present numerical simulations to illustrate the effectiveness of the CHAOTIC MASKING scheme to securely communicate secret message m(t). We choose different functions as m(t) and verify results in each case. Let scaling factor be $k = 5$. For secure communications, we need to achieve only Complete synchronization. Hence we take $\alpha(t) = 1$. Further, $s(t) = m(t) + x_3(t)$. Rest of the values are chosen with same set of values given by: $\tau_1 = 3, \tau_2 = 1, \tau_3 = 2, a = -0.1, b = -0.9, c = -1.15, \sigma = 0.9, \rho = 2.5, \beta = 0.1$, and initial conditions $x_1[t/; t <= 0] == -0.09, x_2[t/; t <= 0] == -0.1, x_3[t/; t <= 0] == 0.0083, y_1[t/; t <= 0] == -0.9, y_2[t/; t <= 0] == 0.1, y_3[t/; t <= 0] == -0.01$. For every signal $m(t)$, which is a continuous function of time 't', secured communication is achieved with high accuracy. For every function $m(t)$, three sets of graphs are drawn that show original message $m(t)$, received message $m_1(t)$ and their asymptotic difference $m(t) - m_1(t)$ respectively.

1. $m(t) = t^3 + t^2 + 1$

2. $m(t) = \log(t + 1) + \cos(t^2 + 1)$

3. $m(t) = \sin(2t^2 + t + 1)$

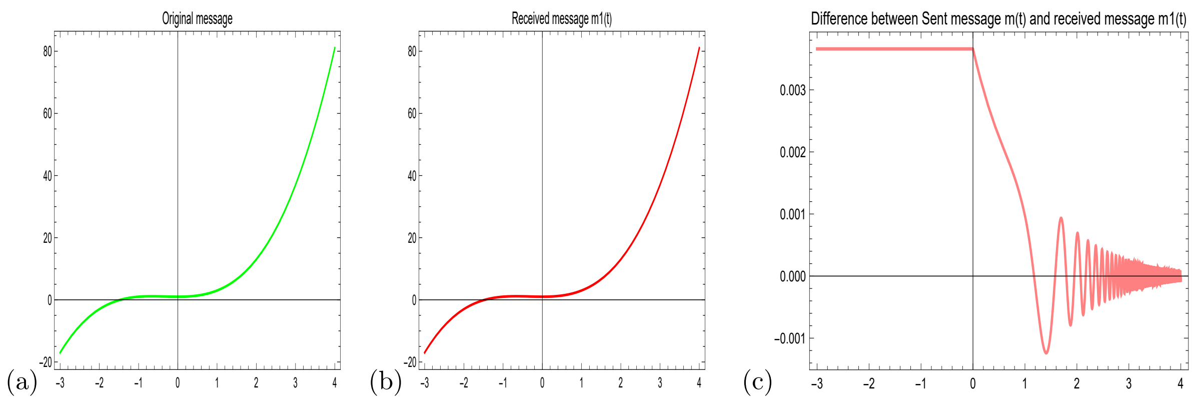


Figure 5: (a) $m(t)$ (b) $m_1(t)$ (c) $m(t) - m_1(t)$

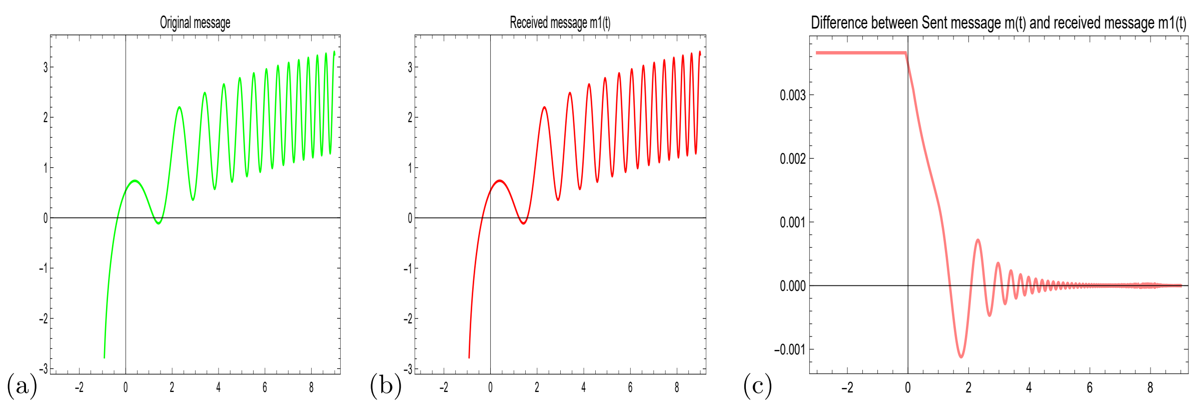


Figure 6: (a) $m(t)$ (b) $m_1(t)$ (c) $m(t) - m_1(t)$

Hence, we observe from these graphs that even for different function values of the messages, the original and received messages are obtained eventually with high level of accuracy.

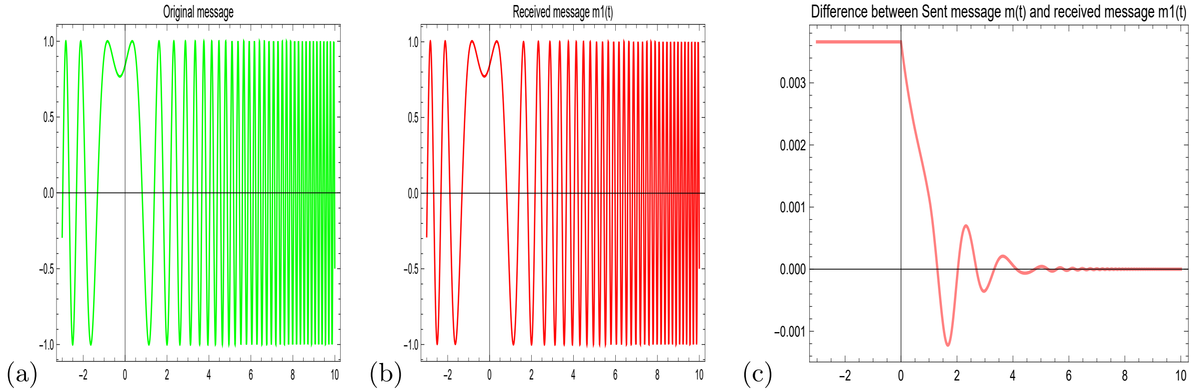


Figure 7: (a) $m(t)$ (b) $m_1(t)$ (c) $m(t) - m_1(t)$

9. Conclusion

In this paper, we are successful in designing an adaptive sliding mode controller containing static feedback and time-varying delayed feedback terms in such a way that the uncertainities brought in by the fully unknown parameters are tackled and the unknown parameters are estimated. Further GFS is realised between two non-identical, chaotic, multi-delayed systems. The proposed control scheme is also applied to synchronize modified genesio and lorenz systems as examples. Furthermore, we have showcased an application of the above synchronization method in obtaining secure communication between the same systems, where the Genesio system is the sender while the Lorenz system is the receiver. The messages are sent as continuous functions of time, and we have modified the chaotic masking scheme to achieve secure communications effectively. We have shown examples of various continuous functions that represent the messages, and all the numerical simulations confirm our analytical findings. In the future, this work can be extended to any continuous function m(t), which represents the signal messages.

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