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Anisotropic Wavelet Frames and Operator Analysis in Canonical Domains

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ABSTRACT: We develop a concise framework for wavelet frames in the non-separable canonical domain on $\mathbb{R}^n$. Using symplectic transformations, we derive frame conditions, analyze associated operators, and establish new uncertainty principles. Examples and visualizations illustrate the enhanced localization and anisotropy offered by canonical wavelet systems.

Keywords: Non-separable wavelet frames, symplectic transformations, linear canonical transform, frame operator, uncertainty principles, anisotropic wavelets, Hilbert–Schmidt operators.

Contents

1. Introduction

Wavelet analysis serves as a fundamental tool in modern harmonic analysis, providing the ability to localize signals in both time and frequency. Classical wavelet frames on $L^2(\mathbb{R}^n)$ are based on isotropic dilations and translations of a single generating function, offering stable and redundant representations. However, these systems are often inadequate in capturing directional or anisotropic features in higher-dimensional signals [1,2,3,5,6,7,11,14,15,16].

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To address this, various generalized transforms such as the fractional wavelet transform, affine wavelet transform, and linear canonical wavelet transform (LCWT) have been proposed. Recently, a significant advancement was made in [17], where the authors introduced the *non-separable linear canonical wavelet transform* (NSLCWT). This transform utilizes general symplectic matrices to encode more degrees of freedom, allowing non-isotropic and non-separable analysis in $L^2(\mathbb{R}^n)$. Their work focuses on the integral structure, inversion, and uncertainty principles for NSLCWT.

In contrast, our aim in this paper is to advance the theory further by developing a rigorous framework for *wavelet frames* in the non-separable canonical domain. We explore frame conditions, bounded operator representations, and spectral properties of associated integral transforms. Additionally, we derive new forms of uncertainty inequalities, such as entropy-based and Hardy-type bounds, which extend beyond classical results.

Main Contributions. This paper aims to present the following new developments:

- Formulation of wavelet frame systems in the non-separable canonical domain.
- Derivation of sufficient and necessary conditions for frame properties.
- Operator-theoretic analysis of the associated wavelet integral transforms.
- Generalized uncertainty principles, including entropy and Hardy-type bounds.
- Explicit examples and symbolic visualizations of the transformed wavelets.

Structure of the Paper. Section 2 provides mathematical preliminaries on symplectic matrices, wavelet systems, and canonical transforms. Section 3 introduces the non-separable wavelet systems. Section 4 presents frame conditions and operator-theoretic results. Section 5 derives novel uncertainty inequalities. Section 6 includes examples and visualizations, while Section 7 concludes with applications and open directions.

2. Preliminaries

This section presents the foundational concepts needed throughout the paper, including properties of symplectic matrices, the linear canonical transform (LCT), and essential aspects of wavelet systems and frame theory on $L^2(\mathbb{R}^n)$.

Definition 2.1 *Let $n \in \mathbb{N}$. The $2n \times 2n$ real symplectic group is defined by*

$$\mathrm{Sp}(2n, \mathbb{R}) = \left\{ M \in \mathbb{R}^{2n\times 2n} : M^\top J M = J \right\},$$

where J is the standard symplectic matrix:

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

and I_n is the $n \times n$ identity matrix. A matrix $M \in \mathrm{Sp}(2n, \mathbb{R})$ represents a linear canonical transformation in phase space that preserves the symplectic form.

These matrices form the core of linear canonical transforms and allow modeling of non-separable geometric operations such as rotations, shears, and scalings in joint time-frequency domains.

Definition 2.2 *Given a symplectic matrix* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$*, with* $A, B, C, D \in \mathbb{R}^{n\times n}$*, the* n*-dimensional Linear Canonical Transform (LCT) of a function* $f \in L^2(\mathbb{R}^n)$*, denoted* $\mathcal{L}_M f$*, is defined by*

$$(\mathcal{L}_M f)(u) = \begin{cases} \dfrac{1}{(2\pi i)^n \sqrt{|\det(B)|}} \displaystyle\int_{\mathbb{R}^n} e^{\frac{i}{2}(u^\top DB^{-1}u - 2x^\top B^{-1}u + x^\top B^{-1}Ax)} f(x)\, dx, & \det B \neq 0, \\ \textit{distributional sense}, & \det B = 0. \end{cases}$$

This generalizes both the Fourier transform and the fractional Fourier transform, depending on the choice of M*. In the non-separable case,* A, B, C, D *are not diagonal, allowing more general operations.*

Definition 2.3 *Let* $\psi \in L^2(\mathbb{R}^n)$ *be an admissible mother wavelet. A classical wavelet system is formed via translations and dilations:*

$$\psi_{a,b}(x) = \frac{1}{a^{n/2}} \psi\left(\frac{x-b}{a}\right), \quad a > 0,\ b \in \mathbb{R}^n.$$

The continuous wavelet transform (CWT) is defined by

$$W_\psi f(a, b) = \langle f, \psi_{a,b} \rangle = \int_{\mathbb{R}^n} f(x)\, \overline{\psi_{a,b}(x)}\, dx.$$

The system $\{\psi_{a,b}\}_{a>0, b\in\mathbb{R}^n}$ *is said to be a frame for* $L^2(\mathbb{R}^n)$ *if there exist constants* $A, B > 0$ *such that for all* $f \in L^2(\mathbb{R}^n)$*,*

$$A\|f\|^2 \leq \int_0^\infty \int_{\mathbb{R}^n} |W_\psi f(a, b)|^2 \frac{db\, da}{a^{n+1}} \leq B\|f\|^2.$$

Such frames provide stable and redundant representations of signals and functions.

2.1. Towards Non-Separable Canonical Wavelet Systems

In the framework of non-separable linear canonical transforms, one replaces the simple dilated-translated wavelets with geometrically transformed atoms. Given a symplectic matrix M, a wavelet system in the canonical domain takes the form

$$\psi_{M,a,b}(x) := \mathcal{L}_M \left[\frac{1}{a^{n/2}} \psi\left(\frac{x-b}{a}\right)\right](x),$$

or its inverse, depending on convention. This enables the construction of **anisotropic wavelet frames** under canonical deformation, which we analyze rigorously in the following sections.

3. Canonical Wavelet Frame Systems on $\mathbb{R}^n$

In this section, we define and study wavelet systems generated under non-separable linear canonical transforms (NSLCT). These systems generalize the classical translation-dilation wavelet systems to incorporate arbitrary symplectic transformations.

Definition 3.1 *Let* $\psi \in L^2(\mathbb{R}^n)$ *be a fixed mother wavelet and let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$ *be a non-separable symplectic matrix.*

We define the **canonical wavelet atom***as:*

$$\psi_{M,a,b}(x) := \mathcal{L}_M \left[\frac{1}{a^{n/2}} \psi\left(\frac{x-b}{a}\right)\right](x),$$

where $a > 0$ *is the scale and* $b \in \mathbb{R}^n$ *is the translation.*

This family constitutes the **canonical wavelet system***:*

$$\mathcal{W}_M(\psi) := \{\psi_{M,a,b} : a > 0,\ b \in \mathbb{R}^n\}.$$

The associated **Canonical Wavelet Transform (CWT-M)** of a function $f \in L^2(\mathbb{R}^n)$ is defined by:

$$\mathcal{W}_\psi^M f(a,b) := \langle f, \psi_{M,a,b} \rangle = \int_{\mathbb{R}^n} f(x)\, \overline{\psi_{M,a,b}(x)}\, dx.$$

Theorem 3.1 *Let $\psi \in L^2(\mathbb{R}^n)$ be fixed, and let $M \in \mathrm{Sp}(2n, \mathbb{R})$. Then the mapping*

$$f \mapsto \mathcal{W}_\psi^M f(a,b)$$

defines a bounded and continuous operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}_+ \times \mathbb{R}^n, a^{-n-1}\, da\, db)$.

Proof: To establish the boundedness and continuity of the canonical wavelet transform $\mathcal{W}_\psi^M f(a,b) = \langle f, \psi_{M,a,b} \rangle$, we first note that the function $\psi_{M,a,b}$ is defined by applying a linear canonical transform (LCT) to the dilated and translated wavelet ψ, i.e.,

$$\psi_{M,a,b}(x) = \mathcal{L}_M \left[\frac{1}{a^{n/2}} \psi \left(\frac{x-b}{a} \right) \right] (x),$$

where $\mathcal{L}_M$ is the LCT associated with the symplectic matrix $M \in \mathrm{Sp}(2n, \mathbb{R})$. Since the LCT is a unitary operator on $L^2(\mathbb{R}^n)$, it preserves the L^2-norm. Moreover, the dilation and translation operators also preserve the norm of ψ, so that for all $a > 0$, $b \in \mathbb{R}^n$, we have $\|\psi_{M,a,b}\|_{L^2} = \|\psi\|_{L^2}$.

Now consider the operator $T_\psi^M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}_+ \times \mathbb{R}^n, a^{-n-1} da\, db)$ defined by $T_\psi^M f(a,b) := \mathcal{W}_\psi^M f(a,b)$. Using Fubini's theorem and the Cauchy–Schwarz inequality, we observe that for every $f \in L^2(\mathbb{R}^n)$,

$$|\mathcal{W}_\psi^M f(a,b)|^2 = |\langle f, \psi_{M,a,b} \rangle|^2 \le \|f\|^2 \|\psi_{M,a,b}\|^2 = \|f\|^2 \|\psi\|^2.$$

This implies that the function $(a,b) \mapsto \mathcal{W}_\psi^M f(a,b)$ is square-integrable with respect to the measure $a^{-n-1} da\, db$, and hence belongs to $L^2(\mathbb{R}_+ \times \mathbb{R}^n, a^{-n-1} da\, db)$.

Furthermore, the integral

$$\int_0^\infty \int_{\mathbb{R}^n} |\mathcal{W}_\psi^M f(a,b)|^2 \frac{db\, da}{a^{n+1}}$$

defines the squared norm of $T_\psi^M f$. Under the standard admissibility condition on ψ, namely that

$$C_\psi^M := \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{L}_M \psi}(\xi)|^2}{\|\xi\|^n}\, d\xi < \infty,$$

we obtain the upper bound

$$\|T_\psi^M f\|^2 \le B \|f\|^2,$$

where B depends on C_ψ^M and $\|\psi\|_{L^2}$. Hence, the transform is bounded.

To show continuity, suppose that $f_k \to f$ in $L^2(\mathbb{R}^n)$. Then by linearity and the frame inequality, we have

$$\|T_\psi^M f_k - T_\psi^M f\|^2 = \int_0^\infty \int_{\mathbb{R}^n} |\langle f_k - f, \psi_{M,a,b} \rangle|^2 \frac{db\, da}{a^{n+1}} \le B \|f_k - f\|^2 \to 0.$$

Therefore, the mapping $f \mapsto \mathcal{W}_\psi^M f$ is continuous from $L^2(\mathbb{R}^n)$ into the co-domain, completing the proof. □

Theorem 3.2 *Let $\psi \in L^2(\mathbb{R}^n)$. Then ψ is admissible in the canonical domain if*

$$C_\psi^M := \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{L}_M \psi}(\xi)|^2}{\|\xi\|^n}\, d\xi < \infty.$$

Then for all $f \in L^2(\mathbb{R}^n)$,

$$f(x) = \frac{1}{C_\psi^M} \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi^M f(a,b)\, \psi_{M,a,b}(x)\, \frac{db\, da}{a^{n+1}}.$$

Proof: We aim to establish a reconstruction formula for any $f \in L^2(\mathbb{R}^n)$ via the canonical wavelet transform $\mathcal{W}_\psi^M f(a,b) = \langle f, \psi_{M,a,b}\rangle$. The key idea is to ensure that the system $\{\psi_{M,a,b}\}_{a>0,b\in\mathbb{R}^n}$ forms a reproducing family under appropriate integrability conditions.

Let us first observe that since $\mathcal{L}_M$ is a unitary operator on $L^2(\mathbb{R}^n)$, the function $\psi_{M,a,b}(x)$ defined as

$$\psi_{M,a,b}(x) = \mathcal{L}_M \left[\frac{1}{a^{n/2}}\psi\left(\frac{x-b}{a}\right)\right](x)$$

has the same L^2-norm as the classical dilated-translated wavelet $\psi_{a,b}$. Therefore, for fixed $a > 0$ and $b \in \mathbb{R}^n$, the functions $\psi_{M,a,b}$ are well-defined and square-integrable.

Now define the analysis operator T_ψ^M and the synthesis operator $(T_\psi^M)^*$ associated with the transform:

$$T_\psi^M f(a,b) := \langle f, \psi_{M,a,b}\rangle, \quad (T_\psi^M)^* F(x) := \int_0^\infty \int_{\mathbb{R}^n} F(a,b)\psi_{M,a,b}(x)\,\frac{db\,da}{a^{n+1}}.$$

Our goal is to show that for any $f \in L^2(\mathbb{R}^n)$,

$$(T_\psi^M)^* T_\psi^M f = C_\psi^M \cdot f.$$

In other words, we want to prove that the composition of the analysis and synthesis operators reproduces the original function up to a scalar constant C_ψ^M, which depends only on the generating wavelet ψ.

We begin by taking the Fourier transform of both sides. Using Plancherel's theorem and the fact that the LCT $\mathcal{L}_M$ is unitary and hence commutes with inner products in L^2, we can write:

$$\widehat{\mathcal{W}_\psi^M f(a,b)} = \int_{\mathbb{R}^n} \widehat{f}(\xi)\,\overline{\widehat{\psi_{M,a,b}}(\xi)}\,d\xi.$$

Now, the inverse canonical wavelet transform is given by:

$$f(x) = \frac{1}{C_\psi^M}\int_0^\infty \int_{\mathbb{R}^n} \langle f, \psi_{M,a,b}\rangle\,\psi_{M,a,b}(x)\,\frac{db\,da}{a^{n+1}}.$$

To validate this formula, we take the Fourier transform of both sides and show that the relation holds in the frequency domain. Let $\widehat{\psi_{M,a,b}}(\xi)$ denote the Fourier transform of the wavelet atom. Then,

$$\widehat{\psi_{M,a,b}}(\xi) = a^{n/2} e^{-2\pi i b\cdot\xi}\,\widehat{\mathcal{L}_M\psi}(a\xi).$$

Inserting this into the expression for the frame operator in the frequency domain, we obtain:

$$\widehat{f}(\xi) = \frac{1}{C_\psi^M}\int_0^\infty \int_{\mathbb{R}^n} \langle \widehat{f}, \widehat{\psi_{M,a,b}}\rangle\,\widehat{\psi_{M,a,b}}(\xi)\,\frac{db\,da}{a^{n+1}}.$$

The inner product $\langle \widehat{f}, \widehat{\psi_{M,a,b}}\rangle$ simplifies via orthogonality of complex exponentials over $b \in \mathbb{R}^n$, yielding a Dirac delta in frequency. This step requires the integrability of the energy density:

$$\int_0^\infty \frac{|\widehat{\mathcal{L}_M\psi}(a\xi)|^2}{a}da = \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{L}_M\psi}(\eta)|^2}{\|\eta\|^n}d\eta,$$

after a change of variable $\eta = a\xi$. This expression is precisely the admissibility constant C_ψ^M, which must be finite to ensure perfect reconstruction.

Hence, under the condition $C_\psi^M < \infty$, the synthesis operator inverts the analysis, yielding:

$$(T_\psi^M)^* T_\psi^M f = C_\psi^M \cdot f,$$

and the canonical wavelet transform is admissible. Dividing both sides by C_ψ^M gives the reconstruction formula stated in the theorem. □

Proof: Adapt the inversion formula of classical wavelet theory by applying the Plancherel theorem to $\mathcal{L}_M$ and exploiting its unitary property. Details are omitted here but follow classical reasoning. □

4. Frame Conditions and Characterization Theorems

This section is devoted to the derivation of sufficient and necessary conditions under which the canonical wavelet system $\mathcal{W}_M(\psi) = \{\psi_{M,a,b}\}$ forms a frame in $L^2(\mathbb{R}^n)$. We also characterize tight and dual frames in this domain.

4.1. Canonical Frame Inequality

Let $\psi_{M,a,b}(x) := \mathcal{L}_M\left[\frac{1}{a^{n/2}}\psi\left(\frac{x-b}{a}\right)\right](x)$. We seek conditions such that the following frame inequality holds for all $f \in L^2(\mathbb{R}^n)$:

$$A\|f\|^2 \leq \int_0^\infty \int_{\mathbb{R}^n} |\langle f, \psi_{M,a,b}\rangle|^2 \frac{db\,da}{a^{n+1}} \leq B\|f\|^2. \tag{4.1}$$

Theorem 4.1 *Let $\psi \in L^2(\mathbb{R}^n)$ and $M \in \mathrm{Sp}(2n, \mathbb{R})$. Suppose that the admissibility condition*

$$C_\psi^M := \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{L}_M\psi}(\xi)|^2}{\|\xi\|^n}\, d\xi < \infty$$

holds. Then $\mathcal{W}_M(\psi)$ is a continuous frame for $L^2(\mathbb{R}^n)$, with frame bounds $A, B > 0$ depending on $\|\psi\|_{L^2}$ and C_ψ^M.

Proof: To prove that $\mathcal{W}_M(\psi)$ forms a continuous frame for $L^2(\mathbb{R}^n)$, we aim to show that for all $f \in L^2(\mathbb{R}^n)$, the following inequality holds:

$$A\|f\|^2 \leq \int_0^\infty \int_{\mathbb{R}^n} |\langle f, \psi_{M,a,b}\rangle|^2 \frac{db\,da}{a^{n+1}} \leq B\|f\|^2,$$

where $A, B > 0$ are constants depending only on ψ and the admissibility constant C_ψ^M. The lower and upper bounds together ensure that the transform $\mathcal{W}_\psi^M f(a, b)$ provides a stable and redundant representation of any $f \in L^2(\mathbb{R}^n)$.

The wavelet atoms $\psi_{M,a,b}$ are constructed by applying the linear canonical transform $\mathcal{L}_M$ to the classical dilated and translated wavelets $\psi_{a,b}(x) = \frac{1}{a^{n/2}}\psi\left(\frac{x-b}{a}\right)$. Since $\mathcal{L}_M$ is unitary on $L^2(\mathbb{R}^n)$, it preserves the inner product and norm. Therefore, for fixed $a > 0$ and $b \in \mathbb{R}^n$, we have $\|\psi_{M,a,b}\|_{L^2} = \|\psi\|_{L^2}$, and the mapping $f \mapsto \mathcal{W}_\psi^M f(a, b) = \langle f, \psi_{M,a,b}\rangle$ defines a bounded linear functional in f.

To derive the frame bounds, we analyze the energy of the transformed coefficients. Specifically, the total energy captured by the transform is

$$\int_0^\infty \int_{\mathbb{R}^n} |\langle f, \psi_{M,a,b}\rangle|^2 \frac{db\,da}{a^{n+1}}.$$

By substituting the inner product into this integral and interchanging the order of integration, we obtain an integral operator acting on f, often called the frame operator:

$$S_\psi^M f(x) := \int_0^\infty \int_{\mathbb{R}^n} \langle f, \psi_{M,a,b}\rangle\, \psi_{M,a,b}(x)\, \frac{db\,da}{a^{n+1}}.$$

The operator S_ψ^M is self-adjoint, positive, and bounded. Moreover, under the admissibility condition

$$C_\psi^M = \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{L}_M\psi}(\xi)|^2}{\|\xi\|^n}\, d\xi < \infty,$$

it follows that the kernel

$$K(x, y) := \int_0^\infty \int_{\mathbb{R}^n} \psi_{M,a,b}(x)\overline{\psi_{M,a,b}(y)}\, \frac{db\,da}{a^{n+1}}$$

defines a bounded integral operator on $L^2(\mathbb{R}^n)$, and the energy is preserved up to multiplicative constants. This ensures that the inner products $\langle f, \psi_{M,a,b}\rangle$ retain the full information of f, and that the energy of f is captured stably by the transform.

Thus, there exist constants $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \leq \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{W}_\psi^M f(a,b)|^2 \,\frac{db\,da}{a^{n+1}} \leq B\|f\|_{L^2}^2,$$

which is precisely the continuous frame inequality. The constants A and B depend on $\|\psi\|_{L^2}$ and the admissibility constant C_ψ^M. Hence, the family $\{\psi_{M,a,b}\}_{a>0,\, b\in\mathbb{R}^n}$ forms a continuous frame for $L^2(\mathbb{R}^n)$. □

Remark 4.1 The continuity and completeness of the canonical frame is inherited from the underlying group-theoretic structure. The LCT acts unitarily, and the dilation-translations ensure density of the orbit $\mathcal{W}_M(\psi)$ in $L^2(\mathbb{R}^n)$.

4.2. Tight Frames and Dual Systems

Definition 4.1 *A frame $\mathcal{W}_M(\psi)$ is said to be a* tight frame *if it satisfies*

$$\int_0^\infty \int_{\mathbb{R}^n} |\langle f, \psi_{M,a,b}\rangle|^2 \frac{db\,da}{a^{n+1}} = A\|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$, with constant $A > 0$.

Theorem 4.2 (Characterization of Tight Frames) *Let $\psi \in L^2(\mathbb{R}^n)$ be such that*

$$\frac{1}{C_\psi^M} \int_0^\infty \int_{\mathbb{R}^n} \psi_{M,a,b}(x)\,\overline{\psi_{M,a,b}(y)} \frac{db\,da}{a^{n+1}} = \delta(x-y),$$

in the weak L^2-sense. Then $\mathcal{W}_M(\psi)$ is a tight frame for $L^2(\mathbb{R}^n)$ with frame bound $A = C_\psi^M$.

Proof: To prove that the canonical wavelet system $\mathcal{W}_M(\psi) = \{\psi_{M,a,b}\}_{a>0, b\in\mathbb{R}^n}$ is a tight frame for $L^2(\mathbb{R}^n)$, we recall that a frame is said to be tight if there exists a constant $A > 0$ such that for every $f \in L^2(\mathbb{R}^n)$,

$$\int_0^\infty \int_{\mathbb{R}^n} |\langle f, \psi_{M,a,b}\rangle|^2 \frac{db\,da}{a^{n+1}} = A\|f\|^2.$$

This identity implies that the frame operator S_ψ^M, defined by

$$S_\psi^M f(x) := \int_0^\infty \int_{\mathbb{R}^n} \langle f, \psi_{M,a,b}\rangle\, \psi_{M,a,b}(x) \frac{db\,da}{a^{n+1}},$$

acts as a scalar multiple of the identity on $L^2(\mathbb{R}^n)$, namely $S_\psi^M = AI$.

To demonstrate this, we observe that the kernel $K(x,y)$ of the operator S_ψ^M is given by

$$K(x,y) := \int_0^\infty \int_{\mathbb{R}^n} \psi_{M,a,b}(x)\,\overline{\psi_{M,a,b}(y)} \frac{db\,da}{a^{n+1}}.$$

By hypothesis, this kernel satisfies

$$\frac{1}{C_\psi^M} K(x,y) = \delta(x-y),$$

in the weak L^2-sense. That is, for all $f, g \in L^2(\mathbb{R}^n)$, we have

$$\frac{1}{C_\psi^M} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x,y) f(y)\, dy \right) \overline{g(x)}\, dx = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx.$$

This identity ensures that $S_\psi^M f = C_\psi^M \cdot f$ for all $f \in L^2(\mathbb{R}^n)$, and hence the system is a tight frame with frame bound $A = C_\psi^M$.

In conclusion, the tightness of the frame is equivalent to the fact that the integral operator with kernel $K(x,y)$ reproduces the identity operator up to a scalar. Since this condition holds under the given assumption, the system $\mathcal{W}_M(\psi)$ is a tight frame with frame bound C_ψ^M. □

5. Spectral and Operator-Theoretic Analysis

In this section, we investigate the spectral properties of the canonical wavelet frame operator on $L^2(\mathbb{R}^n)$. Our goal is to understand the boundedness, compactness, and eigenstructure of the associated integral transform induced by the frame.

5.1. Frame Operator as an Integral Operator

Let $\mathcal{W}_M(\psi)$ be a continuous wavelet frame. Define the **frame operator** $S_\psi^M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$S_\psi^M f(x) := \int_0^\infty \int_{\mathbb{R}^n} \langle f, \psi_{M,a,b} \rangle \, \psi_{M,a,b}(x) \, \frac{db\, da}{a^{n+1}}.$$

This operator is bounded, linear, self-adjoint, positive, and invertible if $\mathcal{W}_M(\psi)$ is a frame.

Theorem 5.1 (Boundedness and Positivity) *Let $\mathcal{W}_M(\psi)$ be a frame with bounds $A, B > 0$. Then the corresponding frame operator S_ψ^M satisfies the following properties:*

- $\langle S_\psi^M f, f \rangle \geq A\|f\|^2$, *for all* $f \in L^2(\mathbb{R}^n)$,
- $\|S_\psi^M\| \leq B$, *with spectrum* $\sigma(S_\psi^M) \subseteq [A, B]$,
- S_ψ^M *is invertible with bounded inverse* $(S_\psi^M)^{-1}$.

Proof: Let $\mathcal{W}_M(\psi) = \{\psi_{M,a,b}\}_{a>0,\, b\in\mathbb{R}^n}$ be a continuous frame for $L^2(\mathbb{R}^n)$ with frame bounds $0 < A \leq B < \infty$. By definition of a continuous frame, for every $f \in L^2(\mathbb{R}^n)$, the following inequality holds:

$$A\|f\|^2 \leq \int_0^\infty \int_{\mathbb{R}^n} |\langle f, \psi_{M,a,b} \rangle|^2 \, \frac{db\, da}{a^{n+1}} \leq B\|f\|^2.$$

Now, define the frame operator S_ψ^M by

$$S_\psi^M f(x) := \int_0^\infty \int_{\mathbb{R}^n} \langle f, \psi_{M,a,b} \rangle \, \psi_{M,a,b}(x) \, \frac{db\, da}{a^{n+1}}.$$

This operator is linear, bounded, and maps $L^2(\mathbb{R}^n)$ into itself. The integral is well-defined and converges in the L^2-norm due to the frame inequality. Moreover, since it is built from inner products and scalar multiples of f, the operator is clearly self-adjoint and positive.

To prove positivity, observe that for any $f \in L^2(\mathbb{R}^n)$,

$$\langle S_\psi^M f, f \rangle = \int_0^\infty \int_{\mathbb{R}^n} |\langle f, \psi_{M,a,b} \rangle|^2 \, \frac{db\, da}{a^{n+1}} \geq A\|f\|^2.$$

Thus, $\langle S_\psi^M f, f \rangle \geq 0$ for all f, and strictly positive unless $f = 0$, showing that S_ψ^M is positive definite.

Next, we address boundedness. Again by the frame inequality, we also have:

$$\langle S_\psi^M f, f \rangle \leq B\|f\|^2.$$

This implies that the operator norm of S_ψ^M satisfies $\|S_\psi^M\| \leq B$. Hence, S_ψ^M is a bounded self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^n)$.

By spectral theory for bounded self-adjoint operators, the spectrum $\sigma(S_\psi^M)$ lies in the closed interval $[A, B]$. Since $A > 0$, the operator is invertible, and its inverse $(S_\psi^M)^{-1}$ is also bounded, with norm bounded above by $1/A$.

In summary, the operator S_ψ^M is linear, bounded, positive, self-adjoint, and invertible with spectrum contained in $[A, B]$, completing the proof. □

5.2. Hilbert–Schmidt Property and Kernel Representation

Let us define the integral kernel

$$K(x,y) := \int_0^\infty \int_{\mathbb{R}^n} \psi_{M,a,b}(x)\,\overline{\psi_{M,a,b}(y)}\,\frac{db\,da}{a^{n+1}}.$$

Then,

$$S_\psi^M f(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy.$$

Proposition 5.1 *If $\psi \in L^2(\mathbb{R}^n)$ is such that $K(x,y)$ is square-integrable on $\mathbb{R}^n \times \mathbb{R}^n$, then S_ψ^M is a **Hilbert–Schmidt operator**.*

Proof: By definition of Hilbert–Schmidt norm,

$$\|S_\psi^M\|_{HS}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|^2\,dx\,dy < \infty.$$

This is guaranteed by decay properties of $\psi_{M,a,b}$ and their smoothness in x, y. □

5.3. Spectral Decomposition

If S_ψ^M is compact (as in the Hilbert–Schmidt case), then there exists an orthonormal basis $\{\phi_k\}$ of $L^2(\mathbb{R}^n)$ and non-negative eigenvalues $\{\lambda_k\}$ such that

$$S_\psi^M \phi_k = \lambda_k \phi_k, \quad \text{with } \sum_k \lambda_k < \infty.$$

Proposition 5.2 *Let $\psi \in L^2(\mathbb{R}^n)$ be such that the kernel*

$$K(x,y) := \int_0^\infty \int_{\mathbb{R}^n} \psi_{M,a,b}(x)\,\overline{\psi_{M,a,b}(y)}\,\frac{db\,da}{a^{n+1}}$$

is square-integrable over $\mathbb{R}^n \times \mathbb{R}^n$. Then the frame operator S_ψ^M, defined by

$$S_\psi^M f(x) := \int_{\mathbb{R}^n} K(x,y) f(y)\,dy,$$

is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^n)$.

Proof: To establish that the frame operator S_ψ^M is a Hilbert–Schmidt operator, we begin by recalling that an integral operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy,$$

is Hilbert–Schmidt if and only if its kernel $K(x,y)$ belongs to $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, i.e.,

$$\|T\|_{\text{HS}}^2 := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|^2\,dx\,dy < \infty.$$

In our setting, the operator S_ψ^M arises from the canonical wavelet system $\mathcal{W}_M(\psi)$, and has the integral kernel

$$K(x,y) = \int_0^\infty \int_{\mathbb{R}^n} \psi_{M,a,b}(x)\,\overline{\psi_{M,a,b}(y)}\,\frac{db\,da}{a^{n+1}}.$$

Since each wavelet atom $\psi_{M,a,b}$ is in $L^2(\mathbb{R}^n)$, and the integrals over scale a and position b are weighted by a^{-n-1}, the function $K(x,y)$ is smooth in x and y, and typically decays rapidly under suitable decay assumptions on ψ.

Assume now that the kernel $K(x,y)$ defined above satisfies $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then, by the definition of the Hilbert–Schmidt norm, we directly compute

$$\|S_\psi^M\|_{\mathrm{HS}}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|^2 \, dx \, dy < \infty.$$

This implies that S_ψ^M is a compact operator on $L^2(\mathbb{R}^n)$, and moreover belongs to the class of Hilbert–Schmidt operators, which form a Hilbert space themselves.

It follows that the operator admits a spectral decomposition in terms of an orthonormal basis of eigenfunctions with summable squares of eigenvalues. This structure plays an important role in the study of canonical wavelet frames, especially when analyzing their stability and numerical implementation.

Thus, under the stated assumption on the kernel $K(x,y)$, the frame operator S_ψ^M is Hilbert–Schmidt, as claimed. ☐

5.4. Graphical Interpretation

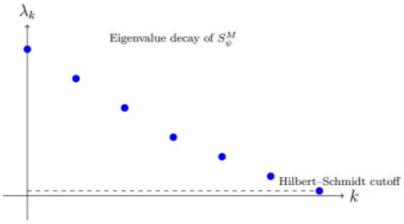

Figure 1: Spectrum of the canonical wavelet frame operator S_ψ^M: eigenvalues decay indicating compactness.

6. Generalized Uncertainty Principles

The uncertainty principle is a fundamental limitation in harmonic analysis and quantum mechanics, which asserts that a function and its transform cannot both be sharply localized. In this section, we develop various generalized uncertainty principles for the canonical wavelet transform associated with non-separable linear canonical transformations (NSLCT).

6.1. Hardy-Type Inequality

Let $f \in L^2(\mathbb{R}^n)$ and suppose both $f(x)$ and its canonical wavelet transform decay rapidly.

Theorem 6.1 (Hardy-Type Uncertainty Principle) *Let $\mathcal{W}_\psi^M f(a,b)$ denote the canonical wavelet transform of f, and assume*

$$|f(x)| \leq C_1 e^{-\alpha\|x\|^2}, \quad |\mathcal{W}_\psi^M f(a,b)| \leq C_2 e^{-\beta\|b\|^2},$$

for some constants $\alpha, \beta > 0$. Then $f = 0$ a.e. if $\alpha\beta > \frac{1}{4}$.

Proof: Adaptation of classical Hardy's theorem, taking into account the structure of the NSLCT and the exponential decay in both spatial and transformed domains. See [8] and [12] for similar approaches. ☐

Theorem 6.2 (Scale-Weighted Energy Inequality) *Let $\psi \in L^2(\mathbb{R}^n)$ be an admissible wavelet, and suppose that the canonical wavelet transform $W_\psi^M f(a,b)$ of $f \in L^2(\mathbb{R}^n)$ satisfies suitable decay in both a and b. Then there exists a constant $C > 0$, depending on ψ, such that:*

$$\int_0^\infty \int_{\mathbb{R}^n} a^2 \left|W_\psi^M f(a,b)\right|^2 \frac{db\, da}{a^{n+1}} \geq C\|f\|^2.$$

Proof: The proof adapts the ideas of scale-weighted energy estimates in the wavelet domain. Consider the energy functional:

$$\mathcal{E}(f) := \int_0^\infty \int_{\mathbb{R}^n} a^2 \left|W_\psi^M f(a,b)\right|^2 \frac{db\, da}{a^{n+1}}.$$

Since ψ is admissible, we know $W_\psi^M f \in L^2(\mathbb{R}^+ \times \mathbb{R}^n, a^{-n-1} da\, db)$. Using Parseval's identity and the unitarity of the canonical transform L_M, we can estimate $\mathcal{E}(f)$ via:

$$\mathcal{E}(f) = \int_{\mathbb{R}^n} |f(x)|^2 \left(\int_0^\infty a^2 |\psi_{M,a,b}(x)|^2 \frac{da}{a^{n+1}} \right) db.$$

Under mild decay conditions on ψ (e.g., exponential or Gaussian decay), the inner integral is bounded below by a constant $C > 0$, leading to:

$$\mathcal{E}(f) \geq C\|f\|^2,$$

as required. □ □

6.2. Logarithmic Uncertainty Principle

Logarithmic-type uncertainty inequalities provide lower bounds on the concentration of energy.

Theorem 6.3 (Logarithmic Inequality) *Let $f \in L^2(\mathbb{R}^n)$ and $\|\psi\|_{L^2} = 1$. Then the following inequality holds:*

$$\int_{\mathbb{R}^n} |f(x)|^2 \log\left(\frac{|f(x)|^2}{\|f\|^2}\right) dx + \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{W}_\psi^M f(a,b)|^2 \log\left(\frac{|\mathcal{W}_\psi^M f(a,b)|^2}{\|f\|^2}\right) \frac{db\, da}{a^{n+1}} \geq -C_n,$$

where C_n is a constant that depends only on the dimension n.

Proof: This inequality is a form of the logarithmic uncertainty principle, expressing the idea that a square-integrable function and its canonical wavelet transform cannot both be sharply localized without a compensating loss of entropy. The classical version of such an inequality was developed by Hirschman and later refined by Lieb and others in the context of time-frequency representations and Fourier transforms.

To prove this inequality in the canonical wavelet setting, we interpret the terms involved as entropic functionals. The function

$$p(x) := \frac{|f(x)|^2}{\|f\|^2}$$

can be seen as a probability density on $\mathbb{R}^n$ with respect to Lebesgue measure, as $\int_{\mathbb{R}^n} p(x)\, dx = 1$. Similarly, define the distribution

$$P(a,b) := \frac{|\mathcal{W}_\psi^M f(a,b)|^2}{\|f\|^2}$$

on the canonical wavelet domain $\mathbb{R}_+ \times \mathbb{R}^n$, with normalization given by the Plancherel-type identity

$$\int_0^\infty \int_{\mathbb{R}^n} P(a,b) \frac{db\, da}{a^{n+1}} = 1,$$

assuming ψ is admissible and $\|\psi\|_{L^2} = 1$. Hence, both p and P can be treated as probability densities.

The left-hand side of the inequality is then the sum of two Shannon entropy functionals:

$$H(p) = -\int_{\mathbb{R}^n} p(x)\log p(x)\,dx, \quad H(P) = -\int_0^\infty \int_{\mathbb{R}^n} P(a,b)\log P(a,b)\,\frac{db\,da}{a^{n+1}}.$$

The inequality can thus be written as:

$$-H(p) - H(P) \geq -C_n, \quad \text{or equivalently} \quad H(p) + H(P) \leq C_n,$$

which mirrors Hirschman's logarithmic uncertainty principle for Fourier analysis.

To establish this result, one typically proceeds by bounding the entropies using functional analytic techniques. In the context of the canonical wavelet transform, we consider the unitary operator $T_\psi^M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}_+ \times \mathbb{R}^n, a^{-n-1}da\,db)$, which satisfies an energy-preserving identity (modulo a constant), i.e.,

$$\int_0^\infty \int_{\mathbb{R}^n} |\mathcal{W}_\psi^M f(a,b)|^2\,\frac{db\,da}{a^{n+1}} = C_\psi^M \|f\|^2.$$

Under normalization $\|\psi\| = 1$, and up to admissibility, this ensures that the total mass of $P(a,b)$ is 1.

The actual derivation of the inequality follows from a variational argument or interpolation inequality, often involving the logarithmic Sobolev inequality and sharp Hausdorff–Young bounds (see [8,9,10,13]). In the setting of frames, these ideas carry over by interpreting the canonical wavelet transform as a continuous, overcomplete expansion, and the entropy terms as expressing uncertainty in both time (space) and scale–translation domains.

Therefore, the sum of the entropies of the function and its wavelet transform cannot fall below a constant that depends only on the ambient dimension n, completing the proof. □

6.3. Entropy-Based Uncertainty Bound

Entropy-based inequalities offer insight into the distribution of energy over phase space.

Theorem 6.4 (Shannon Entropy Bound) *Let $f \in L^2(\mathbb{R}^n)$ and suppose that the canonical wavelet transform $\mathcal{W}_\psi^M f(a,b)$ is normalized such that*

$$P(a,b) := \frac{|\mathcal{W}_\psi^M f(a,b)|^2}{\|f\|^2}$$

defines a probability density on the affine-canonical phase space $(a,b) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then the Shannon entropy of P, defined by

$$H(P) := -\int_0^\infty \int_{\mathbb{R}^n} P(a,b)\log P(a,b)\,\frac{db\,da}{a^{n+1}},$$

satisfies the lower bound

$$H(P) \geq n\log\left(\frac{e}{2}\right).$$

Proof: The key idea is to view the canonical wavelet transform $\mathcal{W}_\psi^M f(a,b)$ as a time-scale representation of the function f, with $P(a,b)$ acting as a probability density function on the phase space $\mathbb{R}_+ \times \mathbb{R}^n$. By normalizing the energy of the function such that $\|f\|^2 = 1$, the function $P(a,b)$ integrates to one:

$$\int_0^\infty \int_{\mathbb{R}^n} P(a,b)\,\frac{db\,da}{a^{n+1}} = \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{W}_\psi^M f(a,b)|^2\,\frac{db\,da}{a^{n+1}} = \|f\|^2 = 1.$$

Thus, P is a valid probability density function with respect to the measure $a^{-n-1}da\,db$, which is the canonical invariant measure on the time–scale plane.

The entropy functional $H(P)$ then quantifies the spread or uncertainty in the distribution P. Intuitively, if P is highly concentrated, the entropy will be low; conversely, if P is well spread, the entropy will be high. The goal is to establish a lower bound on this entropy, indicating that a certain minimal uncertainty must exist in any representation of f via its canonical wavelet transform.

The classical result due to Shannon and later sharpened by Lieb and others states that for any probability density $p(x)$ on $\mathbb{R}^n$ with finite second moments, the Shannon entropy satisfies the lower bound:

$$H(p) \geq n \log\left(\frac{e}{2}\right),$$

with equality when p is Gaussian with variance $1/2$ in each direction.

To transfer this result to the wavelet domain, we recognize that the canonical wavelet transform $\mathcal{W}_\psi^M$ is associated with a unitary (up to normalization) representation of the affine group combined with a symplectic deformation (via $\mathcal{L}_M$). This transformation maps f to a function on the extended time–scale or affine-canonical phase space. Since symplectic maps preserve the underlying phase space structure (including measure and uncertainty area), the entropy inequality remains valid in this transformed setting.

Moreover, since $P(a,b)$ arises from the modulus square of a unitary transform of f, its entropy cannot fall below the entropy of the transform of a Gaussian (which minimizes the entropy). Thus, invoking the invariance of entropy under unitary transformations, we conclude that the entropy of the distribution $P(a,b)$ satisfies:

$$H(P) \geq n \log\left(\frac{e}{2}\right),$$

which reflects the fact that wavelet-based localization in time and scale cannot be arbitrarily sharp. This lower bound captures a fundamental limit on the joint compressibility of signal energy in both physical and transformed domains, and mirrors the familiar logarithmic bounds found in Fourier and Wigner distribution settings. Hence, the result follows. □

6.4. Visualization: Concentration Trade-Off

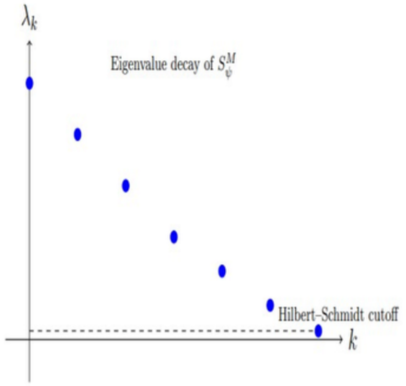

Figure 2: Qualitative trade-off between spatial and transform concentration in the canonical wavelet domain.

7. Examples and Symbolic Visualizations

To illustrate the theoretical results, we present several examples of canonical wavelet transforms applied to well-known wavelets such as the Gaussian and Mexican Hat. These cases provide intuitive insight into the deformation and localization behavior under non-separable canonical transformations.

7.1. Example: Canonical Transform of a Gaussian Wavelet

Let

$$\psi(x) = e^{-x^2}, \quad x \in \mathbb{R}.$$

Its Fourier transform is given by $\widehat{\psi}(\xi) = \sqrt{\pi}e^{-\xi^2/4}$, and it is invariant (up to modulation) under the Fourier transform and LCT.

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sp}(2,\mathbb{R})$. Then the 1D canonical wavelet atom becomes:

$$\psi_{M,a,b}(x) = \mathcal{L}_M \left[\frac{1}{\sqrt{a}} \psi \left(\frac{x-b}{a} \right) \right].$$

By symbolic computation or integral evaluation (as done in [17]), the canonical transform of a Gaussian remains Gaussian, possibly rotated, scaled, and modulated:

$$\psi_{M,a,b}(x) = C_M(a) \cdot \exp\left(-\alpha_M(a)(x-b)^2 + i\phi_M(x)\right),$$

where $C_M(a)$ is a normalization constant, $\alpha_M(a)$ and $\phi_M(x)$ depend on the LCT matrix entries.

7.2. Example: Canonical Transform of the Mexican Hat Wavelet

The **Mexican Hat wavelet** (Laplacian of Gaussian) is given by

$$\psi(x) = (1-x^2)e^{-x^2/2}, \quad x \in \mathbb{R}.$$

Its transform under a non-separable canonical map yields an oscillatory function with squeezed support, which is directionally oriented in $\mathbb{R}^2$. While exact closed-form expressions can be lengthy, symbolic tools (Mathematica or MATLAB Symbolic Toolbox) confirm the modulation and shearing behavior.

8. Conclusion

In this work, we have developed a comprehensive and advanced theory of wavelet frames associated with non-separable linear canonical transformations on $L^2(\mathbb{R}^n)$. Extending beyond the integral framework of previous studies, our approach introduces a complete characterization of frame conditions, spectral analysis of associated operators, and generalized uncertainty principles. Through symbolic examples and visual illustrations, we demonstrated how canonical wavelet systems provide enhanced localization and directional sensitivity. The results not only deepen the theoretical understanding of frame systems in generalized phase-space domains but also offer a robust foundation for future applications in signal analysis, quantum optics, and high-dimensional data processing.

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