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Closed-Form Solutions for 12th-Order Nonlinear Rational Recursive Difference Equations

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ABSTRACT: This study presents direct computations of numerous closed-form solutions for various rational recursive problems. Several results related to the ensuing rational recursive sequences are examined in this article:

$$\eta_{n+1} = \frac{\eta_{n-11}}{\pm 1 \pm \eta_{n-1}\eta_{n-3}\eta_{n-5}\eta_{n-7}\eta_{n-9}\eta_{n-11}}, \quad n = 0, 1, 2, \ldots,$$

where the initial conditions are arbitrary real numbers. A universal closed-form solution for nonlinear rational recursive difference equations is unlikely because of their inherent specificity and complexity; only certain forms can be solved analytically. Researchers therefore focus on particular subclasses of these equations, often transforming them into more tractable linear forms through appropriate substitutions. By analyzing the structure of a given equation and applying the necessary transformations, we can identify a solvable class of nonlinear difference equations and obtain a closed-form solution.

Key Words: Nonlinear rational difference equations, order-twelve difference equations, closed-form solutions, high-order difference equations.

Contents

1. Introduction

This paper investigates the characteristics and stability of rational recursive sequences, providing new insights into their dynamic behavior:

$$\eta_{n+1} = \frac{\eta_{n-11}}{\pm 1 \pm \eta_{n-1}\eta_{n-3}\eta_{n-5}\eta_{n-7}\eta_{n-9}\eta_{n-11}}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

where the initial conditions are arbitrary real numbers.

Over the last decade, research on difference equations has experienced consistent growth, largely due to their role as mathematical models for real-world phenomena across diverse disciplines such as probability theory, queuing systems, statistical analysis, stochastic time series, combinatorics, number theory, geometry, electrical engineering, radiation studies, biological genetics, economics, psychology, and sociology. Today, difference equations form a cornerstone of applied analysis and are expected to continue contributing profoundly to the broader field of mathematics.

Recently, the study of the qualitative properties of rational difference equations has received considerable attention. Comprehensive examinations of both rational and non-rational difference equations can be found in [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33, 34,35,36,37,38,39,40,41,42,43,44] and the references cited therein. Investigating rational difference equations of order higher than one presents both significant challenges and valuable insights, as key results on their global dynamics often serve as foundational examples in extending the general theory of non-linear

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difference equations of the same order. However, no effective general methods have yet been found to address the global behavior of such equations. Accordingly, extending the study of rational difference equations of order exceeding one is both important and timely.

Abdelrahman and Moaaz [1] obtained the solution and investigated the global behavior of the nonlinear rational difference equation

$$\eta_{n+1} = a\eta_{n-k} + \frac{b\eta_{n-k}}{\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i\neq j}^{k} \eta_{n-i}}.$$

Oğul and Şimşek [2] analyzed a nonlinear rational difference equation of order thirty

$$\eta_{n+1} = \frac{\eta_{n-29}}{\pm 1 \pm \eta_{n-5}\eta_{n-11}\eta_{n-17}\eta_{n-23}\eta_{n-29}}, \quad n = 0, 1, 2, \ldots$$

with arbitrary non-zero initial conditions.

Aljoufi et al. [3] investigated the eighteenth-order rational difference equation

$$\eta_{n+1} = \frac{\eta_{n-17}}{\pm 1 \pm \eta_{n-2}\eta_{n-5}\eta_{n-8}\eta_{n-11}\eta_{n-14}\eta_{n-17}}, \quad n = 0, 1, 2, \ldots$$

where the initial conditions are arbitrary real numbers.

Şimşek et al. [4] studied the following fifteenth-order rational difference equation:

$$\eta_{n+1} = \frac{\eta_{n-2}\eta_{n-8}\eta_{n-14}}{\pm \eta_{n-5}\eta_{n-11} \pm \eta_{n-2}\eta_{n-5}\eta_{n-8}\eta_{n-11}\eta_{n-14}},$$

where the initial values are arbitrary positive real numbers.

Aloqeili [5] obtained the solutions of the difference equation

$$\eta_{n+1} = \frac{\eta_{n-1}}{a - \eta_n\eta_{n-1}}.$$

Çinar [6,7,8] investigated the solutions of the following difference equations

$$\eta_{n+1} = \frac{\eta_{n-1}}{1 + a\eta_n\eta_{n-1}}, \quad \eta_{n+1} = \frac{\eta_{n-1}}{-1 + a\eta_n\eta_{n-1}}, \quad \eta_{n+1} = \frac{a\eta_{n-1}}{1 + b\eta_n\eta_{n-1}}.$$

Karataş et al. [9] obtained the form of the solution of the difference equation

$$\eta_{n+1} = \frac{\eta_{n-5}}{1 + \eta_{n-2}\eta_{n-5}}.$$

Here, we recall some notations and results which will be useful in our investigation. Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions

$$\eta_{-k}, \eta_{-k+1}, \eta_{-k+2}, \ldots, \eta_0 \in I,$$

the difference equation

$$\eta_{n+1} = f\left(\eta_n, \eta_{n-1}, \ldots, \eta_{n-k}\right), \quad n = 0, 1, 2, \ldots \tag{1.2}$$

has a unique solution $\{\eta_n\}_{n=-k}^{\infty}$.

Definition 1.1 *(Equilibrium Point) A point $\bar{\eta} \in I$ is called an equilibrium point of Equation (1.2) if*

$$\bar{\eta} = f(\bar{\eta}, \bar{\eta}, \ldots, \bar{\eta}).$$

That is, $\eta_n = \bar{\eta}$ for $n \geq 0$, is a solution of Eq. (1.2), or equivalently, $\bar{\eta}$ is a fixed point of f.

Definition 1.2 *(Stability)*

- *The equilibrium point $\bar{\eta}$ of Eq. (1.2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\eta_{-k}, \eta_{-k+1}, \eta_{-k+2}, \ldots, \eta_0 \in I$, with*

  $$|\eta_{-k} - \bar{\eta}| + |\eta_{-k+1} - \bar{\eta}| + |\eta_{-k+2} - \bar{\eta}| + \cdots + |\eta_0 - \bar{\eta}| < \delta,$$

  we have

  $$|\eta_n - \bar{\eta}| < \varepsilon, \quad \text{for all } n \geq -k.$$

- *The equilibrium point $\bar{\eta}$ of Eq. (1.2) is said to be locally asymptotically stable if it constitutes a locally stable solution and there exists a constant $\gamma > 0$ such that*

  $$|\eta_{-k} - \bar{\eta}| + |\eta_{-k+1} - \bar{\eta}| + |\eta_{-k+2} - \bar{\eta}| + \cdots + |\eta_0 - \bar{\eta}| < \delta$$

  holds for all $\eta_{-k}, \eta_{-k+1}, \eta_{-k+2}, \ldots, \eta_0 \in I$. Under these conditions, we have

  $$\lim_{n \to \infty} \eta_n = \bar{\eta}.$$

- *The equilibrium point $\bar{\eta}$ of Eq. (1.2) is a global attractor if for all $\eta_{-k}, \eta_{-k+1}, \ldots, \eta_0 \in I$ we have*

  $$\lim_{n \to \infty} \eta_n = \bar{\eta}.$$

- *An equilibrium point $\bar{\eta}$ of Eq. (1.2) is considered globally asymptotically stable if it is locally stable and simultaneously serves as a global attractor for the equation.*

- *The equilibrium point $\bar{\eta}$ of Eq. (1.2) is deemed unstable if it fails to exhibit local stability.*

The linearization of Eq. (1.2) around the equilibrium point $\bar{\eta}$ yields the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{\eta}, \bar{\eta}, \ldots, \bar{\eta})}{\partial \eta_{n-i}} y_{n-i}.$$

Theorem 1.1 *Let $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$. The inequality*

$$|p| + |q| < 1,$$

provides a sufficient criterion for the asymptotic stability of the difference equation

$$\eta_{n+1} + p\eta_n + q\eta_{n-k} = 0, \quad n = 0, 1, 2, \ldots.$$

Remark 1.1 This theorem can be directly extended to encompass more general linear equations of the form:

$$\eta_{n+k} + p_1 \eta_{n+k-1} + \cdots + p_k \eta_n = 0, \quad n = 0, 1, 2, \ldots \tag{1.3}$$

where $p_1, p_2, \ldots, p_k \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$. Eq. (1.3) is asymptotically stable if the following condition holds:

$$\sum_{i=0}^{k} |p_i| < 1.$$

Definition 1.3 *(Periodicity) A sequence $\{\eta_n\}_{n=-k}^{\infty}$ is considered periodic with period p if $\eta_{n+p} = \eta_n$ for all $n \geq -k$.*

2. First Case

We introduce a particular representation for the solutions of the rational recursive sequences presented below:

$$\eta_{n+1} = \frac{\eta_{n-11}}{1+\eta_{n-1}\eta_{n-3}\eta_{n-5}\eta_{n-7}\eta_{n-9}\eta_{n-11}}, \quad n=0,1,2,\ldots, \tag{2.1}$$

with initial conditions taken as arbitrary real numbers.

Theorem 2.1 *Assume that $\{\eta_n\}_{n=-11}^{\infty}$ is a solution of Eq. (2.1). Then for $n=0,1,2,\ldots,$ the following holds:*

$$\eta_{12n-11} = \frac{a\prod_{m=0}^{n-1}(1+6m\,kigeca)}{\prod_{m=0}^{n-1}(1+(6m+1)\,kigeca)}, \qquad \eta_{12n-10} = \frac{b\prod_{m=0}^{n-1}(1+6m\,ljhfdb)}{\prod_{m=0}^{n-1}(1+(6m+1)\,ljhfdb)},$$

$$\eta_{12n-9} = \frac{c\prod_{m=0}^{n-1}(1+(6m+1)\,kigeca)}{\prod_{m=0}^{n-1}(1+(6m+2)\,kigeca)}, \qquad \eta_{12n-8} = \frac{d\prod_{m=0}^{n-1}(1+(6m+1)\,ljhfdb)}{\prod_{m=0}^{n-1}(1+(6m+2)\,ljhfdb)},$$

$$\eta_{12n-7} = \frac{e\prod_{m=0}^{n-1}(1+(6m+2)\,kigeca)}{\prod_{m=0}^{n-1}(1+(6m+3)\,kigeca)}, \qquad \eta_{12n-6} = \frac{f\prod_{m=0}^{n-1}(1+(6m+2)\,ljhfdb)}{\prod_{m=0}^{n-1}(1+(6m+3)\,ljhfdb)},$$

$$\eta_{12n-5} = \frac{g\prod_{m=0}^{n-1}(1+(6m+3)\,kigeca)}{\prod_{m=0}^{n-1}(1+(6m+4)\,kigeca)}, \qquad \eta_{12n-4} = \frac{h\prod_{m=0}^{n-1}(1+(6m+3)\,ljhfdb)}{\prod_{m=0}^{n-1}(1+(6m+4)\,ljhfdb)},$$

$$\eta_{12n-3} = \frac{i\prod_{m=0}^{n-1}(1+(6m+4)\,kigeca)}{\prod_{m=0}^{n-1}(1+(6m+5)\,kigeca)}, \qquad \eta_{12n-2} = \frac{j\prod_{m=0}^{n-1}(1+(6m+4)\,ljhfdb)}{\prod_{m=0}^{n-1}(1+(6m+5)\,ljhfdb)},$$

$$\eta_{12n-1} = \frac{k\prod_{m=0}^{n-1}(1+(6m+5)\,kigeca)}{\prod_{m=0}^{n-1}(1+(6m+6)\,kigeca)}, \qquad \eta_{12n} = \frac{l\prod_{m=0}^{n-1}(1+(6m+5)\,ljhfdb)}{\prod_{m=0}^{n-1}(1+(6m+6)\,ljhfdb)},$$

where $\eta_{-11}=a$, $\eta_{-10}=b$, $\eta_{-9}=c$, $\eta_{-8}=d$, $\eta_{-7}=e$, $\eta_{-6}=f$, $\eta_{-5}=g$, $\eta_{-4}=h$, $\eta_{-3}=i$, $\eta_{-2}=j$, $\eta_{-1}=k$, $\eta_0=l$, and $\prod_{m=0}^{-1}A_m=1$.

Proof: We apply induction to prove the result for this rational recursive sequence. It is easy to see that for $n=0$, the result holds. Suppose that $n>0$ and that the assumption is satisfied for $n-1$. That is,

$$\eta_{12n-23} = \frac{a\prod_{m=0}^{n-2}(1+6m\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+1)\,kigeca)}, \qquad \eta_{12n-22} = \frac{b\prod_{m=0}^{n-2}(1+6m\,ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+1)\,ljhfdb)},$$

$$\eta_{12n-21} = \frac{c\prod_{m=0}^{n-2}(1+(6m+1)\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+2)\,kigeca)}, \qquad \eta_{12n-20} = \frac{d\prod_{m=0}^{n-2}(1+(6m+1)\,ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+2)\,ljhfdb)},$$

$$\eta_{12n-19} = \frac{e\prod_{m=0}^{n-2}(1+(6m+2)\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+3)\,kigeca)}, \qquad \eta_{12n-18} = \frac{f\prod_{m=0}^{n-2}(1+(6m+2)\,ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+3)\,ljhfdb)},$$

$$\eta_{12n-17} = \frac{g\prod_{m=0}^{n-2}(1+(6m+3)\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+4)\,kigeca)}, \qquad \eta_{12n-16} = \frac{h\prod_{m=0}^{n-2}(1+(6m+3)\,ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+4)\,ljhfdb)},$$

$$\eta_{12n-15} = \frac{i\prod_{m=0}^{n-2}(1+(6m+4)\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+5)\,kigeca)}, \qquad \eta_{12n-14} = \frac{j\prod_{m=0}^{n-2}(1+(6m+4)\,ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+5)\,ljhfdb)},$$

$$\eta_{12n-13} = \frac{k\prod_{m=0}^{n-2}(1+(6m+5)\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+6)\,kigeca)}, \qquad \eta_{12n-12} = \frac{l\prod_{m=0}^{n-2}(1+(6m+5)\,ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+6)\,ljhfdb)}.$$

From the main Eq. (2.1), it can be derived that:

$$\eta_{12n-11} = \frac{\eta_{12n-23}}{1+\eta_{12n-13}\eta_{12n-15}\eta_{12n-17}\eta_{12n-19}\eta_{12n-21}\eta_{12n-23}}$$

$$= \frac{\frac{a\prod_{m=0}^{n-2}(1+6mkigeca)}{\prod_{m=0}^{n-2}(1+(6m+1)kigeca)}}{1+\left[\begin{array}{c}\left[\frac{k\prod_{m=0}^{n-2}(1+(6m+5)kigeca)}{\prod_{m=0}^{n-2}(1+(6m+6)kigeca)}\right]\left[\frac{i\prod_{m=0}^{n-2}(1+(6m+4)kigeca)}{\prod_{m=0}^{n-2}(1+(6m+5)kigeca)}\right]\\ \left[\frac{g\prod_{m=0}^{n-2}(1+(6m+3)kigeca)}{\prod_{m=0}^{n-2}(1+(6m+4)kigeca)}\right]\left[\frac{e\prod_{m=0}^{n-2}(1+(6m+2)kigeca)}{\prod_{m=0}^{n-2}(1+(6m+3)kigeca)}\right]\\ \left[\frac{c\prod_{m=0}^{n-2}(1+(6m+1)kigeca)}{\prod_{m=0}^{n-2}(1+(6m+2)kigeca)}\right]\left[\frac{a\prod_{m=0}^{n-2}(1+6mkigeca)}{\prod_{m=0}^{n-2}(1+(6m+1)kigeca)}\right]\end{array}\right]}$$

$$= \frac{a\prod_{m=0}^{n-2}(1+6m\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+1)\,kigeca)}\left(\frac{1}{1+\frac{kigeca}{\prod_{m=0}^{n-2}(1+(6m+6)\,kigeca)}\prod_{m=0}^{n-2}(1+6m\,kigeca)}\right)$$

$$= \frac{a\prod_{m=0}^{n-2}(1+6m\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+1)\,kigeca)}\left(\frac{1}{1+\frac{kigeca}{(1+(6n-6)\,kigeca)}}\right)$$

$$= \frac{a\prod_{m=0}^{n-2}(1+6m\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+1)\,kigeca)}\left(\frac{(1+(6n-6)\,kigeca)}{(1+(6n-6)\,kigeca+kigeca)}\right)$$

$$= \frac{a\prod_{m=0}^{n-2}(1+6m\,kigeca)}{\prod_{m=0}^{n-2}(1+(6m+1)\,kigeca)}\left(\frac{(1+(6n-6)\,kigeca)}{(1+(6n-5)\,kigeca)}\right).$$

Consequently, we obtain

$$\eta_{12n-11} = \frac{a\prod_{m=0}^{n-1}(1+6m\,kigeca)}{\prod_{m=0}^{n-1}(1+(6m+1)\,kigeca)}.$$

Likewise, from (2.1), it can be derived that:

$$\eta_{12n-10} = \frac{\eta_{12n-22}}{1+\eta_{12n-12}\eta_{12n-14}\eta_{12n-16}\eta_{12n-18}\eta_{12n-20}\eta_{12n-22}}$$

$$= \frac{\frac{b\prod_{m=0}^{n-2}(1+6mljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+1)ljhfdb)}}{1+\left[\begin{array}{c}\left[\frac{l\prod_{m=0}^{n-2}(1+(6m+5)ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+6)ljhfdb)}\right]\left[\frac{j\prod_{m=0}^{n-2}(1+(6m+4)ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+5)ljhfdb)}\right]\\ \left[\frac{h\prod_{m=0}^{n-2}(1+(6m+3)ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+4)ljhfdb)}\right]\left[\frac{f\prod_{m=0}^{n-2}(1+(6m+2)ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+3)ljhfdb)}\right]\\ \left[\frac{d\prod_{m=0}^{n-2}(1+(6m+1)ljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+2)ljhfdb)}\right]\left[\frac{b\prod_{m=0}^{n-2}(1+6mljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+1)ljhfdb)}\right]\end{array}\right]}$$

$$= \frac{b\prod_{m=0}^{n-2}(1+6mljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+1)ljhfdb)}\left(\frac{1}{1+\frac{ljhfdb}{\prod_{m=0}^{n-2}(1+(6m+6)ljhfdb)}\prod_{m=0}^{n-2}(1+6mljhfdb)}\right)$$

$$= \frac{b\prod_{m=0}^{n-2}(1+6mljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+1)ljhfdb)}\left(\frac{1}{1+\frac{ljhfdb}{(1+(6n-6)ljhfdb)}}\right)$$

$$= \frac{b\prod_{m=0}^{n-2}(1+6mljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+1)ljhfdb)}\left(\frac{(1+(6n-6)ljhfdb)}{(1+(6n-6)ljhfdb+ljhfdb)}\right)$$

$$= \frac{b\prod_{m=0}^{n-2}(1+6mljhfdb)}{\prod_{m=0}^{n-2}(1+(6m+1)ljhfdb)}\left(\frac{(1+(6n-6)ljhfdb)}{(1+(6n-5)ljhfdb)}\right).$$

Thus, we have

$$\eta_{12n-10} = \frac{b\prod_{m=0}^{n-1}(1+6mljhfdb)}{\prod_{m=0}^{n-1}(1+(6m+1)ljhfdb)}.$$

In a similar fashion, the remaining relations can be readily derived. Hence, the proof is complete. □

Theorem 2.2 *Eq. (2.1) possesses a unique equilibrium point at zero, which is not locally asymptotically stable.*

Proof: For the equilibrium points of Eq. (2.1), the following expression holds:

$$\bar{\eta} = \frac{\bar{\eta}}{1+\bar{\eta}^6}$$

Then $\bar{\eta}+\bar{\eta}^7 = \bar{\eta}$ or also $\bar{\eta}^7 = 0$. Thus the equilibrium point of Eq. (2.1) is $\bar{\eta} = 0$.

Let $f : (0,\infty)^6 \to (0,\infty)$ be a function defined by

$$f(r,u,v,w,y,z) = \frac{r}{1+ruvwyz}.$$

This implies that

$$\begin{array}{ll} f_r(r,u,v,w,y,z) = \frac{1}{(1+ruvwyz)^2}, & f_u(r,u,v,w,y,z) = \frac{-r^2vwyz}{(1+ruvwyz)^2}, \\ f_v(r,u,v,w,y,z) = \frac{-r^2uwyz}{(1+ruvwyz)^2}, & f_w(r,u,v,w,y,z) = \frac{-r^2uvyz}{(1+ruvwyz)^2}, \\ f_y(r,u,v,w,y,z) = \frac{-r^2uvwz}{(1+ruvwyz)^2}, & f_z(r,u,v,w,y,z) = \frac{-r^2uvwy}{(1+ruvwyz)^2}. \end{array}$$

We note that

$$f_r(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 1, \quad f_u(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, \quad f_v(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0,$$

$$f_w(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, \quad f_y(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, \quad f_z(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0.$$

The proof is obtained by applying Theorem 1.1. □

Theorem 2.3 *Every positive solution of Eq. (2.1) is bounded and*

$$\lim_{n\to\infty} \eta_n = 0.$$

Proof: It follows from Eq. (2.1) that

$$\eta_{n+1} = \frac{\eta_{n-11}}{1+\eta_{n-1}\eta_{n-3}\eta_{n-5}\eta_{n-7}\eta_{n-9}\eta_{n-11}} \le \eta_{n-11}.$$

Hence, the subsequences $\{\eta_{12n-11}\}_{n=0}^{\infty}, \{\eta_{12n-10}\}_{n=0}^{\infty}, \{\eta_{12n-9}\}_{n=0}^{\infty}, \{\eta_{12n-8}\}_{n=0}^{\infty}$, $\{\eta_{12n-7}\}_{n=0}^{\infty}, \{\eta_{12n-6}\}_{n=0}^{\infty}, \{\eta_{12n-5}\}_{n=0}^{\infty}, \{\eta_{12n-4}\}_{n=0}^{\infty}, \{\eta_{12n-3}\}_{n=0}^{\infty}, \{\eta_{12n-2}\}_{n=0}^{\infty}, \{\eta_{12n-1}\}_{n=0}^{\infty}$, are decreasing, and thus, each is bounded from above by

$$M = \max\{\eta_{-11},\eta_{-10},\eta_{-9},\eta_{-8},\eta_{-7},\eta_{-6},\eta_{-5},\eta_{-4},\eta_{-3},\eta_{-2},\eta_{-1},\eta_0\}.$$

To confirm the results presented in this section, a numerical example is provided to demonstrate a representative type of solution to Eq. (2.1). □

Example 2.1 Consider Eq. (2.1) with the following initial values $\eta_{-11} = 0.9, \eta_{-10} = -1.1, \eta_{-9} = 0.7, \eta_{-8} = -0.6, \eta_{-7} = 1.3, \eta_{-6} = -1.2, \eta_{-5} = 0.8, \eta_{-4} = -1.5, \eta_{-3} = 1, \eta_{-2} = -0.9, \eta_{-1} = 0.6, \eta_0 = -0.7$.

The graph in Figure 1 is presented below.

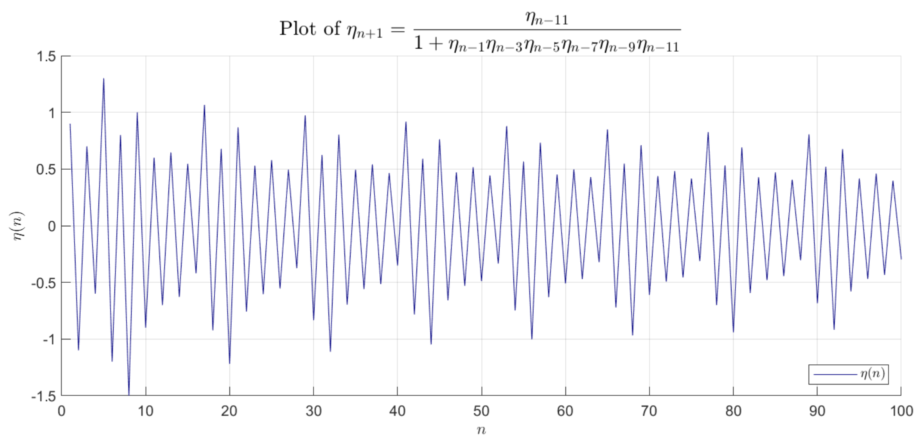

Figure 1:

Example 2.1 illustrates the dynamics of Eq. (2.1) with the given initial conditions.

3. Second Case

The following rational recurrence relations are considered:

$$\eta_{n+1} = \frac{\eta_{n-11}}{1 - \eta_{n-1}\eta_{n-3}\eta_{n-5}\eta_{n-7}\eta_{n-9}\eta_{n-11}}, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

with initial conditions taken as arbitrary real numbers.

A distinct representation of the solutions to Eq. (3.1) is given in the subsequent theorem.

Theorem 3.1 *Assume that* $\{\eta_n\}_{n=-11}^{\infty}$ *is a solution of Eq.* (3.1)*. Then for* $n = 0, 1, 2, \ldots$, *the following holds:*

$$\eta_{12n-11} = \frac{a\prod_{m=0}^{n-1}(1-6m\,kigeca)}{\prod_{m=0}^{n-1}(1-(6m+1)\,kigeca)}, \qquad \eta_{12n-10} = \frac{b\prod_{m=0}^{n-1}(1-6m\,ljhfdb)}{\prod_{m=0}^{n-1}(1-(6m+1)\,ljhfdb)},$$

$$\eta_{12n-9} = \frac{c\prod_{m=0}^{n-1}(1-(6m+1)\,kigeca)}{\prod_{m=0}^{n-1}(1-(6m+2)\,kigeca)}, \qquad \eta_{12n-8} = \frac{d\prod_{m=0}^{n-1}(1-(6m+1)\,ljhfdb)}{\prod_{m=0}^{n-1}(1-(6m+2)\,ljhfdb)},$$

$$\eta_{12n-7} = \frac{e\prod_{m=0}^{n-1}(1-(6m+2)\,kigeca)}{\prod_{m=0}^{n-1}(1-(6m+3)\,kigeca)}, \qquad \eta_{12n-6} = \frac{f\prod_{m=0}^{n-1}(1-(6m+2)\,ljhfdb)}{\prod_{m=0}^{n-1}(1-(6m+3)\,ljhfdb)},$$

$$\eta_{12n-5} = \frac{g\prod_{m=0}^{n-1}(1-(6m+3)\,kigeca)}{\prod_{m=0}^{n-1}(1-(6m+4)\,kigeca)}, \qquad \eta_{12n-4} = \frac{h\prod_{m=0}^{n-1}(1-(6m+3)\,ljhfdb)}{\prod_{m=0}^{n-1}(1-(6m+4)\,ljhfdb)},$$

$$\eta_{12n-3} = \frac{i\prod_{m=0}^{n-1}(1-(6m+4)\,kigeca)}{\prod_{m=0}^{n-1}(1-(6m+5)\,kigeca)}, \qquad \eta_{12n-2} = \frac{j\prod_{m=0}^{n-1}(1-(6m+4)\,ljhfdb)}{\prod_{m=0}^{n-1}(1-(6m+5)\,ljhfdb)},$$

$$\eta_{12n-1} = \frac{k\prod_{m=0}^{n-1}(1-(6m+5)\,kigeca)}{\prod_{m=0}^{n-1}(1-(6m+6)\,kigeca)}, \qquad \eta_{12n} = \frac{l\prod_{m=0}^{n-1}(1-(6m+5)\,ljhfdb)}{\prod_{m=0}^{n-1}(1-(6m+6)\,ljhfdb)},$$

where $r\,kigeca \neq 1$, $r\,ljhfdb \neq 1$ *for* $r = 1, 2, 3, \ldots$.

Proof: We apply induction to prove the result for this rational recursive sequence. It is easy to see that for $n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n - 1$. That is,

$$\eta_{12n-23} = \frac{a\prod_{m=0}^{n-2}(1-6m\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}, \quad \eta_{12n-22} = \frac{b\prod_{m=0}^{n-2}(1-6m\,ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+1)\,ljhfdb)},$$

$$\eta_{12n-21} = \frac{c\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+2)\,kigeca)}, \quad \eta_{12n-20} = \frac{d\prod_{m=0}^{n-2}(1-(6m+1)\,ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+2)\,ljhfdb)},$$

$$\eta_{12n-19} = \frac{e\prod_{m=0}^{n-2}(1-(6m+2)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+3)\,kigeca)}, \quad \eta_{12n-18} = \frac{f\prod_{m=0}^{n-2}(1-(6m+2)\,ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+3)\,ljhfdb)},$$

$$\eta_{12n-17} = \frac{g\prod_{m=0}^{n-2}(1-(6m+3)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+4)\,kigeca)}, \quad \eta_{12n-16} = \frac{h\prod_{m=0}^{n-2}(1-(6m+3)\,ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+4)\,ljhfdb)},$$

$$\eta_{12n-15} = \frac{i\prod_{m=0}^{n-2}(1-(6m+4)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+5)\,kigeca)}, \quad \eta_{12n-14} = \frac{j\prod_{m=0}^{n-2}(1-(6m+4)\,ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+5)\,ljhfdb)},$$

$$\eta_{12n-13} = \frac{k\prod_{m=0}^{n-2}(1-(6m+5)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+6)\,kigeca)}, \quad \eta_{12n-12} = \frac{l\prod_{m=0}^{n-2}(1-(6m+5)\,ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+6)\,ljhfdb)}.$$

By applying the main Eq. (3.1), we obtain

$$\begin{aligned}
\eta_{12n-11} =& \frac{\eta_{12n-23}}{1-\eta_{12n-13}\eta_{12n-15}\eta_{12n-17}\eta_{12n-19}\eta_{12n-21}\eta_{12n-23}} \\
=& \frac{\frac{a\prod_{m=0}^{n-2}(1-6m\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}}{1-\left[\begin{array}{c}\left[\frac{k\prod_{m=0}^{n-2}(1-(6m+5)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+6)\,kigeca)}\right]\left[\frac{i\prod_{m=0}^{n-2}(1-(6m+4)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+5)\,kigeca)}\right] \\ \left[\frac{g\prod_{m=0}^{n-2}(1-(6m+3)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+4)\,kigeca)}\right]\left[\frac{e\prod_{m=0}^{n-2}(1-(6m+2)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+3)\,kigeca)}\right] \\ \left[\frac{c\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+2)\,kigeca)}\right]\left[\frac{a\prod_{m=0}^{n-2}(1-6m\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}\right]\end{array}\right]} \\
=& \frac{a\prod_{m=0}^{n-2}(1-6m\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}\left(\frac{1}{1-\frac{kigeca}{\prod_{m=0}^{n-2}\left(1-(6m+6)\,kigeca\right)}\prod_{m=0}^{n-2}(1-6m\,kigeca)}\right) \\
=& \frac{a\prod_{m=0}^{n-2}(1-6m\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}\left(\frac{1}{1-\frac{kigeca}{(1-(6n-6)\,kigeca)}}\right) \\
=& \frac{a\prod_{m=0}^{n-2}(1-6m\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}\left(\frac{(1-(6n-6)\,kigeca)}{(1-(6n-6)\,kigeca-kigeca)}\right) \\
=& \frac{a\prod_{m=0}^{n-2}(1-6m\,kigeca)}{\prod_{m=0}^{n-2}(1-(6m+1)\,kigeca)}\left(\frac{(1-(6n-6)\,kigeca)}{(1-(6n-5)\,kigeca)}\right).
\end{aligned}$$

Therefore, we have

$$\eta_{12n-11} = \frac{a\prod_{m=0}^{n-1}(1-6m\,kigeca)}{\prod_{m=0}^{n-1}(1-(6m+1)\,kigeca)}.$$

Likewise, by applying the main Eq. (3.1), we obtain

$$
\begin{aligned}
\eta_{12n-10} &= \frac{\eta_{12n-22}}{1-\eta_{12n-12}\eta_{12n-14}\eta_{12n-16}\eta_{12n-18}\eta_{12n-20}\eta_{12n-22}} \\
&= \frac{\frac{b\prod_{m=0}^{n-2}(1-6mljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+1)ljhfdb)}}{1+\left[\begin{array}{c}\left[\frac{l\prod_{m=0}^{n-2}(1-(6m+5)ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+6)ljhfdb)}\right]\left[\frac{j\prod_{m=0}^{n-2}(1-(6m+4)ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+5)ljhfdb)}\right]\\ \left[\frac{h\prod_{m=0}^{n-2}(1-(6m+3)ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+4)ljhfdb)}\right]\left[\frac{f\prod_{m=0}^{n-2}(1-(6m+2)ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+3)ljhfdb)}\right]\\ \left[\frac{d\prod_{m=0}^{n-2}(1-(6m+1)ljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+2)ljhfdb)}\right]\left[\frac{b\prod_{m=0}^{n-2}(1-6mljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+1)ljhfdb)}\right]\end{array}\right]} \\
&= \frac{b\prod_{m=0}^{n-2}(1-6mljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+1)ljhfdb)}\left(\frac{1}{1-\frac{ljhfdb}{\prod_{m=0}^{n-2}(1-(6m+6)ljhfdb)}\prod_{m=0}^{n-2}(1-6mljhfdb)}\right) \\
&= \frac{b\prod_{m=0}^{n-2}(1-6mljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+1)ljhfdb)}\left(\frac{1}{1-\frac{ljhfdb}{(1-(6n-6)ljhfdb)}}\right) \\
&= \frac{b\prod_{m=0}^{n-2}(1-6mljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+1)ljhfdb)}\left(\frac{(1-(6n-6)ljhfdb)}{(1-(6n-6)ljhfdb-ljhfdb)}\right) \\
&= \frac{b\prod_{m=0}^{n-2}(1-6mljhfdb)}{\prod_{m=0}^{n-2}(1-(6m+1)ljhfdb)}\left(\frac{(1-(6n-6)ljhfdb)}{(1-(6n-5)ljhfdb)}\right).
\end{aligned}
$$

Hence, we have

$$
\eta_{12n-10} = \frac{b\prod_{m=0}^{n-1}(1-6mljhfdb)}{\prod_{m=0}^{n-1}(1-(6m+1)ljhfdb)}.
$$

In a similar fashion, the remaining relations can be readily derived. Hence, the proof is complete. □

Theorem 3.2 *Eq. (3.1) possesses a unique equilibrium point at zero, which is not locally asymptotically stable.*

Example 3.1 Consider Eq. (3.1) with initial values $\eta_{-11}=0.4, \eta_{-10}=-0.3, \eta_{-9}=0.2, \eta_{-8}=-0.25, \eta_{-7}=0.3, \eta_{-6}=-0.1, \eta_{-5}=0.15, \eta_{-4}=-0.35, \eta_{-3}=0.1, \eta_{-2}=-0.2, \eta_{-1}=0.05, \eta_0=-0.05$. The graph in Figure 2 is presented below.

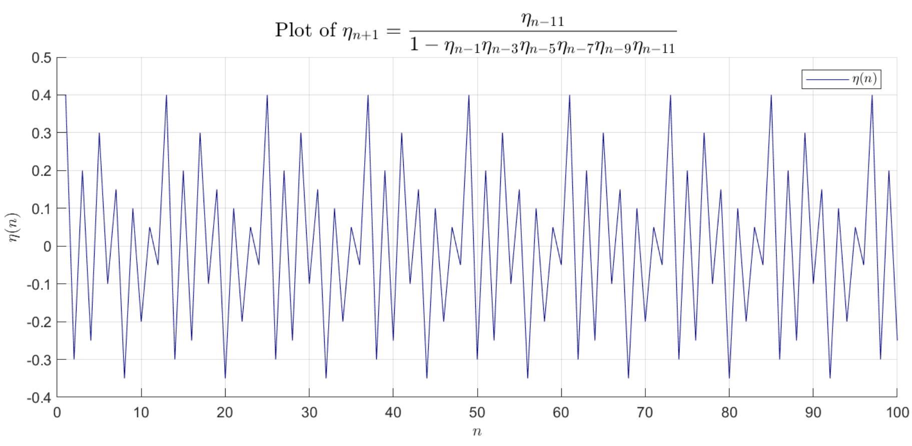

Figure 2:

Example 3.1 illustrates the dynamics of Eq. (3.1) with the given initial conditions.

4. Third Case

Focusing on the rational recursive sequences below, Theorem 4.1 provides a particular representation of their solutions.

$$\eta_{n+1} = \frac{\eta_{n-11}}{-1+\eta_{n-1}\eta_{n-3}\eta_{n-5}\eta_{n-7}\eta_{n-9}\eta_{n-11}}, \quad n=0,1,2,\ldots, \tag{4.1}$$

with the initial conditions chosen as arbitrary real numbers satisfying

$$\eta_{-11}\eta_{-9}\eta_{-7}\eta_{-5}\eta_{-3}\eta_{-1} \neq 1, \quad \eta_{-10}\eta_{-8}\eta_{-6}\eta_{-4}\eta_{-2}\eta_{0} \neq 1.$$

Theorem 4.1 *Let $\{\eta_n\}_{n=-11}^{\infty}$ be a solution of Eq. (4.1). Then Eq. (4.1) has unbounded solutions and for* $n=0,1,2,\ldots,$

$$\begin{aligned}
\eta_{12n-11} &= \frac{a}{(-1+kigeca)^n}, & \eta_{12n-10} &= \frac{b}{(-1+ljhfdb)^n},\\
\eta_{12n-9} &= c(-1+kigeca)^n, & \eta_{12n-8} &= d(-1+ljhfdb)^n,\\
\eta_{12n-7} &= \frac{e}{(-1+kigeca)^n}, & \eta_{12n-6} &= \frac{f}{(-1+ljhfdb)^n},\\
\eta_{12n-5} &= g(-1+kigeca)^n, & \eta_{12n-4} &= h(-1+ljhfdb)^n,\\
\eta_{12n-3} &= \frac{i}{(-1+kigeca)^n}, & \eta_{12n-2} &= \frac{j}{(-1+ljhfdb)^n},\\
\eta_{12n-1} &= k(-1+kigeca)^n, & \eta_{12n} &= l(-1+ljhfdb)^n.
\end{aligned}$$

Proof: The result holds for $n=0$. Now assume that $n>0$ and that the result is true for $n-1$. That is,

$$\begin{aligned}
\eta_{12n-23} &= \frac{a}{(-1+kigeca)^{n-1}}, & \eta_{12n-22} &= \frac{b}{(-1+ljhfdb)^{n-1}},\\
\eta_{12n-21} &= c(-1+kigeca)^{n-1}, & \eta_{12n-20} &= d(-1+ljhfdb)^{n-1},\\
\eta_{12n-19} &= \frac{e}{(-1+kigeca)^{n-1}}, & \eta_{12n-18} &= \frac{f}{(-1+ljhfdb)^{n-1}},\\
\eta_{12n-17} &= g(-1+kigeca)^{n-1}, & \eta_{12n-16} &= h(-1+ljhfdb)^{n-1},\\
\eta_{12n-15} &= \frac{i}{(-1+kigeca)^{n-1}}, & \eta_{12n-14} &= \frac{j}{(-1+ljhfdb)^{n-1}},\\
\eta_{12n-13} &= k(-1+kigeca)^{n-1}, & \eta_{12n-12} &= l(-1+ljhfdb)^{n-1}.
\end{aligned}$$

As a consequence of Eq. (4.1), we have

$$\begin{aligned}
\eta_{12n-11} &= \frac{\eta_{12n-23}}{-1+\eta_{12n-13}\eta_{12n-15}\eta_{12n-17}\eta_{12n-19}\eta_{12n-21}\eta_{12n-23}}\\
&= \frac{\frac{a}{(-1+kigeca)^{n-1}}}{-1+\left[\begin{array}{c}\left[k(-1+kigeca)^{n-1}\right]\left[\frac{i}{(-1+kigeca)^{n-1}}\right]\left[g(-1+kigeca)^{n-1}\right]\\ \left[\frac{e}{(-1+kigeca)^{n-1}}\right]\left[c(-1+kigeca)^{n-1}\right]\left[\frac{a}{(-1+kigeca)^{n-1}}\right]\end{array}\right]}\\
&= \frac{\frac{a}{(-1+kigeca)^{n-1}}}{-1+kigeca}
\end{aligned}$$

Hence, we have

$$\eta_{12n-11} = \frac{a}{(-1+kigeca)^n}.$$

Similarly, as a consequence of Eq. (4.1), we have

$$\begin{aligned}\eta_{12n-10} &= \frac{\eta_{12n-22}}{-1+\eta_{12n-12}\eta_{12n-14}\eta_{12n-16}\eta_{12n-18}\eta_{12n-20}\eta_{12n-22}} \\ &= \frac{\frac{b}{(-1+ljhfdb)^{n-1}}}{-1+\left[\begin{array}{c}\left[l(-1+ljhfdb)^{n-1}\right]\left[\frac{j}{(-1+ljhfdb)^{n-1}}\right]\left[h(-1+ljhfdb)^{n-1}\right] \\ \left[\frac{f}{(-1+ljhfdb)^{n-1}}\right]\left[d(-1+ljhfdb)^{n-1}\right]\left[\frac{b}{(-1+ljhfdb)^{n-1}}\right]\end{array}\right]} \\ &= \frac{\frac{b}{(-1+ljhfdb)^{n-1}}}{-1+ljhfdb}\end{aligned}$$

So, we have

$$\eta_{12n-10} = \frac{b}{(-1+ljhfdb)^n}$$

In a similar fashion, the remaining relations can be readily derived. Hence, the proof is complete. □

Theorem 4.2 *All three of the equilibrium points in Eq. (4.1) are not locally asymptotically stable and are* $0, \pm\sqrt[6]{2}$.

Proof: The argument follows the same reasoning as in the proof of Theorem 2.2 and is therefore omitted. □

Theorem 4.3 *Eq. (4.1) admits a periodic solution of period twelve if and only if* $kigeca = ljhfdb = 2$. *The solution then takes the form*

$$\{b, a, d, c, f, e, h, g, j, i, l, k, b, a, d, c, f, e, h, g, j, i, l, k, \ldots\}.$$

Proof: First suppose that there exists a prime period twelve solution

$$b, a, d, c, f, e, h, g, j, i, l, k, b, a, d, c, f, e, h, g, j, i, l, k, \ldots.$$

of Eq. (4.1), we see from Eq. (4.1) that

$$\begin{array}{ll} b = \frac{b}{(-1+ljhfdb)^n}, & h = h(-1+ljhfdb)^n \\ a = \frac{a}{(-1+kigeca)^n}, & g = g(-1+kigeca)^n \\ d = d(-1+ljhfdb)^n, & j = \frac{j}{(-1+ljhfdb)^n} \\ c = c(-1+kigeca)^n, & i = \frac{i}{(-1+kigeca)^n} \\ f = \frac{f}{(-1+ljhfdb)^n}, & l = l(-1+ljhfdb)^n \\ e = \frac{e}{(-1+kigeca)^n}, & k = k(-1+kigeca)^n \end{array}$$

or

$$(-1+ljhfdb)^n = 1, \quad (-1+kigeca)^n = 1.$$

Then

$$ljhfdb = 2, \quad kigeca = 2.$$

Secondly, assume that $ljhfdb = kigeca = 2$. Then, from Eq. (4.1), we obtain $\eta_{12n} = l, \eta_{12n-1} = k, \eta_{12n-2} = j, \eta_{12n-3} = i, \eta_{12n-4} = h, \eta_{12n-5} = g, \eta_{12n-6} = f, \eta_{12n-7} = e, \eta_{12n-8} = d, \eta_{12n-9} = c, \eta_{12n-10} = b, \eta_{12n-11} = a$.

This establishes a period-twelve solution, completing the proof. □

Example 4.1 Let us examine Eq. (4.1) with $\eta_{-11} = 0.1, \eta_{-10} = 0.2, \eta_{-9} = 0.15, \eta_{-8} = 0.25, \eta_{-7} = 0.2, \eta_{-6} = 0.3, \eta_{-5} = 0.25, \eta_{-4} = 0.35, \eta_{-3} = 0.3, \eta_{-2} = 0.4, \eta_{-1} = 0.35, \eta_0 = 0.45$. The graph in Figure 3 is presented below.

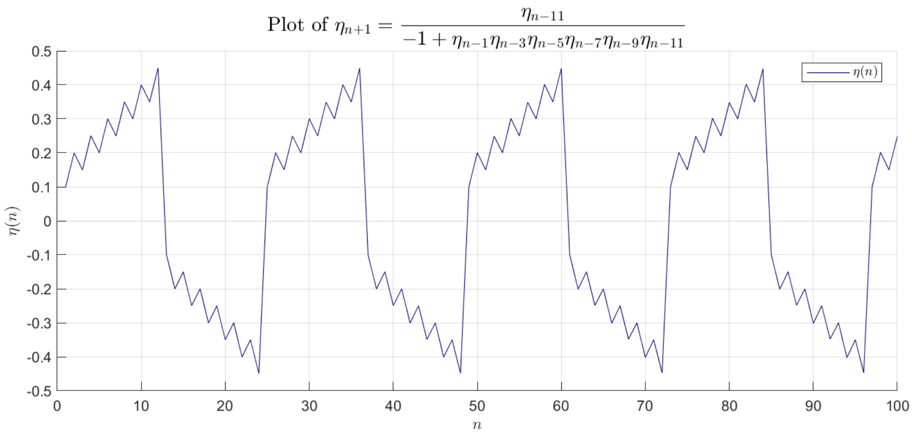

Figure 3:

Example 4.1 illustrates the dynamics of Eq. (4.1) with the given initial conditions.

5. Forth Case

Here, we consider the final case, given by:

$$\eta_{n+1} = \frac{\eta_{n-11}}{-1 - \eta_{n-1}\eta_{n-3}\eta_{n-5}\eta_{n-7}\eta_{n-9}\eta_{n-11}}, \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

where the initial conditions are arbitrary real numbers satisfying

$$\eta_{-11}\eta_{-9}\eta_{-7}\eta_{-5}\eta_{-3}\eta_{-1} \neq -1, \quad \eta_{-10}\eta_{-8}\eta_{-6}\eta_{-4}\eta_{-2}\eta_0 \neq -1.$$

Theorem 5.1 presents a comprehensive characterization of the solutions.

Theorem 5.1 *Let $\{\eta_n\}_{n=-11}^{\infty}$ be a solution of Eq. (5.1). It follows that Eq. (5.1) admits unbounded solutions for $n = 0, 1, 2, \ldots,$ and*

$$\eta_{12n-11} = \frac{(-1)^n a}{(1 + kigeca)^n}, \quad \eta_{12n-10} = \frac{(-1)^n b}{(1 + ljhfdb)^n},$$
$$\eta_{12n-9} = (-1)^n c(1 + kigeca)^n, \quad \eta_{12n-8} = (-1)^n d(1 + ljhfdb)^n,$$
$$\eta_{12n-7} = \frac{(-1)^n e}{(1 + kigeca)^n}, \quad \eta_{12n-6} = \frac{(-1)^n f}{(1 + ljhfdb)^n},$$
$$\eta_{12n-5} = (-1)^n g(1 + kigeca)^n, \quad \eta_{12n-4} = (-1)^n h(1 + ljhfdb)^n,$$
$$\eta_{12n-3} = \frac{(-1)^n i}{(1 + kigeca)^n}, \quad \eta_{12n-2} = \frac{(-1)^n j}{(1 + ljhfdb)^n},$$
$$\eta_{12n-1} = (-1)^n k(1 + kigeca)^n, \quad \eta_{12n} = (-1)^n l(1 + ljhfdb)^n.$$

Proof: The result holds for $n = 0$. Now assume that $n > 0$ and that the result is true for $n-1$. That is,

$$\begin{aligned}
\eta_{12n-23} &= \frac{(-1)^{n-1}a}{(1+kigeca)^{n-1}}, \quad \eta_{12n-22} = \frac{(-1)^{n-1}b}{(1+ljhfdb)^{n-1}},\\
\eta_{12n-21} &= (-1)^{n-1}c(1+kigeca)^{n-1}, \quad \eta_{12n-20} = (-1)^{n-1}d(1+ljhfdb)^{n-1},\\
\eta_{12n-19} &= \frac{(-1)^{n-1}e}{(1+kigeca)^{n-1}}, \quad \eta_{12n-18} = \frac{(-1)^{n-1}f}{(1+ljhfdb)^{n-1}},\\
\eta_{12n-17} &= (-1)^{n-1}g(1+kigeca)^{n-1}, \quad \eta_{12n-16} = (-1)^{n-1}h(1+ljhfdb)^{n-1},\\
\eta_{12n-15} &= \frac{(-1)^{n-1}i}{(1+kigeca)^{n-1}}, \quad \eta_{12n-14} = \frac{(-1)^{n-1}j}{(1+ljhfdb)^{n-1}},\\
\eta_{12n-13} &= (-1)^{n-1}k(1+kigeca)^{n-1}, \quad \eta_{12n-12} = (-1)^{n-1}l(1+ljhfdb)^{n-1}.
\end{aligned}$$

Now, it follows from Eq. (5.1) that

$$\begin{aligned}
\eta_{12n-11} &= \frac{\eta_{12n-23}}{-1-\eta_{12n-13}\eta_{12n-15}\eta_{12n-17}\eta_{12n-19}\eta_{12n-21}\eta_{12n-23}}\\
&= \frac{\frac{(-1)^{n-1}a}{(1+kigeca)^{n-1}}}{-1-\left[\begin{array}{c}\left[(-1)^{n-1}k(1+kigeca)^{n-1}\right]\left[\frac{(-1)^{n-1}i}{(1+kigeca)^{n-1}}\right]\left[(-1)^{n-1}g(1+kigeca)^{n-1}\right]\\ \left[\frac{(-1)^{n-1}e}{(1+kigeca)^{n-1}}\right]\left[(-1)^{n-1}c(1+kigeca)^{n-1}\right]\left[\frac{(-1)^{n-1}a}{(1+kigeca)^{n-1}}\right]\end{array}\right]}\\
&= \frac{\frac{(-1)^{n-1}a}{(1+kigeca)^{n-1}}}{-1-kigeca}
\end{aligned}$$

Hence, we have

$$\eta_{12n-11} = \frac{(-1)^{n}a}{(1+kigeca)^{n}}.$$

Similary, it follows from Eq. (5.1) that

$$\begin{aligned}
\eta_{12n-10} &= \frac{\eta_{12n-22}}{-1-\eta_{12n-12}\eta_{12n-14}\eta_{12n-16}\eta_{12n-18}\eta_{12n-20}\eta_{12n-22}}\\
&= \frac{\frac{(-1)^{n-1}b}{(1+ljhfdb)^{n-1}}}{-1-\left[\begin{array}{c}\left[(-1)^{n-1}l(1+ljhfdb)^{n-1}\right]\left[\frac{(-1)^{n-1}j}{(1+ljhfdb)^{n-1}}\right]\left[(-1)^{n-1}h(1+ljhfdb)^{n-1}\right]\\ \left[\frac{(-1)^{n-1}f}{(1+ljhfdb)^{n-1}}\right]\left[(-1)^{n-1}d(1+ljhfdb)^{n-1}\right]\left[\frac{(-1)^{n-1}b}{(1+ljhfdb)^{n-1}}\right]\end{array}\right]}\\
&= \frac{\frac{(-1)^{n-1}b}{(1+ljhfdb)^{n-1}}}{-1-ljhfdb}
\end{aligned}$$

Hence, we have

$$\eta_{12n-10} = \frac{(-1)^{n}b}{(1+ljhfdb)^{n}}.$$

In a similar fashion, the remaining relations can be readily derived. Hence, the proof is complete. □

Theorem 5.2 *The two equilibrium points of Eq. (5.1), specifically* 0 *and* $-\sqrt[6]{2}$*, do not exhibit local asymptotic stability.*

Theorem 5.3 *Eq. (5.1) admits a periodic solution of period twelve if and only if* $kigeca = ljhfdb = -2$*. The solution then takes the form*

$$\{b, a, d, c, f, e, h, g, j, i, l, k, b, a, d, c, f, e, h, g, j, i, l, k, \dots\}.$$

Example 5.1 For Eq. (5.1) consider $\eta_{-11} = 0.05, \eta_{-10} = -0.06, \eta_{-9} = 0.04, \eta_{-8} = -0.03, \eta_{-7} = 0.02, \eta_{-6} = -0.01, \eta_{-5} = 0.01, \eta_{-4} = -0.2, \eta_{-3} = 0.03, \eta_{-2} = -0.04, \eta_{-1} = 0.06, \eta_0 =$ -0.05 . See Figure 4

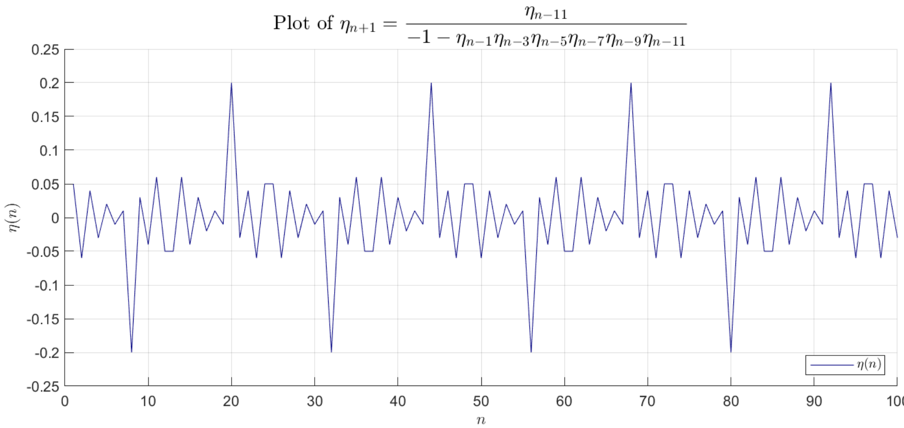

Figure 4:

Example 5.1 illustrates the dynamics of Eq. (5.1) with the given initial conditions.

6. Conclusion

This study addressed a specific rational nonlinear difference equation of the twelfth order defined by arbitrary real initial conditions. By constructing explicit solutions, we were able to capture the intricate recursive structure inherent in such high-order formulations. The initial conditions being arbitrary real numbers ensures the generality of the results and allows for the examination of the sequence behavior across a broad spectrum of starting values. Four distinct instances of the equation—differing in the sign configurations within the recurrence—were explored in depth. For each case, graphical analysis was conducted to visualize the qualitative behavior of the sequences, revealing varied dynamics as a consequence of the sign alternations. The findings enrich the theoretical framework surrounding high-order rational difference equations and open avenues for future exploration concerning their long-term behavior and structural properties.

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