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Delay-Induced Dynamics in a Stage-Structured Food Chain Model

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ABSTRACT: An investigation is performed into a non-linear mathematical model with a stage structure for a predator. The model shows that predators may be classified as both immature or mature. A time delay represents the age of adulthood, and immature predators are no longer capable of attacking prey. Standards for the system's boundedness are installed. The presence of the equilibrium point and the stability of the model are analyzed using the concepts of ordinary differential equations (ODEs). Additionally, the factors affecting system persistence are determined. A bifurcation analysis is done to assess the system's stability and instability in the presence of delay. The system's global stability is likewise verified graphically. Numerical simulations reveal that to govern the middle predator (pest) in the prey (plant), changes to each maturation delay and maturation rate are necessary.

Key Words: Mathematical model, stage-structure, ratio-dependent, Hopf-bifurcation, persistence.

Contents

1. Introduction

In epidemiology and demography, stage-structured models are employed to describe population systems in which individuals are classified according to their developmental or lifehistory stages. This modelling framework reflects biological realism more accurately, as birth, mortality, and interaction rates often differ substantially between immature and mature individuals. Incorporating stage structure has therefore become an important approach for
understanding population regulation, extinction mechanisms, disease spread, and trophic interactions in ecological systems.

A pioneering contribution to this area was made by Aiello and Freedman [1990], who introduced stage-structured model with a maturation delay for a single species. In their formulation, the recruitment of the mature population depends on the historical density of immature individuals, highlighting the importance

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of time delay in population growth. Since then, such models have been widely extended and applied to more complex biological scenarios.

In recent decades, increasing attention has been paid to stage-structured competitive and predator-prey models, where the inclusion of maturation processes and delays leads to rich dynamical behaviour. Several researchers have shown that maturation delays can significantly influence system stability, permanence, and oscillatory behaviour [Peng et al., 2020; Wang et al., 2020]. Ratio-dependent predator-prey systems with stage-structured prey populations have also been investigated, particularly in cases where prey is divided into immature and mature classes [Xu et al., 2022]. These studies demonstrate that life-stage differentiation alters predator efficiency and can either stabilize or destabilize the ecosystem.

Further studies have considered stage structure in both predator and prey populations, revealing that immature predator dynamics play a crucial role in shaping long-term system behaviour [Zhang et al., 2023]. In particular, the presence of maturation delays in predators has been shown to induce Hopf bifurcations and sustained oscillations [Liu et al., 2021]. Nonlinear ecological mechanisms such as the Allee effect, which describes reduced population growth at low densities, have also been incorporated into stage-structured predator-prey models, providing insights into resilience, extinction thresholds, and recovery dynamics [Li et al., 2021].

Food chain models extend traditional predator-prey frameworks by accounting for interactions across multiple trophic levels. While early investigations focused mainly on two-species systems, it has been recognized that food chains involving three or more trophic levels exhibit qualitatively new dynamics, including complex boundary equilibria and persistence conditions [Freedman & Ruan, 2021]. Ratio-dependent food chain models have attracted significant interest due to their ability to capture realistic predator interference and biological control mechanisms [Wang et al., 2020; Liu et al., 2021]. These models can effectively describe extinction scenarios and complex population transitions that are not adequately represented in

prey-dependent formulations. Motivated by these developments, the present work proposes a stage-structured ratio-dependent food chain model consisting of three trophic levels. In this framework, the prey population represents a plant species, the intermediate predator is a pest population divided into immature and mature stages, and the top predator corresponds to the natural enemy of the pest. Only mature predators are assumed to be capable of feeding and reproduction, whereas immature predators do not contribute directly to predation or population growth. A discrete time delay is incorporated to describe the maturation process from immature to mature stages, capturing the biological time lag inherent in development.

This study first formulates the mathematical model and outlines its underlying biological assumptions. The boundedness of solutions is then analysed to ensure biological feasibility. Subsequently, the existence and stability of equilibria are examined, including both interior equilibria and top-predator-free boundary equilibria. The bifurcation behaviour of the system is investigated to identify qualitative changes induced by variations in key parameters, particularly the maturation delay. Conditions for persistence and extinction of populations are derived, providing insight into long-term ecosystem survival. Numerical simulations are finally presented to support the analytical results. The paper concludes with a discussion of the ecological significance of the findings and suggestions for future research directions.

2. Mathematical Model

This study uses the following differential equations to simulate a ratio-dependent food chain model with a stage structure for the middle predator.

$$\dot{x}(t) = x(t)(1-x(t)) - \frac{c_1 x(t) y_m(t)}{x(t)+y_m(t)},$$
$$\dot{y}_i(t) = \alpha y_m(t) - d_1 y_i(t) - \alpha e^{-d_1\tau} y_m(t-\tau),$$
$$\dot{y}_m(t) = \alpha e^{-d_1\tau} y_m(t-\tau) - d_2 y_m^2(t) - \frac{c_2 y_m(t) z(t)}{y_m(t)+z(t)} + \frac{p_1 x(t) y_m(t)}{x(t)+y_m(t)},$$
$$\dot{z}(t) = z(t)\left(-d_3 + \frac{p_2 y_m(t)}{y_m(t)+z(t)}\right).$$
$$y_m(t) = \emptyset_m(t) \geq 0, \quad -\tau \leq t < 0 \& y_i(0) > 0, x(0) > 0, z(0) > 0.$$

where, $x(t), y_i(t)$ and $y_m(t)$ represent the densities of prey, immature and mature middle predator populations at time t respectively, $z(t)$ denotes the top predator density at time t.

Using the following presumptions, the model (1) is developed:
(H1): Middle predator population's history is divided into tiers: immature and mature. τ indicate the period a predator species needs to reach maturity. So, we assume that mature predators feed the prey species while top predators catch the mature predator species. Immature predators do not feed on prey and are unable to breed.
(H2): At time $t > 0$, the immature predator population's birth rate is always proportional to the current birth rate' of the mature predator population, with the proportionality remaining constant with $\alpha > 0$. After giving birth, the immature predator population will transition to the mature prey class throughout a maturity period τ. Subsequently, we make the idea that the immature people born at that time $t-\tau$ who live to the present time tare members of the immature population who later join the mature population at time t. Under is how this will be calculated:

If $N(t)$ is a given population at time t, then the number that survives from t_1 to t_2 is given by

$$N(t_2) = N(t_1)\, e^{-d_1(t_2-t_1)} \tag{2}$$

Hence if $t_1 = t-\tau$ and $t_2 = t$, then $N(t) = N(t-\tau)e^{-d_1\tau}$, where $e^{-d_1\tau}$ denotes the survival rate of immature species to reach maturity. The term $\alpha e^{-d_1\tau} y_m(t-\tau)$ that appears in the first and second equations of system (1) represents the immature predator population born at time $(t-\tau)$ and still surviving at the time t and therefore represents the transformation from immature predator population to mature predator population.
(H3): The natural death rate ($d_1 > 0$) is present in the population of immature predators. The death rate of a population of mature predators is proportional to the square of the current population of mature predators with a proportionately constant $d_2 > 0$. Positive constants c_i and $p_i, (i = 1, 2)$ stand for capturing rate of the predators and the conversion rate for the predation of predators, respectively. The mature predator consumes prey with the ratio $\frac{p_1 x(t) y_m(t)}{x(t)+y_m(t)}$. The top predator population has the natural death rate d_3.
So, in order to maintain the starting conditions, we need

$$y_i(0) = \int_{-\tau}^{0} \alpha e^{d_1 s} \emptyset_m(s) ds \tag{3}$$

the total surviving immature population from the observed births on $-\tau \le t < 0$.
The solution to the second equation of system (1) can be expressed in terms of a solution for $y_m(t)$ by using equation (3) as follows:

$$y_i(t) = \int_{t-\tau}^{t} \alpha e^{-d_1(t-s)} y_m(s) ds \tag{4}$$

Equations (3) and (4) show that, mathematically speaking, the system (1) does not require knowledge of the history of $y_i(t)$ since, if we know the qualities of $y_m(t)$, we may derive the properties of $y_i(t)$ from (3) and (4).

As a result, in the remainder of this paper, we will only address the next model.
$\dot{x}(t) = x(t)(1-x(t)) - \frac{c_1 x(t) y_m(t)}{x(t)+y_m(t)}$,
$\dot{y}_m(t) = \alpha e^{-d_1\tau} y_m(t-\tau) - d_2 y_m^2(t) - \frac{c_2 y_m(t) z(t)}{y_m(t)+z(t)} + \frac{P_1 x(t) y_m(t)}{x(t)+y_m(t)}$,
$\dot{z}(t) = z(t)\left(-d_3 + \frac{p_2 y_m(t)}{y_m(t)+z(t)}\right)$.
$y_m(t) = \emptyset_m(t) \ge 0, \quad -\tau \le t < 0$ andx $(0) > 0, z(0) > 0$.

3. Boundedness of Solutions

Theorem (3.1): For all $t \ge 0$, all solutions to system (5) with initial conditions (3) and (4) are bounded.

Proof: Starting with the system's first equation (5),
$\dot{x}(t) \le x(1-x)$.
Using the usual comparison principle, we find

$\lim_{t\to\infty} \sup x(t) \leq 1$ for $t \geq 0$.
We get the following from the second equation of system (5),
$\dot{y_m}(t) \leq \alpha e^{-d_1\tau} y_m(t-\tau) - d_2 y_m^2(t)$.
According to lemma (3.1) from [9] and comparison principle $y_m(t) \leq \frac{\alpha e^{-d_1\tau}}{d_2}$ for $t \to \infty$.
Let, $w(t) = \frac{p_1}{c_1} x(t) + y_m(t) + \frac{c_2}{p_2} z(t)$.
Then, we have

$$\begin{aligned}\frac{dw}{dt} =& \frac{p_1}{c_1} x(1-x) + \alpha e^{-d_1\tau} y_m(t-\tau) - d_2 y_m^2(t) - d_3 \frac{c_2}{p_2} z(t) \\ \leq& \frac{2p_1}{c_1} - \min\{1, \varepsilon, d_3\} \left(\frac{p_1}{c_1} x(t) + y_m(t) + \frac{c_2}{p_2} z(t) \right) + \alpha e^{-d_1\tau} y_m(t-\tau) - d_2 y_m^2(t) \\ &+ \varepsilon y_m(t)\end{aligned}$$

As a result, there are positive constants M and T such that
$\frac{dw}{dt} \leq M - \delta w(t)$ for $t \geq T$, where $\delta = \min\{1, \varepsilon, d_2\}$.
Therefore, $\frac{dw}{dt} + \delta w(t) \leq M$.
It is obtained $0 \leq w(x,y,z) \leq \frac{M}{\delta} + \frac{w(x(0),y(0),z(0))}{e^{\delta t}}$, and for $t \to \infty$,
$0 \leq w(t) \leq \frac{M}{\delta}$.
As result, the entire solution set for system (5) enters the region:
$\mathrm{B} = \left\{ (\mathrm{x,y,z}) : 0 \leq \mathrm{w} \leq \frac{M}{\delta} + \xi, \text{ for any } \xi > 0 \right\}$.
Hence complete the theorem.

4. Equilibria and their Stabilities

By setting $\dot{x} = \dot{y}_m = \dot{z}$ in system (5) and solving the following equations, we may get the following equilibria
$x\left(1 - x - \frac{c_1 y_m}{x + y_m}\right) = 0$,
$\alpha e^{-d_1\tau} y_m - d_2 y_m^2 - \frac{c_2 y_m z}{y_m + z} + \frac{P_1 x y_m}{x + y_m} = 0$,
$z\left(-d_3 + \frac{p_2 y_m}{y_m + z}\right) = 0$.
The system (5) has four non-negative equilibrium points.
(1) Equilibrium point $E_0 = (0,0,0)$ is always exist.
(2) In the absence of predators, population of prey reaches its carrying capacity, the equilibrium point $E_1(1,0,0)$ always exists.

The middle predator can subsist on its prey if the top predator is not present. In light of this, the equilibrium point $\widehat{E}\left(\hat{x}, \hat{y}_m, 0\right)$ in the $x - y_m$ plane, where $\hat{x}$ and $\hat{y}_m$ are given by $\hat{y}_m = \frac{\hat{x}(1-\hat{x})}{c_1 - (1-\hat{x})}$, $\hat{x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where
$a = p_1 + d_2 c_1, b = c_1\left(\alpha e^{-d_1\tau} - d_2\right) + 2\left(c_1 - 1\right) p_1$, and $c = c_1\left(c_1 - 1\right) \alpha e^{-d_1\tau} + \left(c_1 - 1\right)^2 p_1$.
(3) The equilibrium point $E^*\left(x^*, y_m^*, z^*\right)$ exists in the interior of the first octant. Where $y_m^* = \frac{x^*(1-x^*)}{c_1 - (1-x^*)}, z^* = \left(\frac{p_2 - d_3}{d_3}\right) y_m^*$. For y_m^* and z^* if it follows that

$$0 < x^* < 1 \text{ and } p_2 > d_3.$$

5. Stability Analysis

Assume $V\left(x, y_m, z\right)$ represent the variational matrix of system (5) at the point $\left(x, y_m, z\right)$ then

$$V\left(x, y_m, z\right) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

where,

$a_{11} = \left(1 - x - \frac{c_1 y_m}{x + y_m}\right) + \left(-1 + \frac{c_1 y_m}{(x + y_m)^2}\right)$, $a_{12} = \frac{-c_1 x^2}{(x + y_m)^2}$, $a_{21} = \frac{p_1 y_m^2}{(x + y_m)^2}$, $a_{22} = \alpha e^{-(d_1+\lambda)\tau} - 2d_2 y_m - \frac{c_2 z^2}{(y_m + z)^2} + \frac{P_1 x^2}{(x + y_m)^2}$, $a_{23} = \frac{-c_2 y_m^2}{(y_m + z)^2}$, $a_{32} = \frac{p_2 z^2}{(y_m + z)^2}$, $a_{33} = \left(d_3 + \frac{p_2 y_m}{y_m + z}\right) - \frac{p_2 y_m z}{(y_m + z)^2}$

System (5) cannot be linearized at $E_0(0,0,0)$ and $E_1(1,0,0)$, therefore local stability of E_0 and E_1 cannot be studied. So, we find the stability of the boundary equilibrium point $\widehat{E}(\hat{x}, \hat{y}, 0)$.Hence equilibrium point $\widehat{E}(\hat{x}, \hat{y}, 0)$ has variational matrix $V(\widehat{E})$, given by

$$V(\widehat{E}) = \begin{bmatrix} \widehat{x}\left(-1 + \frac{c_1 \hat{y}_m}{(\widehat{x}+\hat{y}_m)^2}\right) & \frac{-c_1 \hat{x}^2}{(\widehat{x}+\hat{y}_m)^2} & 0 \\ \frac{p_1 \hat{y}_m^2}{(\widehat{x}+\hat{y}_m)^2} & \alpha e^{-(d_1+\lambda)\tau} - 2d_2 \hat{y}_m + \frac{P_1 \hat{x}^2}{(\widehat{x}+\hat{y}_m)^2} & -c_2 \\ 0 & 0 & p_2 - d_3 \end{bmatrix}$$

The characteristic equation is given by
$\left[\lambda^2 + B_1\lambda + B_2 - (B_3\lambda + B_4)\, e^{-\lambda\tau}\right](p_2 - d_3 - \lambda) = 0.$
where,
$B_1 = \hat{x} + 2d_2\hat{y}_m - \frac{\hat{x}(c_1\hat{y}_m + P_1\hat{x})}{(\hat{x}+\hat{y}_m)^2}$,
$B_2 = 2d_2\hat{y}_m\hat{x} + \frac{c_1 p_1 \hat{x}^2 \hat{y}_m(\hat{x}+\hat{y}_m)}{(\hat{x}+\hat{y}_m)^4} - \frac{P_1\hat{x}^3 + 2d_2 c_1 \hat{x}\hat{y}_m^2}{(\hat{x}+\hat{y}_m)^2}$,
$B_3 = \alpha e^{-d_1\tau}(> 0)$,
$B_4 = \left(\hat{x} - \frac{c_1\hat{x}\hat{y}_m}{(\hat{x}+\hat{y}_m)^2}\right)\alpha e^{-d_1\tau}$.
Clearly, eigen value $\lambda = p_2 - d_3$ is positive in $z-$ direction, since one of the eigen values are positive. Therefore, equilibrium point $\widehat{E}$ is unstable.

The variational matrix $V(E^*)$ corresponding to interior equilibrium point $E^*(x^*, y_m^*, z^*)$ is given by

$$V(E^*) = \begin{bmatrix} x^*\left(-1 + \frac{c_1 y_m^*}{(x^*+y_m^*)^2}\right) & \frac{-c_1 x^{*2}}{(x^*+y_m^*)^2} & 0 \\ \frac{p_1 y_m^2}{(x^*+y_m^*)^2} & J_{22} & -\frac{c_2 y_m^{*2}}{(y_m^*+z^*)^2} \\ 0 & \frac{p_2 z^{*2}}{(y_m^*+z^*)^2} & -\frac{p_2 y_m^* z^*}{(y_m^*+z^*)^2} \end{bmatrix}$$

where $J_{22} = \alpha e^{-(d_1+\lambda)\tau} - \alpha e^{-d_1\tau} - d_2 y_m^* - \frac{p_1 x^* y_m^*}{(x^*+y_m^*)^2} + \frac{c_2 y_m^* z^*}{(y_m^*+z^*)^2}$.
The characteristic equation of the equilibrium point E^* is
$\lambda^3 + M_1\lambda^2 + M_2\lambda + M_3 - \left(M_4\lambda^2 + M_5\lambda + M_6\right) e^{-\lambda\tau} = 0.$
where,

$$\begin{aligned}
M_1 =& x^* + \frac{(p_1 - c_1)\, x^* y_m^*}{\left(x^* + y_m^2\right)^2} + \frac{(p_2 - c_2)\, z^* y_m^*}{\left(z^* + y_m^2\right)^2} + \alpha e^{-d_1 \tau} + d_2 y_m^* (> 0)\\
M_2 =& \frac{(p_2 - c_2)\, x^* y_m^* z^*}{\left(y_m^* + z^*\right)^2} + \frac{\left[p_2 \left(p_1 - c_1\right) + c_2 c_1\right] x^* y_m^{*2}}{\left(y_m^* + z^*\right)^2 \left(x^* + y_m^*\right)^2} + \frac{p_2 y_m^* z^* \left(\alpha e^{-d\tau} + d_2 y_m^*\right)}{\left(y_m^* + z^*\right)^2}\\
& + x^* \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right) + \frac{p_1 x^* y_m^*}{\left(x^* + y_m^*\right)^2} - \frac{c_1 y_m^* x^* \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right)}{\left(x^* + y_m^*\right)^2} (> 0)\\
M_3 =& \frac{p_1 p_2 x^{*2} y_m^{*2} z^*}{\left(y_m^* + z^*\right)\left(x^* + y_m^*\right)^2} + \frac{p_2 x^* y_m^* z^* \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right)\left[\left(x^* + y_m^*\right)^2 - c_1 y_m^*\right]}{\left(y_m^* + z^*\right)^2 \left(x^* + y_m^*\right)^2} (> 0)\\
M_4 =& \alpha e^{-d_1 \tau} (> 0)\\
M_5 =& x^* \alpha e^{-d_1 \tau} + \frac{p_2 z^* y_m^* \alpha e^{-d_1 \tau}}{\left(y_m^* + z^*\right)^2} - \frac{c_1 x^* y_m^* \alpha e^{-d_1 \tau}}{\left(x^* + y_m^*\right)^2} (> 0)\\
M_6 =& \frac{p_2 x^* z^* y_m^* \alpha e^{-d_1 \tau}\left[\left(x^* + y_m^*\right)^2 - c_1 y_m^*\right]}{\left(y_m^* + z^*\right)^2 \left(x^* + y_m^*\right)^2} (> 0)
\end{aligned} \tag{9}$$

Let $\psi(\lambda, \tau) = \lambda^3 + M_1 \lambda^2 + M_2 \lambda + M_3 - \left(M_4 \lambda^2 + M_5 \lambda + M_6\right) e^{-\lambda \tau} = 0$.

To show that the interior equilibrium $E^*\left(x^*, y_m^*, z^*\right)$ is locally asymptotically stable for $\tau > 0$, we use the following theorem

Theorem (4.1): A set of necessary and sufficient conditions for $E * \left(x^*, y_m^*, z^*\right)$ to be asymptotically stable for all $\tau \geq 0$ is

(i) The real parts of all roots of $\psi(\lambda, 0) = 0$ are negative.

(ii) For all real ω_0 and $\tau \geq 0, \psi\left(i\omega_0, \tau\right) \neq 0$ where $i = \sqrt{-1}$.

Theorem (4.2): Assume that $p_1 > c_1, p_2 > c_2$ and $p_2 z^*\left(x^* + y_m^*\right)^2 > c_1 x^*\left(y_m^* + z^*\right)^2$. Then the positive equilibrium of system (5) is asymptotically stable provided that

$$\begin{aligned}
& x^{2^*} + d_2^2 y_m^{2*} + 2 d_2 y_m^* \alpha e^{-d_1 \tau} + \frac{\left(p_1 - c_1\right)^2 x^{*2} y_m^{*2}}{\left(x^* + y_m^*\right)^4} + \frac{\left(p_2 - c_2\right)^2 z^{*2} y_m^{*2}}{\left(y_m^* + z^*\right)^4}\\
+ & \quad \frac{2 p_1 \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right) x^* y_m^*}{\left(x^* + y_m^*\right)^2}\\
> & \frac{2 c_2 \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right) z^* y_m^* \left(x^* + y_m^*\right)^2 + 2 c_1 x^{*2} y_m^* \left(y_m^* + z^*\right)^2 + 2 p_1 c_2 x^* y_m^{*2} z^*}{\left(x^* + y_m^*\right)^2 \left(y_m^* + z^*\right)^2}
\end{aligned}$$

$$\begin{aligned}
& x^{2^*} d_2^2 y_m^{2^*} + 2 x^2 d_2 y_m^* \alpha e^{-d_1 \tau} + \frac{\left(p_2 - c_2\right)^2 x^{2^* z^2 y_m^2 + p_2^2 d_2^2 y_m^4 z^2 + 2 d_2 p_2^2 y_m^3 z^2 z^* \alpha e^{-d_1 \tau}}}{\left(y_m^* + z^*\right)^4}\\
& \frac{\left\{p_2 \left(p_1 - c_1\right) + c_2 c_1\right\}^2 x^{2^*} y_m^{4^* z^2} + 2 p_2^2 p_1 x^* y^{3^*} z^{2^*} \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right)\left(x^* + y_m^*\right)^2 + 2 c_2 c_1 p_1 x^{3^*} y^{3^*} z^*}{\left(x^* + y_m^*\right)^4 \left(y_m^* + z^*\right)^4}\\
+ & \frac{p_1 x^{4^*} y_m^{2*} + c_1^2 x^{2^*} \left(d_2^2 y_m^{2^*} + 2 d_2 y_m^* \alpha e^{-d_1 \tau}\right) + 2 p_1 y_m^* x^{3^*} \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right)\left(x^* + y_m^*\right)^2}{\left(x^* + y_m^*\right)^4} >\\
& \frac{2 p_1 c_1 x^{3^*} y_m^{2^*} \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right)}{\left(x^* + y_m^*\right)^4} + \frac{2 c_2 x^{2^*} y_m^* z^* \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right)}{\left(y_m^* + z^*\right)^2} + \frac{2 c_1 d_2 x^{2^*} y_m^* \left(2 \alpha e^{-d_1 \tau} + d_2 y_m^*\right)}{\left(x^* + y_m^*\right)^2}\\
& \frac{\left(p_2 - c_2\right)^2 c_1 x^{2^*} z^{2^*} y_m^{3^*} \left(x^* + y_m^*\right)^2 + 2 c_2 c_1^2 y_m^{3^*} x^{2^*} z^* \left(\alpha e^{-d_1 \tau} + d_2 y_m^*\right)\left(y_m^* + z^*\right)^2}{\left(x^* + y_m^*\right)^4 \left(y_m^* + z^*\right)^4}
\end{aligned}$$

Proof. We now apply theorem (4.1) to prove the theorem (4.2). We prove this theorem in two steps.

Step 1. Substituting $\tau = 0$ in equation (7), we get
$\psi(\lambda, 0) = \lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 = 0$,
where,

$$P_1 = x^* + \frac{(p_1 - c_1)\, x^* y_{m0}^*}{(x^* + y_{m0}^*)^2} + \frac{(p_2 - c_2)\, z^* y_{m0}^*}{(z^* + y_{m0}^*)^2} + d_2 y_{m0}^* (> 0), P_2$$
$$= x^* d_2 y_{m0}^* + \frac{(p_2 - c_2)\, x^* y_{m0}^* z^*}{(y_{m0}^* + z^*)^2} + \frac{(p_2 (p_1 - c_1) + c_2 c_1)\, x^* y_{m0}^{2*} z^*}{(y_m^* + z^*)^2 (x^* + y_{m0}^*)^2}$$
$$+ \frac{p_1 x^{*2} y_{m0}^*}{(x^* + y_{m0}^*)^2} + \frac{d_2 {y_{m0}^*}^2 \left\{ p_2 z^* (x^* + y_{m0}^*)^2 - c_1 x^* (y_{m0}^* + z^*)^2 \right)}{(y_{m0}^* + z^*)^2} (> 0)$$
$$P_3 = \frac{p_2 d_2 x^* z^* {y_{m0}^*}^2 \left\{ (x^* + y_{m0}^*)^2 - c_1 y_{m0}^* \right\} + p_2 p_1 x^{*2} z^* {y_{m0}^*}^2}{(x^* + y_{m0}^*)^2 (y_{m0}^* + z^*)^2} (> 0)$$

Here we have used $y_{m0}^* = y_m^* \mid \tau = 0$.
Therefore, by Routh - Hurwitz criterion, all roots of equation (10) have negative real parts. Hence, condition (i) of theorem (4.1) is satisfied and E^* is a locally asymptotically stable equilibrium in the absence of delay.

Step 2. Suppose that $\psi(i\omega_0, \tau) = 0$, holds for some real ω_0.

When $\omega_0 = 0$, we have
$\psi(0, \tau) = M_3 - M_6 \neq 0$.
Now suppose $\omega_0 \neq 0$,
$\psi(i\omega_0, \tau) = -i\omega_0^3 - M_1\omega_0^2 + iM_2\omega_0 + M_3 - \left(-M_4\omega_0^2 + iM_5\omega_0 + M_6\right) e^{-i\omega_0\tau} = 0$.

Equating real and imaginary parts of equation (10), we get
$-M_1\omega_0^2 + M_3 = \left(-M_4\omega_0^2 + M_6\right)\cos\omega_0\tau + M_5\omega_0 \sin\omega_0\tau$,
$-\omega_0^3 + M_2\omega_0 = M_5\omega_0\cos\omega_0\tau - \left(-M_4\omega_0^2 + B_6\right)\sin\omega_0\tau$.

Squaring and adding equations (12) and (13), we get
$\omega_0^6 + \left(M_1^2 - 2M_2 - M_4^2\right)\omega_0^4 + \left(M_2^2 - 2M_1M_3 + 2M_4M_6 - M_5^2\right)\omega_0^2 + \left(M_3^2 - M_6^2\right) = 0$.

$$M_1^2 - 2M_2 - M_4^2 = x^{2^*} + d_2^2 {y_m^2}^* + 2d_2 y_m^* \alpha e^{-d_1\tau} + \frac{(p_1 - c_1)^2 x^{2^*} {y_m^2}^*}{(x^* + y_m^*)^4} + \frac{(p_2 - c_2)^2 z^{2^*} {y_m^2}^*}{(y_m^* + z^*)^4} \tag{14}$$

$+\frac{2p_1\left(\alpha e^{-d_1\tau} + d_2 y_m^*\right) x^* y_m^*}{(x^* + y_m^*)^2} - \frac{2c_2\left(\alpha e^{-d_1\tau} + d_2 y_m^*\right) z^* y_m^* (x^* + y_m^*)^2 + 2c_1 x^{2^*} y_m^* (y_m^* + z^*)^2 + 2p_1 c_2 x^* y_m^2 z^*}{(x^* + y_m^*)^2 (y_m^* + z^*)^2} (> 0)$, if condition (i) holds.

$$M_3 - M_6 = \frac{p_2 d_2 x^* z^* y_m^{*2}\left\{(x^*+y_m^*)^2 - c_1 y_m^*\right\} + p_2 p_1 x^{*2} z^* y_m^{*2}}{(x^*+y_m^*)^2 (y_m^*+z^*)^2} (>0) \Rightarrow M_3^2 - M_6^2 > 0.$$

$$M_2^2 - 2M_1M_3 + 2M_4M_6 - M_5^2$$

$$= x^{2^*} d_2^2 y_m^{2^*} + 2x^2 d_2 y_m^* \alpha e^{-d_1\tau} - \frac{2p_1c_1 x^{3^*} y_m^{2\,*}\left(\alpha e^{-d_1\tau} + d_2 y_m^*\right)}{(x^*+y_m^*)^4}$$

$$\frac{\{p_2(p_1-c_1)+c_2c_1\}^2 x^{2^*} y_m^{4^* z^2} + 2p_2^2 p_1 x^* y^{3^*} z^{2^*}\left(\alpha e^{-d_1\tau} + d_2 y_m^*\right)(x^*+y_m^*)^2 + 2c_2c_1p_1 x^{3^*} y^{3^*} z^*}{(x^*+y_m^*)^4 (y_m^*+z^*)^4}$$

$$+\frac{(p_2-c_2)^2 x^{2^* z^{2^*} y_m^{2\;2} + p_2^2 d_2^2 y_m^{4^* z^2} + 2d_2 p_2^2 y_m^{3^* z^2} \alpha e^{-d_1\tau}}}{(y_m^*+z^*)^4}$$

$$-\frac{2c_2 x^{2^*} y_m^* z^*\left(\alpha e^{-d_1\tau} + d_2 y_m^*\right)}{(y_m^*+z^*)^2} - \frac{2c_1 d_2 x^{2^*} y_m^*\left(2\alpha e^{-d_1\tau} + d_2 y_m^*\right)}{(x^*+y_m^*)^2}$$

$$+\frac{p_1 x^{4^*} y_m^{2*} + c_1^2 x^{2^*}\left(d_2^2 y_m^{2^*} + 2d_2 y_m^* \alpha e^{-d_1\tau}\right) + 2p_1 y_m^* x^{3^*}\left(\alpha e^{-d_1\tau} + d_2 y_m^*\right)(x^*+y_m^*)^2}{(x^*+y_m^*)^4}$$

$$\frac{(p_2-c_2)^2 c_1 x^{2^* z^2 y_m^3 *}(x^*+y_m^*)^2 + 2c_2 c_1^2 y_m^{3^*} x^{2^* z^*}\left(\alpha e^{-d_1\tau} + d_2 y_m^*\right)(y_m^*+z^*)^2}{(x^*+y_m^*)^4 (y_m^*+z^*)^4}$$

provided that both conditions of theorem (4.2) hold.
Hence, we have $M_1^2 - 2M_2 - M_4^2 > 0, M_2^2 - 2M_1M_3 + 2M_4M_6 - M_5^2 > 0$ and $M_3^2 - M_6^2 > 0$.
It follows that
$\omega_0^6 + \left(M_1^2 - 2M_2 - M_4^2\right)\omega_0^4 + \left(M_2^2 - 2M_1M_3 + 2M_4M_6 - M_5^2\right)\omega_0^2 + \left(M_3^2 - M_6^2\right) > 0$.
This contradicts with (14). Hence $\varphi(i\omega_0, \tau) \neq 0$. For any real ω_0, it satisfies condition (ii) of theorem (4.1). Therefore, the unique positive equilibrium $E^*(x^*, y_m^*, z^*)$ is locally asymptotically stable for all $\tau \geq 0$ and the delay is harmless in this case.

6. Bifurcation Analysis

Separating real and imaginary parts after putting $\lambda = \alpha(\tau) + i\beta(\tau)$ in equation (7), we obtain following transcendental equations
$e^{-\alpha\tau}\left[M_4\left(\alpha^2-\beta^2\right) + \alpha M_5 + M_6\right]\cos\beta\tau + e^{-\alpha\tau}\left(2\alpha\beta M_4 + \beta M_5\right)\sin\beta\tau = \alpha^3 - 3\alpha\beta^2 + M_1\left(\alpha^2-\beta^2\right) + M_2\beta + M_3$,

$$e^{-\alpha\tau}\left[M_4\left(\alpha^2-\beta^2\right) + \alpha M_5 + M_6\right]\sin\beta\tau - e^{-\alpha\tau}\left(2\alpha\beta M_4 + \beta M_5\right)\cos\beta\tau = \beta^3 - 3\alpha^2\beta - \quad (15)$$

$2M_1\alpha\beta - M_2\beta$.

Now, we will discuss about the change of stability of E^* at the values of τ for which $\alpha = 0$ and $\beta \neq 0$.Let τ^* be such that for which $\alpha(\tau^*) = 0$ and $\beta(\tau^*) = \beta^* \neq 0$. Then equations (15) and (16) can be written as
$\left(M_6 - M_4\beta^{*2}\right)\cos\beta^*\tau^* + M_5\beta^*\sin\beta^*\tau^* = M_3 - M_1\beta^{*2}$,
$\left(M_6 - M_4\beta^{*2}\right)\sin\beta^*\tau^* - M_5\beta^*\cos\beta^*\tau^* = M_2\beta^* - \beta^{*3}$.

Now eliminating τ^* from (17) and (18), we get
$\beta^{*6} + S_1\beta^{*4} + S_2\beta^{*2} + S_3 = 0$.

Where $S_1 = M_1^2 - 2M_2 - M_4^2, S_2 = M_2^2 - 2M_1M_3 + 2M_4M_6 - M_5^2, S_3 = M_3^2 - M_6^2$.
Now, we examine the sign of $\frac{d\alpha}{d\tau}$ as α crosses zero. Differentiating equations (15) and (16) with respect to τ and putting $\tau = \tau^*, \alpha = 0$ and $\beta = \beta^*$, we get
$\theta_1\frac{d\alpha}{d\tau}(\tau^*) + \theta_2\frac{d\beta}{d\tau}(\tau^*) = g$,
$-\theta_2\frac{d\alpha}{d\tau}(\tau^*) + \theta_1\frac{d\beta}{d\tau}(\tau^*) = h$,
where,

$$\theta_1 = \left(M_6 - M_4\beta^{*2}\right)\tau^* \cos\beta^*\tau^* - M_5\cos\beta^*\tau^* + M_5\beta^*\tau^*\sin\beta^*\tau^* + M_2 - 3\beta^{*2} \tag{22}$$

$-2M_4\beta^*\sin\beta^*\tau^*$,

$$\theta_2 = \left(M_6 - M_4\beta^{*2}\right)\tau^* \sin\beta^*\tau^* - M_5\sin\beta^*\tau^* - M_5\beta^*\tau^*\cos\beta^*\tau^* + M_2 - 2M_1\beta^* \tag{23}$$

$+2M_4\beta^*\sin\beta^*\tau^*$,
$g = M_5\beta^{*2}\cos\beta^*\tau^* - \left(M_6 - M_4\beta^{*2}\right)\beta^*\sin\beta^*\tau^*$,
$h = -M_5\beta^{*2}\sin\beta^*\tau^* - \left(M_6 - M_4\beta^{*2}\right)\beta^*\cos\beta^*\tau^*$.

Solving (20) and (21), we get
$\frac{d\alpha}{d\tau}(\tau^*) = \frac{g\theta_1 - h\theta_2}{\theta_1^2+\theta_2^2}$.

Equation (26) implies that $\frac{d\alpha}{d\tau}(\tau^*)$ has the same sign as $(g\theta_1 - h\theta_2)$.
Solving equations (17) and (18) with using equations (22)-(25), we get
$g\theta_1 - h\theta_2 = \beta^{*2}\left[3\beta^{*4} + 2S_1\beta^{*2} + S_2\right]$.

After assuming $\beta^{*2} = u$,equation (19) can be written as
$G(u) = u^3 + S_1u^2 + S_2u + S_3 = 0$.

Now
$\frac{dG\left(\beta^{*2}\right)}{du} = \frac{g\theta_1 - h\theta_2}{\beta^{*2}} = \frac{\theta_1^2+\theta_2^2}{\beta^{*2}}\frac{d\alpha}{d\tau}(\tau^*)$.
This implies that,
$\frac{d\alpha}{d\tau}(\tau^*) = \frac{\beta^{*2}}{\theta_1^2+\theta_2^2}\frac{dG\left(\beta^{*2}\right)}{du}$.

From equation (28), we have the following theorems
Theorem (5.1): For $\tau = 0$, if E^* is unstable with $S_3 < 0$, it will remain unstable for $\tau > 0$.
Theorem (5.2): For $\tau = 0$, if E^* is asymptotically stable with $S_3 < 0$, it is not possible that it remains stable for $\tau > 0$. Hence there exists a $\tau^* > 0$, such that for $\tau < \tau^*, E^*$ is asymptotically stable and for $\tau > \tau^*, E^*$ is unstable, and as τ increases together with $\tau^*.E^*$ bifurcates into small amplitude periodic solutions of Hopf type[11]. The value of τ^* is given by the following equation

$$\tau^* = \frac{1}{\beta^*}\sin^{-1}\left[\frac{\left(\beta^{*3} - M_2\beta^*\right)\left(M_6 - M_4\beta^{*2}\right) - \left(M_1\beta^{*2} - M_3\right)M_5\beta^*}{\left(M_6 - M_4\beta^{*2}\right)^2 + M_5^2\beta^{*2}}\right]$$

7. Persistence

Theorem (6.1): The system is permanent with conditions $\alpha e^{-d_1\tau} > c_2, p_2 > d_3$ and $c_1 < 1$.
Proof: We can write from equations of the system (5),

$$\begin{aligned}\dot{y}_m(t) &\geq \alpha e^{-d_1\tau}y_m(t-\tau) - d_2y_m^2(t) - \frac{c_2y_m(t)z(t)}{y_m(t)+z(t)} \\ &\geq \alpha e^{-d_1\tau}y_m(t-\tau) - d_2y_m^2(t) - c_2y_m(t)\end{aligned}$$

Using comparison principle and lemma [3.1], we get
$\lim_{t\to\infty}\inf y_m(t) \geq \frac{\alpha e^{-d_1\tau} - c_2}{d_2}(> 0)$.
First equation of the system (5) can be written as,

$$\begin{aligned}\dot{x}(t) &\geq x\left(1 - x - \frac{c_1y_{\max}}{x + y_{\min}}\right) \\ &\geq x\left(1 - x - c_1 + c_1\frac{x\delta}{M}\right)\end{aligned}$$

This implies that
$\lim_{t\to\infty}\inf x(t) \geq \frac{(1-c_1)M}{M - c_1\delta}(> 0)$.
We get $c_1 < 1$.
Using the third equation of system (5), we have
$\dot{z}(t) \geq -d_3z + p_2z - \frac{p_2d_2z^2}{\alpha e^{-d_1\tau} - c_2}$,

$\lim_{t\to\infty} \inf z(t) \geq \frac{(p_2-d_3)p_2d_2}{\alpha e^{-d_1\tau}-c_2}(>0)$.
This completes the evidence of the theorem.

8. Remark

(i) If $c_1 > 1$, condition of permanence will not be satisfied, and this implies that if the predator's capturing rate is greater than one, the population of prey population tends to extinction.
(ii) If c_2 (capturing rate of top predators) is larger, then the condition of permanence will not be satisfied.
(iii) If d_3, the death rate of top predators is greater than the conversion rate for predation by top predators then the system is not permanent.

9. Numerical Simulation

In this study, system (7) is numerically integrated using MATLAB with a selected set of consistent parameters. This approach facilitates understanding the mathematical conclusions through numerical simulation. The parameter values for the numerical analysis are set as
$\alpha = 2.5, d_1 = 0.2, \tau = 1.5, c_1 = 0.6, c_2 = 1.5, d_2 = 3.1, d_3 = 1.3, p_1 = 2, p_2 = 2.4.$

For the above-assumed set of parameters, the equilibrium point E^* is provided by

$$x^* = 0.7003741, y_m^* = 0.6986284, z^* = 0.591147108$$

All the conditions for stability and permanence specified in Theorems (4.1) and (6.1) are satisfied. Consequently, the equilibrium point E^* is locally asymptotically stable for $\tau \geq 0$. This means that small perturbations around the equilibrium point will decay over time, and the system will return to the equilibrium state. In this case, the time delay τ, which typically adds complexity and can lead to instability or oscillations, does not adversely affect the system's stability.

Figures have been plotted between variables and time for various parameter values to demonstrate population changes through time under varied settings. Interior equilibrium points existence, stability, and persistence τ, c_1, c_2, p_1 and p_2 are all listed as key parameters. Figure (1) displays the numerical simulation results, with the prey, intermediate, and top predator populations displayed against time. According to the diagram, for given initial values, each population will converge to their respective equilibrium points E^* and coexist in a steady state, ensuring local stability of E^*.

Now, if we define the system parameters (7) as, $\alpha = 2.5, d_1 = 0.2, \tau = 1.5, c_1 = 0.6, c_2 = 1.5, p_1 = 2, p_2 = 2.4, d_2 = 3.1, d_3 = 2.4$ and initial values of x, y, z as $0.12, 0.3$ and 0.5 respectively.

In Figure 2 it is illustrates that the top predator populations become extinct as the first predator's population grows and prey population decreases. As a result, equilibrium point E^* is asymptotically stable at the local level. Figure 3(a-c) shows the evolution of prey, mature intermediate predators, and top predators through time for varied maturation time delays. It is observed that as maturation time delay rises, equilibrium value of mature prey populations' also rises while decreases in mature mid-predator and top predator populations. From this behavior of figure (3a) it can be interpreted that if the time taken by the immature (larva) to become adult rises, then prey population (plant) increases because only mature predator (pest) preys on the prey (plant) population. From fig.(3b) and (3c), we observe that the mature intermediate predator's population decreases while with the rise in τ (maturation period) the top predator population lacks. This is obvious as for larger values of τ if the population of middle predator's declines and so the population of top predators (natural enemies of pests) that depends only on mature predators. Figure 4(a-c) depicts the variation of x, y_m and z over time for various maturation rates of the middle predator population. From the figures, it can be interpreted that the equilibrium level of the three populations increases with the rises in rate of maturation of mature predator. Figures 5(a-c) show how populations of prey, mature intermediate predators, and top predators change over a period for various values of c_1. The graphic shows that each population declines with a rise in c_1. It follows that when prey population decline, so does the population of intermediate and top predators. Prey population tends to go extinct at $c_1 = 1.2$. Figure 6$(\mathrm{a}-\mathrm{c})$ shows that y_m and z are decreasing functions of c_2, the captures rate of top predators, but x is increasing function of c_2, which is clear since as middle predator population decreases, prey population increases, and therefore top predator population decreases. Figure 7(a-c) shows that when p_1 grows, mature middle predators and top predator populations rise and eventually reach their equilibrium levels. Prey population decline. The population of prey declines as the pace of conversion for predation of prey by mature predators grows, whereas the number of top and midrange predators rises. Figure 8(a-c) illustrates the changes in prey population, mature predator population, and top predator populations over time for different values of p_2. The graph shows that as the pace of conversion of mature predator predated by top predator grows, the populations of mature predators, top predators, and prey increase. The variance between the people is presented in Figure 9(a)-(c) for various starting starts of 1, 2, 3, and 4. The graph shows that whatever is the initial condition, the solution converges to the equilibrium point E^*. The variance of mature populations with a mature predator to prey, prey to top predator, and mature predator to top predator populations is presented in Figure 9(a)-(c) for various initial conditions of $1, 2, 3$, and 4 . The graph indicates that at different starting points, the solution converges to the equilibrium point, displaying global stability E^*.

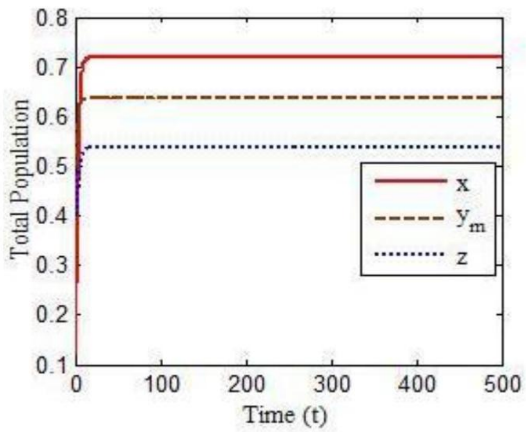


Fig. 1 Depicts the steady behavior of ' x, y_m and z '

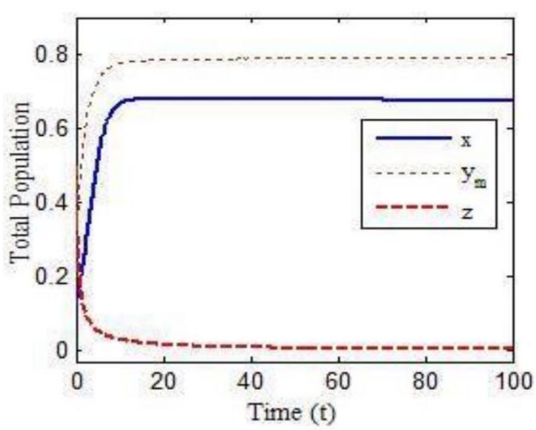
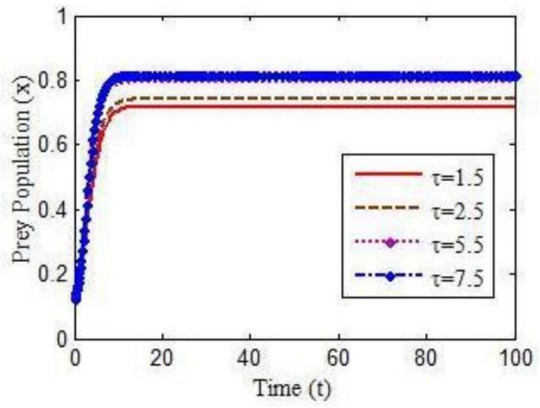
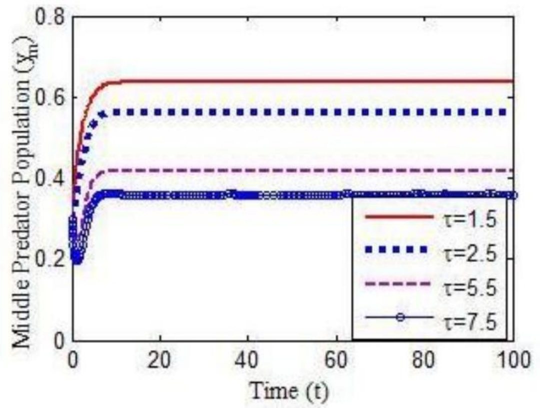


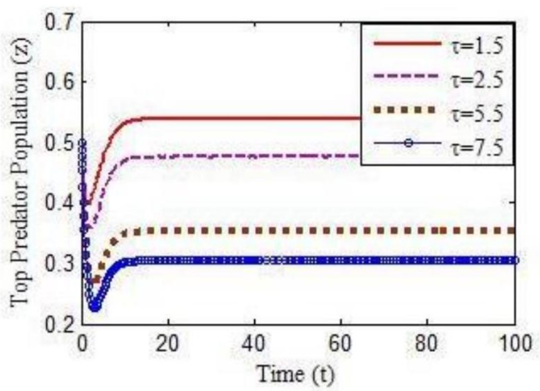
Fig. 2 Depicts how ' x, y_m ' populations tend their equilibrium levels in time, while the ' z ' extinct.



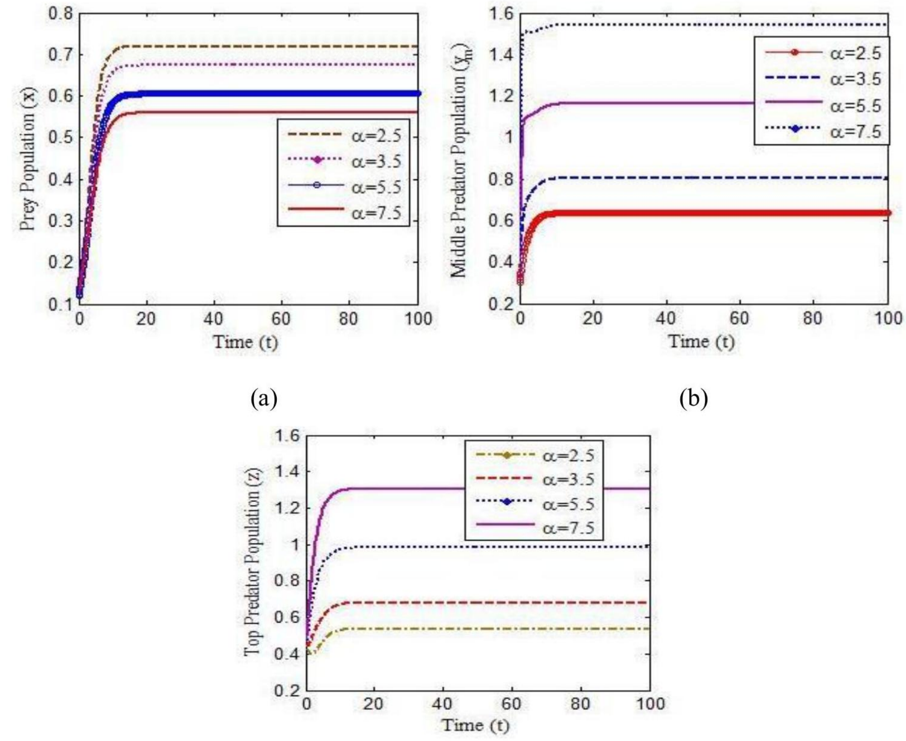
3(a): Populations change over time for varying values of ' τ '



3 (b): Populations change over time for varying values of ' τ '



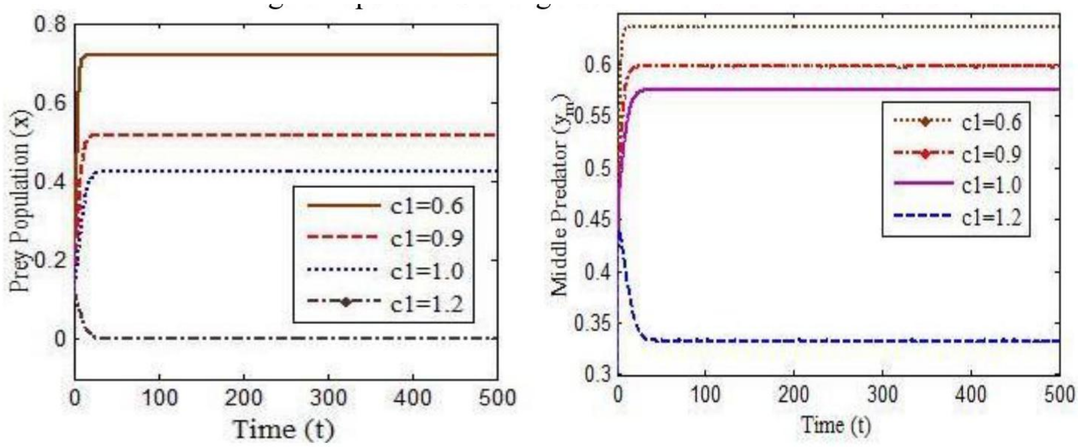
3 (c): Populations change over time for varying values of ' τ '



(a) (b)

(c)

Fig.4. Populations change over time for different values of ' α '.



(a) (b)

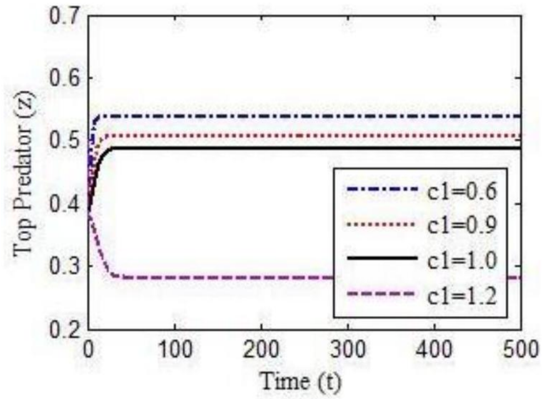


Fig. 5(a-c). Populations change over time for varying values of ' c_1 '.

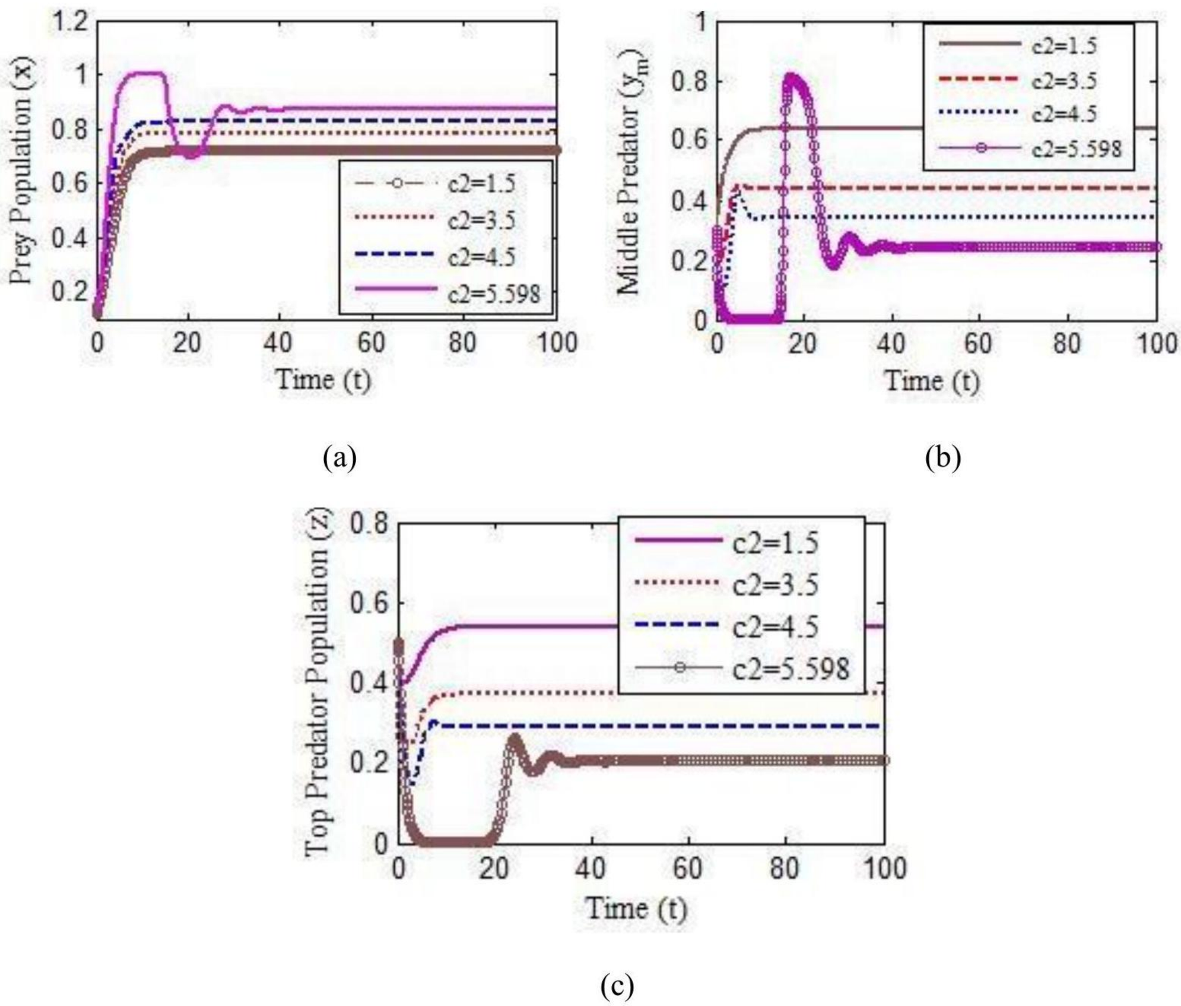


Fig. 6(a-c). Populations through time for varying values of ' c_2 ' while other parameter values remain the same.

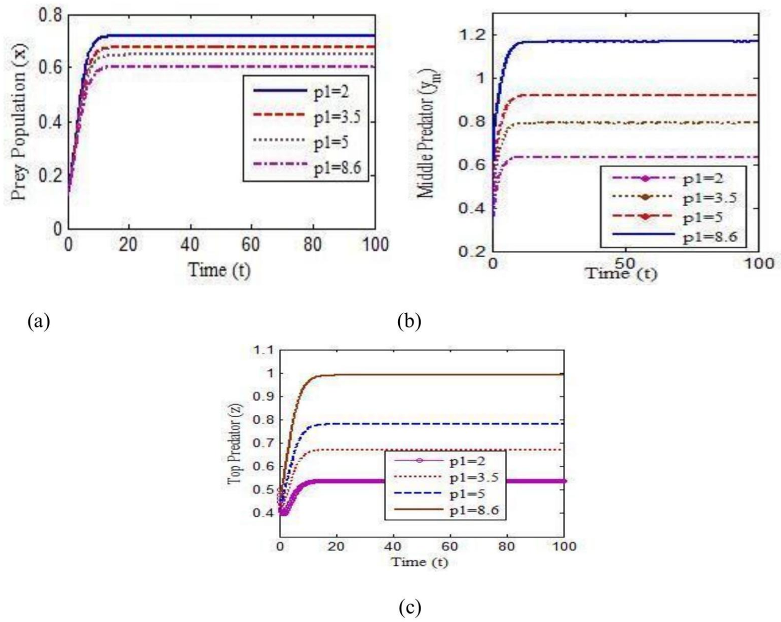


Fig. 7(a-c). Populations through time for varying values of ' p_1 ', while other parameter values remain the same.

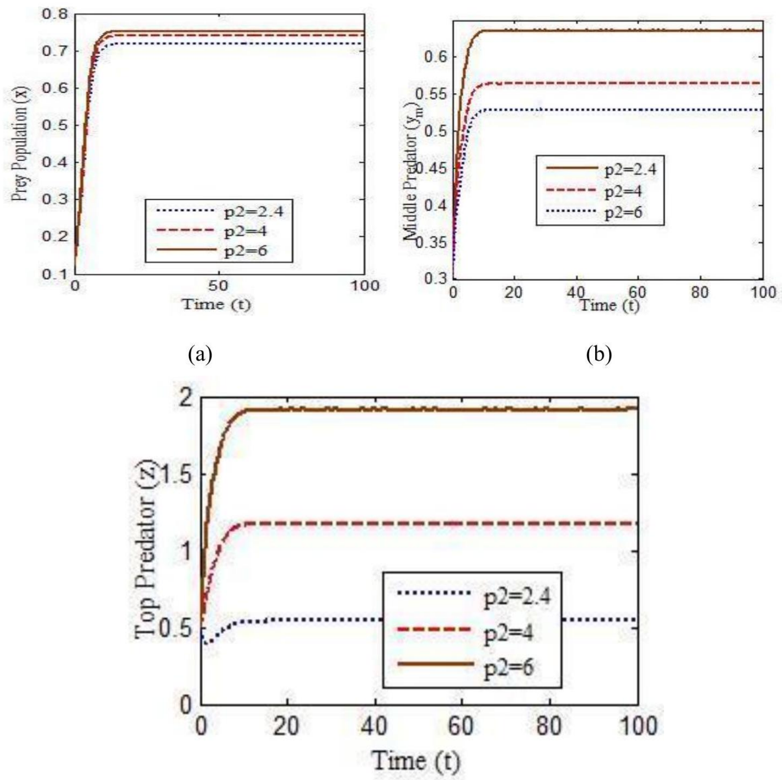


Fig. 8(a-c). Populations through time for varying values of ' p_2 ', while all other parameter values remain the same.

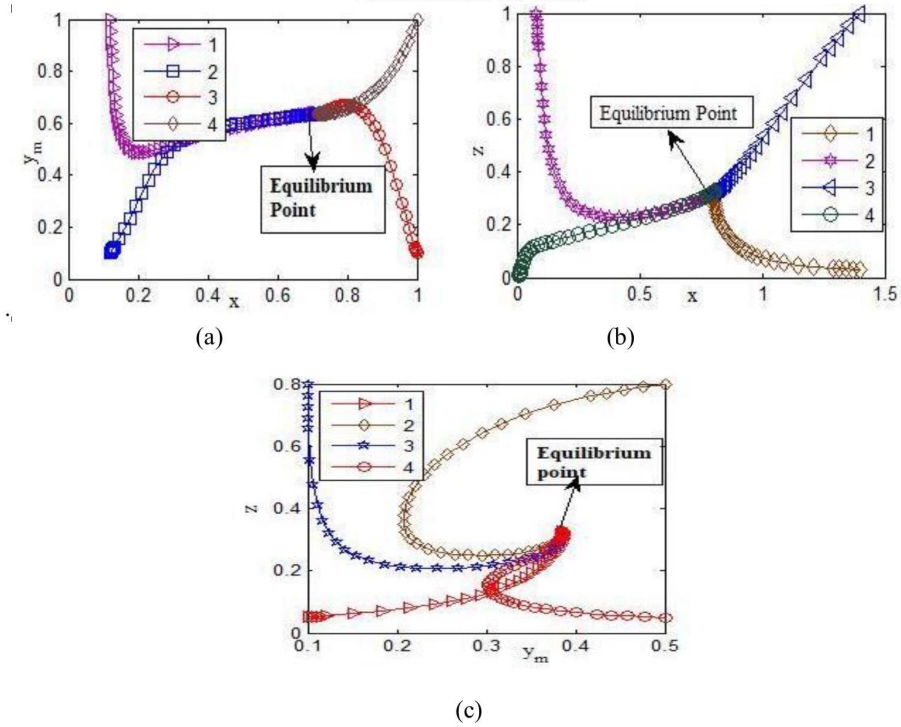


Fig. 9(a-c). Variations of ' xv/sy_m ', ' y_mv/sz' and ' xv/sz ' for various beginning starts for the set of parameters provided in (30).

10. Conclusion

This study presents a food chain model that incorporates a middle predator, divided into two stages-immature and adult-separated by a fixed time delay. The dynamics of the system are described by a set of four nonlinear differential equations. The model explores the interactions between prey, the middle predator in its two stages, and possibly a top predator. The investigation focuses on determining the equilibrium stability, where populations remain constant, and the persistence criteria, which assess the long-term survival and coexistence of species within the food chain.

This study proposes and discusses a food chain model that includes a middle predator divided into two parts by a set time delay: immature and adult. A set of four nonlinear differential equations is used to explain the situation. Equilibrium stability and persistence criteria were used in the investigation.

After investigating the system's boundedness of solutions, equilibria, and their stability, the existence conditions for the system's equilibrium points are established. The linear stability approach is used to analyze the stability and instability of the equilibrium points. Also, a bifurcation study of the system is performed to determine its stability and instability under delay. The system's criteria for long-term survival (population persistence) are biologically understood, and the conditions that impact the persistence of all populations are derived. $\alpha e^{-d_1\tau} > c_2, c_1 < 1$ and $p_2 > d_3$. These findings suggest that the top predator's capturing, conversion, and mortality rates all have an essential impact in the long-term sustainability of the solutions.

It is found through computer simulations that if maturation time increases, the population of prey rises, and the population of mid and top predators lacks. Moreover, the system becomes unstable, and the prey population is more likely to go extinct when the middle predator's capture rate exceeds by 1 . Hence, to control mature middle predators (pest) in prey (plant), attempt should be made to increase the value of maturation delay and decrease the maturation rate of a mature middle predator, which enhances the fitness of the prey population. It is also observed that large value parameter p_2 and c_2 Increasing the number of prey and parameter is a critical parameter that we must regulate to keep the prey population from extinction.

11. Declarations

Ethical Approval: Not Required

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Conflicts of Interest: Authors declare that they have no conflicts of interest to report regarding the present study.

Authors' contributions: All the authors have equally contributed.

Data and Code Availability Not Applicable

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