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Convergence of Multivalued Martingales: Application to Multivalued Uniform Amarts

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ABSTRACT: In this work, we prove a convergence theorem for bounded martingales taking values in a Banach space Y, without requiring Y to have the Radon Nikodym Property. We then extend this result to martingales with values in $ccb(Y)$ (the set of nonempty, convex, closed, and bounded subsets of Y) under several topologies. As applications, we show the convergence of multivalued uniform asymptotic martingales with values in $ccb(Y)$.

Keywords: Mutifunction, martingale, uniform amart, Radon Nikodym Property and Mosco-convergence.

Contents

1. Introduction

The convergence of martingales has been the subject of extensive study. One of the early results in this direction is due to Neveu, who proved the almost sure convergence of real-valued bounded martingales in L^1 [16]. This was later extended by Chatterji to bounded vector-valued martingales in L^1_Y, where Y is a Banach space with the Radon Nikodym Property (RNP) [6]. Bourgin subsequently established a similar result for bounded vector-valued martingales in L^1_Y, where Y does not necessarily have the RNP, provided the martingale takes values in a convex, closed, and bounded subset $H \subset Y$ that satisfies the RNP [2]. In the present work, we further extend these results to bounded vector-valued martingales in L^1_Y taking values in a convex, closed subset $H \subset Y$, where Y lacks the RNP and H satisfies the RNP but is not necessarily bounded. Furthermore, this result is extended to the convergence of multivalued martingales with values in $ccb(Y)$, under both Mosco convergence and convergence in the linear topology. The results presented here extend several theorems established in the literature, including Theorem 2.8 in [7] and Theorem 3.3 in [17]. For more results of multivalued martingales, see [1,7,11,10,21]. As applications, we establish the convergence again in the sense of Mosco and the linear topology of multivalued uniform amarts taking values in $ccb(Y)$. This improves Theorem 4.2 from [8], theorem 3.3 from [9] and theorem 3.1 from [19].

2. Preliminaries

Throughout this paper, $(\Omega, \mathcal{A}, \mathbf{P})$ denotes a complete probability space and $(\mathcal{A}_n)_{n\geq 1}$ a nondecreasing sequence of sub-σ-algebras of $\mathcal{A}$ such that $\mathcal{A}$ is generated by $\bigcup_{n\geq 1} \mathcal{A}_n$. Y denotes a linear separable Banach space, and Y^* its topological dual. We denote by $\overline{B}_Y$ (respectively $\overline{B}_{Y^*}$) the closed unit ball of Y (respectively of Y^*), and by 2^Y the collection of all subsets of Y. In what follows, we use the following notations :

$$cc(Y) = \{B \subseteq Y : \text{nonempty convex and closed }\},$$
$$ccb(Y) = \{B \subseteq Y : \text{nonempty, convex, closed and bounded }\}$$

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and

$$cwk(Y) = \{B \subseteq Y : \text{nonempty, convex and weakly compact }\}.$$

Also for each $B \in 2^Y \setminus \{\emptyset\}$, $\overline{co}B$ denote the *norm-closed convex hull* of B.
The *distance function* associated with a set $B \subset Y$ is defined by

$$d(y, B) := \inf_{b \in B} \|y - b\|, \qquad y \in Y,$$

while the *support function* of B is given by

$$\delta^*(y^*, B) := \sup_{b \in B} \langle y^*, b\rangle, \qquad y^* \in Y^*,$$

and the *Hausdorff norm* of B is defined by

$$|B| = \sup_{b \in B} \|b\|.$$

A multivalued mapping $F : \Omega \to cc(Y)$ is said to be $\mathcal{A}$-measurable multifunction if, for every open set $O \subset Y$, the set

$$F^-(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \text{ belongs to } \mathcal{A}.$$

A function $f : \Omega \to Y$ is called an $\mathcal{A}$-measurable selector of F if

$$f(\omega) \in F(\omega), \qquad \text{for all } \omega \in \Omega.$$

A sequence $(g_n)_{n\geq 1}$ of $\mathcal{A}$-measurable selectors of F is called a *Castaing representation* of F if, for every $\omega \in \Omega$,

$$F(\omega) = \overline{\{g_n(\omega) : n \in \mathbb{N}^*\}}^{\|.\|}.$$

Let $H \in cc(Y)$. L^1_Y (resp. L^1_H) (resp. $L^1_{\mathbb{R}}$) denotes the space of $\mathcal{A}$-measurable and Bochner integrable functions defined from Ω to Y (resp. from Ω to H) (resp. from Ω to $\mathbb{R}$) .
F is integrable if $d(0, F(.))$ is in $L^1_{\mathbb{R}}$.
Let $\mathcal{L}^0_{ccb(Y)}(\mathcal{A})$ be the set of all $\mathcal{A}$-measurable multifunctions from $(\Omega, \mathcal{A})$ to $ccb(Y)$.

$$\mathcal{L}^{d_1}_{ccb(Y)}(\mathcal{A}) = \{F \in \mathcal{L}^0_{ccb(Y)}(\mathcal{A}) \quad \text{such that} \quad d(0, F(.)) \in L^1_{\mathbb{R}}\},$$
$$\mathcal{L}^1_{ccb(Y)}(\mathcal{A}) = \{F \in \mathcal{L}^0_{ccb(Y)}(\mathcal{A}) \quad \text{such that} \quad |F| \in L^1_{\mathbb{R}}\},$$

$$S^1_F(\mathcal{A}) = \{g \in L^1_Y(\mathcal{A}) \quad \text{such that} \quad g(\omega) \in F(\omega) \quad \text{a.s.}\}.$$

A multifunction F is said to be integrable if and only if $S^1_F(\mathcal{A}) \neq \emptyset$.
Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{A}$, and let $g : \Omega \to Y$ be an integrable random variable. The conditional expectation of g with respect to $\mathcal{B}$ is defined as the (almost surely unique) $\mathcal{B}$-measurable and integrable random variable h satisfying

$$\int_A g(\omega)\, d\mathbf{P}(\omega) = \int_A h(\omega)\, d\mathbf{P}(\omega), \qquad \text{for all } A \in \mathcal{B}.$$

Whenever such a conditional expectation exists, it is denoted by $E^{\mathcal{B}}g$. In the multivalued case, the existence of the conditional expectation of
$F \in \mathcal{L}^{d_1}_{ccb(Y)}(\mathcal{A})$ is defined by

$$S^1_{E^{\mathcal{B}}F}(\mathcal{B}) = \overline{\{E^{\mathcal{B}}g : g \in S^1_F(\mathcal{A})\}},$$

where the closure is taken in the space L^1_Y. Further properties of integrable multifunction are discussed in [13,20].
The Hausdorff metric $h(.,.)$ between the sets C_1, $C_2 \in ccb(Y)$ is defined as a uniform distance between the functions $d(., C_1)$ and $d(., C_2)$ by

$$h(C_1, C_2) = \sup_{y \in Y} |d(y, C_1) - d(y, C_2)|.$$

By corollary 3.2.8 of [4] for any $C_1, C_2 \in ccb(Y)$, the *Hausdorff distance* between C_1 and C_2 is also defined by

$$h(C_1, C_2) = \sup_{y^* \in \overline{B}_{Y^*}} |\delta^*(y^*, C_1) - \delta^*(y^*, C_2)|.$$

Definition 2.1 *Let $\sigma : \Omega \to \overline{\mathbb{N}^*}$ a function. This function is a stopping time relative to $(\mathcal{A}_n)_{n\geq 1}$ if for all $n \geq 1$*

$$[\sigma = n] \in \mathcal{A}_n.$$

We specify $\mathcal{A}_\sigma$ by

$$\mathcal{A}_\sigma = \{A \in \mathcal{A} : A \cap [\sigma = n] \in \mathcal{A}_n \quad \text{for all} \quad n \geq 1\}.$$

And

$$\text{for each } \omega \in \Omega,\ F_\sigma(\omega) = F_{\sigma(\omega)}(\omega).$$

Definition 2.2 *$(F_n, \mathcal{A}_n)_{n\geq 1}$ is an adapted sequence if, $\forall n \in \mathbb{N}^*$, F_n is measurable relative to the σ-algebra $\mathcal{A}_n$.*

Definition 2.3 *An adapted sequence $(g_n, \mathcal{A}_n)_{n\geq 1}$ in L^1_Y is said to be a* martingale *if $g_n = E^{\mathcal{A}_n} g_{n+1}$ a.s. for each $n \in \mathbb{N}^*$.*

Definition 2.4 *An adapted sequence $(F_n, \mathcal{A}_n)_{n\geq 1}$ in $\mathcal{L}^{d_1}_{ccb(Y)}(\mathcal{A})$ is called a* multivalued martingale *if for every $n \geq 1$,*

$$F_n = E^{\mathcal{A}_n} F_{n+1} \quad a.s.$$

Here, $E^{\mathcal{A}_n} F_{n+1}$ denotes the conditional expectation of F_{n+1} with respect to the σ-algebra $\mathcal{A}_n$.

The following definition provides the natural generalisation of the concept of vector-valued uniform amart to that of set valued uniform amart (see Luu [14]).

Definition 2.5 *An adapted sequence of multivalued functions $(F_n, \mathcal{A}_n)_{n\geq 1}$ in $\mathcal{L}^{d_1}_{ccb(Y)}(\mathcal{A})$ is said to be a multivalued uniform amart if*

$$\lim_{\tau \in T} \sup_{\sigma \geq \tau} H(E^{\mathcal{A}_\tau} F_\sigma, F_\tau) = 0.$$

Where T is the set of all bounded stopping time.
Recall that

$$H(E^{\mathcal{A}_\tau} F_\sigma, F_\tau) = \int_\Omega h(E^{\mathcal{A}_\tau} F_\sigma, F_\tau) d\mathbf{P}.$$

A sequence $(F_n)_{n\geq 1}$ in $\mathcal{L}^1_{cc(Y)}(\mathcal{A})$ is said to be bounded if the sequence $(|F_n|)_{n\geq 1}$ is bounded in $L^1_{\mathbb{R}}(\mathcal{A})$. Let $(B_n)_{n\geq 1}$ in $cc(Y)$ and B_∞ in $cc(Y)$. $(B_n)_{n\geq 1}$ Mosco-converges to B_∞ and denoted $M-\lim_{n\to\infty} B_n = B_\infty$ if

$$B_\infty = s - liB_n = w - lsB_n.$$

where

$$s - liB_n = \{y \in Y : \|y_n - y\| \to 0; y_n \in B_n\}$$

and

$$w - lsB_n = \{y \in Y : y = w - \lim_{j\to\infty} y_{n_j}; y_{n_j} \in B_{n_j}\}$$

s (resp. w) denotes the strong (resp. weak) topology in Y.
For more information of Mosco-convergence see [15].
A sequence $(B_n)_{n\geq 1}$ in $ccb(Y)$ converges weakly to B_∞ if

$$\lim_{n\to\infty} \delta^*(y^*, B_n) = \delta^*(y^*, B_\infty),\ \forall y^* \in Y^*,$$

and $(B_n)_{n\geq 1}$ converges in Wijsmann topology to B_∞ if

$$\lim_{n\to\infty} d(y, B_n) = d(y, B_\infty),\ \forall y \in Y.$$

A sequence $(B_n)_{n\geq 1}$ in $ccb(Y)$ converges to $B_\infty \in ccb(Y)$ with respect to the linear topology τ_L if the following conditions hold :

- $\lim_{n\to\infty} \delta^*(y^*, B_n) = \delta^*(y^*, B_\infty),\ \forall y^* \in Y^*$;
- $\lim_{n\to\infty} d(y, B_n) = d(y, B_\infty),\ \forall y \in Y$.

In this case, we write

$$B_\infty = \tau_L\text{-}\lim_{n\to\infty} B_n.$$

Further properties of this topology are discussed in theorem 3.4 of [3].

Remark 2.1 *Beer proved in ([3] theorem 5.1) that the Mosco topology τ_M is weaker then the topology τ_L.*

Definition 2.6 *Let H in $cc(Y)$. H has the Radon-Nikodym Property (RNP) if for each Y-valued measure m on $\mathcal{A}$ which is finite total variation and satisfies $m << \mathbf{P}$ and whose average range $Ar(m)$ is contained in H, there exists $f \in L^1_Y(\Omega, \mathcal{A}, \mathbf{P})$ such that*

$$m(A) = \int_A f d\mathbf{P}.$$

Where
$$Ar(m) = \{m(A)/\mathbf{P}(A) : \mathbf{P}(A) > 0\}.$$

3. Main Results

3.1. Convergence of martingales with values in not RNP Banach space

The following theorem is an enhaced version of Chatterji's theorem [6] and Bourgin's theorem [2]. In fact here Y lacks the RNP and a sequence of martingale $(f_n, \mathcal{A}_n)_{n\geq 1}$ is with values in a closed convex subset H satisfies the RNP but is not necessarily bounded.

Theorem 3.1 *Let H be a nonempty convex closed subset of Y, possessing the Radon-Nikodym property. Let $(l_n, \mathcal{A}_n)_{n\geq 1}$ be a bounded martingale in $L^1_H(\mathcal{A})$. Consequently, there exists an H-valued integrable function $l \in L^1_H(\mathcal{A})$ such that*

$$\lim_{n\to\infty} l_n(\omega) = l(\omega) \qquad a.s.$$

Proof: Let's fix $\lambda > 0$ and let $\sigma : \Omega \longrightarrow \mathbb{N} \cup \{+\infty\}$ defined as follows :

$$\sigma(\omega) = \begin{cases} n & \text{if} \quad \|l_i(\omega)\| < \lambda \quad \text{for} \quad i = 1, ..., n-1 \quad \text{and} \quad \|l_n(\omega)\| \geq \lambda \\ \infty & \text{if} \quad \|l_i(\omega)\| < \lambda \quad \text{for} \quad i = 1, 2, ... \end{cases}$$

See that $\{\sigma = n\} \in \mathcal{A}_n$, for each n.
Set $n \wedge \sigma(\omega) = \inf(n, \sigma(\omega))$, then $l_{n\wedge\sigma}$ is $\mathcal{A}_n$-measurable for each n.
By using the fact that $(l_n, \mathcal{A}_n)_{n\geq 1}$ is a martingale and the technique due to Chatterji, see Theorem 2.2.7 in [2] is not hard to show that $(l_{n\wedge\sigma}, \mathcal{A}_n)_{n\geq 1}$ is a martingale with values in H and the function g defined on Ω by

$g(\omega) = \sup_{n\geq 1} \|l_{n\wedge\sigma}(\omega)\|$ is measurable and integrable.
In fact, from the definition of σ:
$g(\omega) = \lim_{n\to\infty} \|l_{n\wedge\sigma}(\omega)\|$ if $\omega \in \{\sigma < \infty\}$ and $g(\omega) \leq \lambda$ if $\omega \in \{\sigma = \infty\}$

$$\text{Then } \int_\Omega g(\omega)d\mathbf{P} = \int_{\{\sigma=\infty\}} g(\omega)d\mathbf{P} + \int_{\{\sigma<\infty\}} g(\omega)d\mathbf{P}.$$
$$\leq \quad \lambda \quad + \int_{\{\sigma<\infty\}} g(\omega)d\mathbf{P}.$$

By Fatou's Lemma $\int_{\{\sigma<\infty\}} g(\omega)d\mathbf{P} \leq \liminf_{n\to\infty} \int_\Omega \|l_{n\wedge\sigma}\|d\mathbf{P}$,

$$\text{and } \int_\Omega \|l_{n\wedge\sigma}\|d\mathbf{P} = \int_{\{\sigma<n\}} \|l_{n\wedge\sigma}(\omega)\|d\mathbf{P} + \int_{\{\sigma\geq n\}} \|l_{n\wedge\sigma}(\omega)\|d\mathbf{P}$$
$$= \sum_{i=1}^{n-1} \int_{\{\sigma=i\}} \|l_i(\omega)\|d\mathbf{P} + \int_{\{\sigma\geq n\}} \|l_n(\omega)\|d\mathbf{P}$$

Since $(\|l_n\|, \mathcal{A}_n)_{n\geq 1}$ is a submartingale, then
$\int_{\{\sigma=i\}} \|l_i(\omega)\|d\mathbf{P} \leq \int_{\{\sigma=i\}} \|l_n(\omega)\|d\mathbf{P}$, $\forall i < n$.
So, $\int_\Omega \|l_{n\wedge\sigma}\|d\mathbf{P} \leq \int_\Omega \|l_n\|d\mathbf{P} \leq \sup_{n\geq 1} \int_\Omega \|l_n\|d\mathbf{P} = \alpha < \infty$.
Finally $\int_\Omega g(\omega)d\mathbf{P} \leq \quad \lambda \quad + \quad \alpha \quad < \infty$.
Since g is integrable then for all $\frac{\epsilon}{2} > 0$, $\exists \delta > 0$ such that for a measurable set $A \in \mathcal{A}$ verifying $\mathbf{P}(A) \leq \delta \Longrightarrow \int_A g(\omega)d\mathbf{P} \leq \frac{\epsilon}{2}$.
For $\delta > 0$ and $A \in \mathcal{A}$ there exists $p \in \mathbb{N}$ and $B \in \mathcal{A}_p$ such that $\mathbf{P}(A\Delta B) < \delta$.
Set $m_n(A) = \int_A l_{n\wedge\sigma}(\omega)d\mathbf{P}$.
Let's show that $(m_n(A))_{n\geq 1}$ is a Cauchy sequence in Y.
For $k \geq n > p$, we have :
$\|m_k(A) - m_n(A)\| = \| \int_A l_{k\wedge\sigma}(\omega)d\mathbf{P} - \int_A l_{n\wedge\sigma}(\omega)d\mathbf{P}\|$
By introducing the set $B \in \mathcal{A}_p$, we have :

$$\|m_k(A) - m_n(A)\|$$
$$\leq \| \int_B l_{k\wedge\sigma}(\omega)d\mathbf{P} - \int_B l_{n\wedge\sigma}(\omega)d\mathbf{P}\| + \int_{A\Delta B} \|l_{k\wedge\sigma}(\omega)\|d\mathbf{P} + \int_{A\Delta B} \|l_{n\wedge\sigma}(\omega)\|d\mathbf{P}$$
$$\leq 2\int_{A\Delta B} g(\omega)d\mathbf{P} \quad < \quad \epsilon.$$

Then for each $A \in \mathcal{A}$, $m(A) = \lim_{n\to\infty} m_n(A)$ exists.
According to the Hahn-Vitali-Saks theorem [2], m is a Y-valued measure on $\mathcal{A}$.
m is absolutely continuous with respect to $\mathbf{P}$, $\forall A \in \mathcal{A}$ and finite partition $(A_i)_{1\leq i\leq n}$ of A
$\|m(A_i)\| \leq \lim_{n\to\infty} \int_{A_i} \|l_{n\wedge\sigma}\|d\mathbf{P} \leq \int_{A_i} gd\mathbf{P} < \infty, \quad \forall A_i$.
So the total variation $|m|$ of m is finite.
Let $y^* \in Y^*$, two cases are to be considered :

If $\delta^*(y^*, H) = \infty$, $\quad \langle y^*, \frac{m(A)}{\mathbf{P}(A)}\rangle \leq \delta^*(y^*, H)$.

$$\text{If } \delta^*(y^*, H) < \infty, \quad \langle y^*, m(A)\rangle = \lim_{n\to\infty} \langle y^*, \int_A l_{n\wedge\sigma}d\mathbf{P}\rangle$$
$$= \lim_{n\to\infty} \int_A \langle y^*, l_{n\wedge\sigma}\rangle d\mathbf{P}$$
$$\leq \delta^*(y^*, H) \cdot \mathbf{P}(A).$$

Then

$$\langle y^*, \frac{m(A)}{\mathbf{P}(A)}\rangle \leq \delta^*(y^*, H), \quad \forall y^* \in Y^*.$$

Since $H \in cc(Y)$ then $\dfrac{m(A)}{\mathbf{P}(A)} \in H$, so $Ar(m) \subset H$.

Then by definition 2.6, there exists $l \in L^1_Y$ such that :

$$\forall A \in \mathcal{A} : m(A) = \int_A l d\mathbf{P}.$$

$\forall n \geq 1$, $\forall k \geq n$ and $\forall A \in \mathcal{A}_n$:

$m_k(A) = \int_A l_{k\wedge\sigma}(\omega) d\mathbf{P} = \int_A l_{n\wedge\sigma}(\omega) d\mathbf{P} = m_n(A)$.

So : $m(A) = \int_A l(\omega) d\mathbf{P} = \int_A l_{n\wedge\sigma}(\omega) d\mathbf{P} = m_n(A)$.

Then

$$\int_A l(\omega) d\mathbf{P} = \int_A l_{n\wedge\sigma}(\omega) d\mathbf{P}.$$

According to the uniqueness of the conditional expectation

$$E^{\mathcal{A}_n} l(.) = l_{n\wedge\sigma}(.) \text{ a.s. } \forall n \geq 1.$$

Now, from Levy's theorem, we can conclude that $(l_{n\wedge\sigma})_{n\geq 1}$ converges almost surely to l.

(i.e. $\lim_{n\to\infty} l_{n\wedge\sigma}(\omega) = l(\omega)$ a.s.).

Finally, we show that $\mathbf{P}(\{\sigma = \infty\}) \longrightarrow 1$ as $\lambda \to \infty$.

$\forall \omega \in \{\sigma < \infty\}$ there exists an integer $n \geq 1$ such that $\|l_n(\omega)\| \geq \lambda$.

So $\{\sigma < \infty\} \subset \{\sup_{n\geq 1} \|l_n\| \geq \lambda\} \Longrightarrow \mathbf{P}(\{\sigma < \infty\}) \leq \mathbf{P}(\sup_{n\geq 1} \|l_n\| \geq \lambda)$.

According to the maximal inequality :

$\mathbf{P}(\sup_{n\geq 1} \|l_n\| \geq \lambda) \leq \dfrac{1}{\lambda} \sup_{n\geq 1} E\|l_n\| \longrightarrow 0$ as $\lambda \longrightarrow \infty$.

Since $l_{n\wedge\sigma}(\omega) = l_n(\omega)$ when $\omega \in \{\sigma = \infty\}$ then $(l_n)_{n\geq 1}$ converges almost surely to l. So, there exists a negligeable $N \in \mathcal{A}$ such that $\forall \omega \in \Omega \setminus N : \lim_{n\to\infty} l_n(\omega) = l(\omega)$. □

3.2. Convergence of set valued martingales

We begin by recalling some preliminary results that will be needed after.

Lemma 3.1 *For any $X \in \mathcal{L}^{d_1}_{cc(Y)}$ and sub-σ-algebras $\mathcal{G} \subset \mathcal{A}$,*

(i) [21] $|E^{\mathcal{G}} X| \leq E^{\mathcal{G}} |X|$ *a.s.*

(ii) [12] $d(y, E^{\mathcal{G}} X) \leq E^{\mathcal{G}}(d(y, X))$ *a.s.* , $y \in Y$.

(iii) [18] $\delta^*(y^*, E^{\mathcal{G}} X) = E^{\mathcal{G}}(\delta^*(y^*, X))$ *a.s.* , $y^* \in Y^*$.

Proposition 3.1 [7] *Let $(X_n, \mathcal{A}_n)_{n\geq 1}$ be a martingale in $\mathcal{L}^{d_1}_{cc(Y)}$. Then there exists a family of adapted sequences $(f^i_n)_{(n\geq 1, i\geq 1)}$ in L^1_Y such that*

$\forall i \geq 1$, $(f^i_n, \mathcal{A}_n)_{n\geq 1}$ is a selection martingale of $(X_n, \mathcal{A}_n)_{n\geq 1}$.

$\forall n \geq 1$, $(f^i_n)_{i\geq 1}$ is a Castaing representation of X_n.

The lemma below plays a fundamental role in the proof of the multivalued extension of Theorem 3.1. It provides a key tool for establishing the almost sure convergence of the support functions of the multivalued martingales $(X_n)_{n\geq 1}$ denoted by $(\delta^*(\cdot, X_n))_{n\geq 1}$ to the support function $\delta^*(\cdot, X)$. This convergence is obtained using only martingale selections of $(X_n)_{n\geq 1}$ without invoking any separability condition on the dual space Y^* or any compactness assumption.

Lemma 3.2 *Let H in $cc(Y)$, possessing the Radon-Nikodym property and $(X_n, \mathcal{A}_n)_{n\geq 1}$ a multivalued martingale in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$ verifying :*

$$\sup_{n\geq 1} E|X_n| < \infty,$$

Then there exists a measurable multifunction $X \in \mathcal{L}^1_{ccb(H)}(\mathcal{A})$ *such that :*

(a) $\forall y \in Y, \quad \limsup_{n \longrightarrow \infty} d(y, X_n) \leq d(y, X) \quad a.s.$

(b) $\lim_{n \longrightarrow \infty} |X_n| = |X| \quad a.s.$

(c) $\forall y^* \in Y^*, \quad \lim_{n \longrightarrow \infty} \delta^*(y^*, X_n) = \delta^*(y^*, X) \quad a.s.$

Proof: By proposition 3.1, there is a collection of adapted sequences $(f^i_n)_{(n\geq 1, i\geq 1)}$ in L^1_Y verifying:
$\forall i \in \mathbb{N}^*$, $(f^i_n, \mathcal{A}_n)_{n\geq 1}$ is a selection martingale of $(X_n, \mathcal{A}_n)_{n\geq 1}$.
(a) See that $\sup_{n\geq 1} E|f^i_n| \leq \sup_{n\geq 1} E|X_n| < \infty$.
So, $(f^i_n, \mathcal{A}_n)_{n\geq 1}$ is a bounded martingale with values in H.
By theorem 3.1, $\forall i \geq 1$ there exists $f^i \in L^1_H$ such that $(f^i_n)_{n\geq 1}$ converges a.s. to f^i. Then, there exists a negligible N such that $\forall i \geq 1$, $\forall \omega \in \Omega \setminus N$: $\lim_n f^i_n(\omega) = f^i(\omega)$.

Let's put : $X(\omega) = \begin{cases} \overline{co}\{f^i(\omega), \forall i \geq 1\} & \forall \ \omega \in \Omega \setminus N \\ \{0\} & \text{if } \omega \in N. \end{cases}$

X is a $\mathcal{A}$-measurable multifunction.
So, by lemma 3.1 (ii) we conclude that $(d(y, X_n), \mathcal{A}_n)_{n\geq 1}$ is a bounded sub-martingale, $\forall y \in Y$.
Then $(d(y, X_n))_{n\geq 1}$ converge a.s.
Now by using the same technique as in [21], we show (a) and (b)

$$\begin{aligned}
\limsup_{n \longrightarrow \infty} d(y, X_n) &= \lim_{n \longrightarrow \infty} \{ \inf_{i_1,\dots,i_k \in \mathbb{N}^*, a_{i_1},\dots,a_{i_k} \in [0,1]} d(y, \sum_{j=1}^k a_{i_j} f^{i_j}_n) \} \\
&\leq \{ \inf_{i_1,\dots,i_k \in \mathbb{N}^*, a_{i_1},\dots,a_{i_k} \in [0,1]} \lim_{n \longrightarrow \infty} d(y, \sum_{j=1}^k a_{i_j} f^{i_j}_n) \} \\
&= \{ \inf_{i_1,\dots,i_k \in \mathbb{N}^*, a_{i_1},\dots,a_{i_k} \in [0,1]} d(y, \sum_{j=1}^k a_{i_j} f^{i_j}) \} \\
&= d(y, X) \text{ a.s} \qquad \text{where } \sum_{j=1}^k a_{i_j} = 1.
\end{aligned}$$

$$\text{So, } \limsup_{n \longrightarrow \infty} d(y, X_n) \leq d(y, X) \text{ a.s.}$$

(b) Since for each $i \in \mathbb{N}^*$, the sequence $(f^i_n, \mathcal{A}_n)_{n\geq 1}$ is a martingale then $(\| f^i_n \|, \mathcal{A}_n)_{n\geq 1}$ is a submartingale satisfying $\sup_{n\geq 1} \int \sup_{i\in\mathbb{N}^*} ||f^i_n|| d\mathbf{P} < \infty$. Then, by a convergence result of Neveu ([16], lemma V.2.9), we get :

$$\begin{aligned}
\lim_{n\to\infty} |X_n| &= \lim_{n\to\infty} \sup_{i\geq 1} ||f^i_n|| \\
&= \sup_{i\geq 1} \lim_{n \longrightarrow \infty} ||f^i_n|| \\
&= \sup_{i\geq 1} ||f^i|| \text{ a.s} \\
&= \sup_{i_1,\dots,i_k \in \mathbb{N}^*, a_{i_1},\dots,a_{i_k} \in [0,1]} || \sum_{j=1}^k a_{i_j} f^{i_j} || = |X| \text{ a.s.}
\end{aligned}$$

$$\text{Then } \lim_{n\to\infty} |X_n| = |X| \text{ a.s.}$$

By Fatou's Lemma :
$E|X| \leq \liminf_{n\to\infty} E|X_n| \leq \sup_{n\geq 1} E|X_n| < \infty$. Then $X \in \mathcal{L}^1_{ccb(H)}(\mathcal{A})$.
(c) For any fixed $y^* \in Y^*$, $(< y^*, f^i_n >, \mathcal{A}_n)_{n\geq 1}$ is a martingale so : a submartingale satisfying the condition $\sup_{n\geq 1} \int \sup_{i\in\mathbb{N}^*} |< y^*, f^i_n >| \, d\mathbf{P} < \infty$.
By the result of Neveu ([16], lemma V.2.9), we have :

$\sup_{i\geq 1} < y^*, f_n^i > \quad \longrightarrow \quad \sup_{i\geq 1} < y^*, f^i >$ a.s. as $n \longrightarrow \infty$.

Then, $\lim_{n\longrightarrow\infty} \delta^*(y^*, X_n) = \lim_{n\longrightarrow\infty} \sup_{i\in\mathbb{N}^*} < y^*, f_n^i >$

$$= \sup_{i\in\mathbb{N}^*} \lim_{n\longrightarrow\infty} < y^*, f_n^i >$$

$$= \sup_{i\in\mathbb{N}^*} < y^*, f^i > \quad = \quad \delta^*(y^*, X) \text{ a.s.}$$

$$\text{So, } \lim_{n\longrightarrow\infty} \delta^*(y^*, X_n) = \delta^*(y^*, X) \text{ a.s.}$$

□

Building on Theorem 3.1 and Proposition 3.1, we derive new convergence results for martingales taking values in the family of closed, bounded, and convex subsets of a Banach space Y, not necessarily possessing the Radon Nikodym Property. The results presented here can be viewed as a multivalued extension of Theorem 3.1. Moreover, they refine and generalize classical results found in the literature for example, those in references [7] and [17].

Theorem 3.2 *Assume that H is a nonempty convex closed subset of Y, possessing the Radon-Nikodym property and $(X_n, \mathcal{A}_n)_{n\geq 1}$ a multivalued martingale in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$ verifying :*

$$\sup_{n\geq 1} E|X_n| < +\infty.$$

Let X be the same multivalued random function defined in lemma 3.2, if $X \in \mathcal{L}^1_{cwk(H)}(\mathcal{A})$ then :

$$M - \lim_{n\to\infty} X_n = X \ a.s.$$

Proof: Let $(\alpha_k^*)_{k\geq 1}$ be a dense sequence in Y^* with respect to the Mackey topology $\tau(Y^*, Y)$. By lemma 3.2 (c), there exists a negligeable N_1 such that $\forall \omega \in \Omega \setminus N_1$, $\forall k \geq 1 : \lim_{n\longrightarrow\infty} \delta^*(\alpha_k^*, X_n(\omega)) = \delta^*(\alpha_k^*, X(\omega))$.
Let $\omega \in \Omega \setminus N_1$ and let $y \in w - lsX_n(\omega)$, there exists $(y_{n_j})_{j\geq 1} \in (X_{n_j}(\omega))_{j\geq 1}$ such that : $y = w - \lim y_{n_j}$
So : $\langle \alpha_k^*, y\rangle = \lim_{n\longrightarrow\infty} \langle \alpha_k^*, y_{n_j}\rangle \leq \lim_{n\longrightarrow\infty} \delta^*(\alpha_k^*, X_n(\omega)) = \delta^*(\alpha_k^*, X(\omega))$.
Since $X(\omega) \in cwk(Y)$, according to lemma III-34 in [5], $y \in X(\omega)$. Then w-$lsX_n(\omega) \subset X(\omega)$ a.s.
By construction of X and convexity of X_n, it easy to see that

$$X(\omega) \subset s - liX_n(\omega) \text{ a.s.}$$

Then : $M - \lim_{n\to\infty} X_n = X$ a.s. □

When Y^* is strongly separable, the following theorem guarantees the convergence of a bounded martingale $(X_n)_{n\geq 1}$ to X in linear topology which is stronger than Mosco topology.

Theorem 3.3 *Let Y be a separable Banach space, Y^* its strongly separable dual and H a nonempty convex closed subset of Y possessing the Radon-Nikodym property. Assume that $(X_n, \mathcal{A}_n)_{n\geq 1}$ is a multivalued martingale in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$ verifying*

$$\sup_{n\geq 1} E|X_n| < +\infty.$$

Then there exists $X \in \mathcal{L}^1_{ccb(H)}(\mathcal{A})$ such that

$$\tau_L - \lim_{n\to\infty} X_n = X \ a.s.$$

Proof: Let $D_1^* = (y_j^*)_{j\in\mathbb{N}^*}$ be a countable dense sequence in $\overline{B}_{Y^*}$.
From lemma 3.2 (c), there exists a negligeable subset N' such that for all $\omega \in \Omega \setminus N'$:
$\lim_{n\longrightarrow\infty} \delta^*(y_j^*, X_n) = \delta^*(y_j^*, X)$.
And by the same lemma (b) $\lim_{n\longrightarrow\infty} |X_n| = |X|$ a.s, then $\sup_n |X_n| < \infty$ a.s.
So, $(\delta^*(., X_n))_{n\geq 1}$ is equicontinuous, then there exists a negligible N'' such that $\forall \omega \in \Omega \setminus N''$

$$\lim_{n\to\infty} \delta^*(y^*, X_n(\omega)) = \delta^*(y^*, X(\omega)),\ \forall y^* \in Y^*.$$

$\forall y \in Y$, we have :

$$\begin{aligned}\liminf_{n\longrightarrow\infty} d(y, X_n) &= \liminf_{n\longrightarrow\infty} \sup_{y_j^*\in D_1^*} (\langle y_j^*, y\rangle - \delta^*(y_j^*, X_n)) \\ &\geq \sup_{y_j^*\in D_1^*} \liminf_{n\longrightarrow\infty} (\langle y_j^*, y\rangle - \delta^*(y_j^*, X_n)) \\ &= \sup_{y_j^*\in D_1^*} \lim_{n\longrightarrow\infty} (\langle y_j^*, y\rangle - \delta^*(y_j^*, X_n)) \\ &= \sup_{y_j^*\in D_1^*} (\langle y_j^*, x\rangle - \delta^*(y_j^*, X)) \\ &= d(y, X) \text{ a.s.}\end{aligned}$$

By lemma 3.2 (a), we have $\limsup_{n\longrightarrow\infty} d(y, X_n) \leq d(y, X)$ a.s, $\forall y \in Y$.
So : $\lim_{n\to\infty} d(y, X_n) = d(y, X)$ a.s.
Since $\sup_n |X_n| < \infty$ a.s., $(d(., X_n))_{n\geq 1}$ is equicontinuous.
Then $\lim_{n\to\infty} d(y, X_n) = d(y, X)$ a.s. $\forall y \in Y$.
So : $\tau_L - \lim_{n\to\infty} X_n = X$ a.s. □

More relations between X_n and X are defined by the following result.

Theorem 3.4 *Assume that Y^* is the strongly separable dual of Y, H a nonempty convex closed subset of Y possessing the Radon-Nikodym property and $(X_n, \mathcal{A}_n)_{n\geq 1}$ a bounded martingale in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$. Then there exists $X \in \mathcal{L}^1_{ccb(H)}(\mathcal{A})$ such that*

(i) $\tau_L - \lim_{n\to\infty} E^{\mathcal{A}_n} X = X$ *a.s.*

(ii) $\lim_{n\to\infty} h(X_n, E^{\mathcal{A}_n} X) = 0$ *a.s.*

Proof: (i) We will proceed in 2 steps.
Step 1. Let X be the multifunction constructed in lemma 3.2, since for all $y^* \in Y^*$ $\delta^*(y^*, X)$ is integrable by Levy's theorem :

$$\lim_{n\longrightarrow\infty} \delta^*(y^*, E^{\mathcal{A}_n} X) = E^{\mathcal{A}_n} \delta^*(y^*, X) = \delta^*(y^*, X) \quad \text{a.s.}$$

Let $(\beta_j^*)_{j\in\mathbb{N}^*}$ be a countable dense sequence in Y^*, then $\forall j \geq 1$:

$$\lim_{n\longrightarrow\infty} \delta^*(\beta_j^*, E^{\mathcal{A}_n} X) = \delta^*(\beta_j^*, X) \quad \text{a.s.}$$

Let $y^* \in Y^*$ and $\omega \in \Omega$,

$$\begin{aligned}&| \delta^*(y^*, E^{\mathcal{A}_n} X(\omega)) - \delta^*(y^*, X(\omega)) | \\ &\quad \leq | \delta^*(x^*, E^{\mathcal{A}_n} X(\omega)) - \delta^*(\beta_j^*, E^{\mathcal{A}_n} X(\omega)) | + | \delta^*(e_j^*, E^{\mathcal{A}_n} X(\omega)) - \delta^*(\beta_j^*, X(\omega)) | \\ &\qquad\qquad + | \delta^*(\beta_j^*, X(\omega)) - \delta^*(x^*, X(\omega)) | \\ &\quad \leq \| x^* - \beta_j^* \| \cdot E^{\mathcal{A}_n} |X(\omega)| + | \delta^*(\beta_j^*, E^{\mathcal{A}_n} X(\omega)) - \delta^*(\beta_j^*, X(\omega)) | + \| \beta_j^* - x^* \| \cdot |X(\omega)|.\end{aligned}$$

By lemma 3.2, $|X| < \infty$ a.s. and by Levy's theorem applied to $(E^{\mathcal{A}_n}|X|)_{n\geq 1}$, $\sup_n E^{\mathcal{A}_n}|X| < \infty$ a.s.

Then there exists a negligible N such that $\forall \omega \in \Omega \setminus N$:
$\lim_{n\to\infty} \delta^*(y^*, E^{\mathcal{A}_n} X(\omega)) = \delta^*(y^*, X(\omega))$.

Step 2.

$$\begin{aligned}\limsup_{n\longrightarrow\infty} d(y, E^{\mathcal{A}_n} X) &\leq \limsup_{n\longrightarrow\infty} E^{\mathcal{A}_n}(d(y, X)) \\ &= \lim_{n\longrightarrow\infty} E^{\mathcal{A}_n}(d(y, X)) \\ &= d(y, X) \text{ a.s.}\end{aligned}$$

And

$$\begin{aligned}\liminf_{n\longrightarrow\infty} d(y, E^{\mathcal{A}_n} X) &= \liminf_{n\longrightarrow\infty} \sup_{y^*\in\overline{B}_{Y^*}} (\langle y^*, y\rangle - \delta^*(y^*, E^{\mathcal{A}_n} X)) \\ &\geq \sup_{y^*\in\overline{B}_{Y^*}} \liminf_{n\longrightarrow\infty} (\langle y^*, y\rangle - \delta^*(y^*, E^{\mathcal{A}_n} X)) \\ &= \sup_{y^*\in\overline{B}_{Y^*}} \lim_{n\longrightarrow\infty} (\langle y^*, y\rangle - \delta^*(y^*, E^{\mathcal{A}_n} X))\end{aligned}$$

$$= \sup_{y^* \in \overline{B}_{Y^*}} (\langle y^*, y\rangle - \delta^*(y^*, X))$$
$$= d(y, X) \text{ a.s.}$$

Since $\sup_n |E^{\mathcal{A}_n} X| < \infty$ a.s., the convergence in Wijsmann topology is required from the equicontinuity of $(d(., E^{\mathcal{A}_n} X))_{n\geq 1}$.

We conclude that $\quad \tau_L - \lim_{n\to\infty} E^{\mathcal{A}_n} X = X$ a.s.

(ii) $\sup_n \int_\Omega h(X_n, E^{\mathcal{A}_n} X) d\mathbf{P} \leq \sup_n \int_\Omega |X_n| d\mathbf{P} + \int_\Omega |X| d\mathbf{P} < \infty.$

Let $(y_k^*)_{k\geq 1}$ be a dense sequence in the closed unit ball $\overline{B}_{Y^*}$, so :

$$h(X_n, E^{\mathcal{A}_n} X) = \sup_{k\geq 1} |\delta^*(y_k^*, X_n) - \delta^*(y_k^*, E^{\mathcal{A}_n} X)|.$$

Since $\big((\delta^*(y_k^*, X_n) - \delta^*(y_k^*, E^{\mathcal{A}_n} X)), \mathcal{A}_n\big)_{n\geq 1}$ is martingale then $\big(|\delta^*(y_k^*, X_n) - \delta^*(y_k^*, E^{\mathcal{A}_n} X)|, \mathcal{A}_n\big)_{n\geq 1}$ is a sub-martingale satisfying $\sup_{n\geq 1} \int_\Omega \sup_{k\geq 1} |\delta^*(y_k^*, X_n) - \delta^*(y_k^*, E^{\mathcal{A}_n} X)| d\mathbf{P} < \infty$.

Then by [16], lemma V.2.9 :

$$\lim_{n\to\infty} h(X_n, E^{\mathcal{A}_n} X) = \lim_{n\to\infty} \sup_{k\geq 1} |\delta^*(y_k^*, X_n) - \delta^*(y_k^*, E^{\mathcal{A}_n} X)|$$
$$= \sup_{k\geq 1} \lim_{n\to\infty} |\delta^*(y_k^*, X_n) - \delta^*(y_k^*, E^{\mathcal{A}_n} X)|$$
$$= 0 \text{ a.s.}$$

□

3.3. Application to the convergence of mutivalued uniform amart

We start this part with the following lemma. It shows the different relationships between the convergence of closed and bounded subset of Y in several topologies and will serve in the proof of the main results presented in the following section.

Lemma 3.3 *Let $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ two sequences in $ccb(Y)$ such that :*

$$\lim_{n\to\infty} h(A_n, B_n) = 0,$$

let $A \in ccb(Y)$,

(a) If $\tau_L - \lim_{n\to\infty} A_n = A \implies \tau_L - \lim_{n\to\infty} B_n = A.$

(b) If $\lim_{n\to\infty} |A_n| = |A| \implies \lim_{n\to\infty} |B_n| = |A|.$

(c) If $M - \lim_{n\to\infty} A_n = A \implies M - \lim_{n\to\infty} B_n = A.$

Proof: (a) Since, $\forall y^* \in B_{Y^*} \quad \delta^*(y^*, B_n) \leq \delta^*(y^*, A_n) + h(A_n, B_n)$, then

$$\lim_{n\to\infty} \delta^*(y^*, B_n) \leq \delta^*(y^*, A),$$

but $\quad \delta^*(y^*, A_n) \leq \delta^*(y^*, B_n) + h(A_n, B_n)$, then

$$\lim_{n\to\infty} \delta^*(y^*, B_n) \geq \delta^*(y^*, A),$$

therefore

$$\tau_W - \lim_{n\to\infty} B_n = A.$$

Since $\quad d(y, B_n) \leq d(y, A_n) + h(A_n, B_n)$, then

$$\lim_{n\to\infty} d(y, B_n) \leq d(y, A),$$

but $\quad d(y, A_n) \leq d(y, B_n) + h(A_n, B_n)$, hence

$$\lim_{n\to\infty} d(y, B_n) \geq d(y, A),$$

then $$\lim_{n\to\infty} d(y, B_n) = d(y, A).$$
Finally $$\tau_L - \lim_{n\to\infty} B_n = A.$$
(b) Since $|A_n| \leq |B_n| + h(A_n, B_n)$, then

$$\lim_{n\to\infty} |B_n| \geq |A|,$$

but $|B_n| \leq |A_n| + h(A_n, B_n)$, then

$$\lim_{n\to\infty} |B_n| \leq |A|.$$

Finally $$\lim_{n\to\infty} |B_n| = |A|.$$
(c) follow from proposition 3.1 in [1], but for a completeness of the reader, we give the proof.
Let $y \in A = M - \lim_{n\to\infty} A_n$, so there exists a sequence $y_n \in A_n$ such that $y = \lim_{n\to\infty} y_n$.
We have : $d(y, B_n) \leq \| y - y_n \| + d(y_n, B_n) \leq \| y - y_n \| + h(A_n, B_n)$.
By $\lim_{n\to\infty} h(A_n, B_n) = 0$, we deduce that $\lim_{n\to\infty} d(y, B_n) = 0$.
So $y \in s - liB_n$.
Consequently

$$A \subset s - liB_n.$$

Now let $y \in w - lsB_n$. Then there exists a subsequence $(y_{n_j})_{j\geq 1}$ with $y_{n_j} \in B_{n_j}$ such that $y = w - \lim_{j\to\infty} y_{n_j}$, for all $j \in \mathbb{N}$.
Let us choose a subsequence $z_{n_j} \in A_{n_j}$ such that

$$\| y_{n_j} - z_{n_j} \| \leq h(A_{n_j}, B_{n_j}) + \frac{1}{j}.$$

And knowing that $z_{n_j} = (z_{n_j} - y_{n_j}) + y_{n_j}$, we have $w - \lim_{j\to\infty} z_{n_j} = y \in A$.
Therefore

$$w - lsB_n \subset A.$$

Finally $$M - \lim_{n\to\infty} B_n = A.$$ □

Since every set valued martingale $(X_n, \mathcal{A}_n)_{n\geq 1}$ is a set valued uniform amart, the next result presents a generalised version of lemma 3.2 for set valued uniform amart.

Theorem 3.5 *Assume that H is a nonempty convex closed subset of Y, possessing the Radon-Nikodym property and $(X_n, \mathcal{A}_n)_{n\geq 1}$ an adapted sequence of uniform amart in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$ verifying :*

$$\sup_{n\geq 1} E|X_n| < \infty.$$

Then there exists a measurable multifunction $X \in \mathcal{L}^1_{ccb(H)}(\mathcal{A})$ such that
(a) $\forall y \in Y,\quad \limsup_{n\longrightarrow\infty} d(y, X_n) \leq d(y, X)$ *a.s.*
(b) $\lim_{n\longrightarrow\infty} |X_n| = |X|$ *a.s.*
(c) $\forall y^* \in Y^*,\quad \lim_{n\longrightarrow\infty} \delta^*(y^*, X_n) = \delta^*(y^*, X)$ *a.s.*

Proof: From [14] Proposition 2.6, there is a martingale $(M_n)_{n\geq 1}$ in $\mathcal{L}^1_{ccb(Y)}(\mathcal{A})$ such that the sequence $(P_n)_{n\geq 1}$ given by
(1) $P_n(\omega) = h(X_n(\omega), M_n(\omega))$ is a nonnegative uniform potential, where the martingale $(M_n)_{n\geq 1}$ can be defined by

(2) $\lim_{m\to\infty} H(E^{\mathcal{A}_n}X_m, M_n) = 0.$

From (2) $\lim_{m\to\infty}\int_\Omega h(E^{\mathcal{A}_n}X_m(\omega), M_n(\omega))d\mathbf{P} = 0,\ 1 \le n \le m.$

Set $(g_m)_{m\ge n} = (h(E^{\mathcal{A}_n}X_m, M_n))_{m\ge n}$.

There exists a subsequence $(g_{m_k})_{k\ge n}$ such that $\lim_{k\to\infty} g_{m_k}(\omega) = 0$ a.s.

Then $\lim_{k\to\infty} h(E^{\mathcal{A}_n}X_{m_k}(\omega), M_n(\omega)) = 0$ a.s, $1 \le n \le m_k$.

Which implies that :

$\lim_{k\to\infty}\sup_{y^*\in\overline{B}_{E^*}} |\delta^*(y^*, E^{\mathcal{A}_n}X_{m_k}(\omega)) - \delta^*(y^*, M_n(\omega))| = 0$ a.s

So there exists a negligible set N_2 such that $\forall\omega\in\Omega\setminus N_2$ and $\forall y^*\in Y^*$:

$\lim_{k\longrightarrow\infty}\delta^*(y^*, E^{\mathcal{A}_n}X_{m_k}(\omega)) = \delta^*(y^*, M_n(\omega))$ (3.5.1)

By hypothesis $X_{m_k}(.) \subset H$ a.s. so, $E^{\mathcal{A}_n}X_{m_k}(.) \subset H$ a.s, $\forall k \ge n$.

By (3.5.1) $\delta^*(y^*, M_n(.)) \le \delta^*(y^*, H)$ a.s, $\forall y^* \in Y^*$.

Since H is convex and closed then, $M_n(.) \subset H$ a.s.

Because $(P_n)_{n\ge1}$ is an uniform potential (i.e $\lim_{\tau\in T}\int_\Omega P_\tau d\mathbf{P} = 0$) so :

$\lim_{n\to\infty} h(X_n, M_n) = 0$ a.s.

Since : $\int |M_n|d\mathbf{P} \le \int |X_n|d\mathbf{P} + \int h(X_n, M_n)d\mathbf{P}.$ (3.5.2)

So $(M_n, \mathcal{A}_n)_{n\ge1}$ is a bounded martingale. By lemma 3.2, there exists a measurable multifunction $X \in \mathcal{L}^1_{ccb(H)}(\mathcal{A})$ such that : $\forall y\in Y,\quad \limsup_{n\longrightarrow\infty} d(y, M_n) \le d(y, X)$ a.s.

$(P_n)_{n\ge1}$ is a nonnegative uniform potential : $\lim_{n\to\infty} h(X_n, M_n) = 0$ a.s. Then

(a) $\forall y \in Y,\quad \limsup_{n\longrightarrow\infty} d(y, X_n) \le \limsup_{n\longrightarrow\infty} d(y, M_n) \le d(y, X)$ a.s.

(b) By lemma 3.2 (b) $\lim_{n\longrightarrow\infty} |M_n| = |X|$ a.s.

By lemma 3.3 (b) and the fact that $\lim_{n\to\infty} h(X_n, M_n) = 0$ a.s,

$$\lim_{n\longrightarrow\infty} |X_n| = |X| \quad \text{a.s.}$$

(c) Since $\lim_{n\to\infty} h(X_n, M_n) = 0$ a.s, then

$\lim_{n\longrightarrow\infty}\delta^*(y^*, X_n) = \lim_{n\longrightarrow\infty}\delta^*(y^*, M_n) = \delta^*(y^*, X)$ a.s, $\forall y^* \in Y^*$. □

If X is in $cwk(Y)$, the following theorem provides the Mosco-convergence of a bounded uniform amart $(X_n, \mathcal{A}_n)_{n\ge1}$. It's also an improved version of theorem 3.1 in [19].

Theorem 3.6 *Assume that H is a nonempty convex closed subset of Y, possessing the Radon-Nikodym property and $(X_n, \mathcal{A}_n)_{n\ge1}$ an adapted sequence of uniform amart in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$ verifying :*

$$\sup_{n\ge1} E|X_n| < +\infty.$$

Let X be the multifunction constructed in lemma 3.2, if $X \in \mathcal{L}^1_{cwk(H)}(\mathcal{A})$ then :

$$M - \lim_{n\to\infty} X_n = X \ a.s.$$

Proof: From [14] Proposition 2.6 and the proof of theorem 3.5, there is a martingale $(M_n, \mathcal{A}_n)_{n\ge1}$ in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$ such that $\lim_{n\to\infty} h(X_n(\omega), M_n(\omega)) = 0$ a.s,

since $\sup_{n\ge1} E|M_n| < +\infty$, by theorem 3.2 there exist $X \in \mathcal{L}^1_{cwk(H)}(\mathcal{A})$ such that

$$M - \lim_{n\to\infty} M_n = X \text{ a.s.}$$

Then applying lemma 3.3 (c) :

$$M - \lim_{n\to\infty} X_n = X \text{ a.s.}$$ □

The result below can be seen as both an extension of theorem 3.3 and 3.4 to uniform amarts and further generalizes results previously established in theorem 4.2 of [8] and theorem 4.3 in [9].

Theorem 3.7 *Let Y^* be the strongly separable dual of the Banach space Y. Let H a nonempty convex closed subset of Y possessing the Radon-Nikodym property and $(X_n, \mathcal{A}_n)_{n\geq 1}$ a uniform amart in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$.*
Then there exists $X \in \mathcal{L}^1_{ccb(H)}(\mathcal{A})$ such that

(i) $\tau_L - \lim_{n\to\infty} E^{\mathcal{A}_n} X = X$ *a.s.*

(ii) $\lim_{n\to\infty} h(X_n, E^{\mathcal{A}_n} X) = 0$ *a.s.*

(iii) $\tau_L - \lim_{n\to\infty} X_n = X$ *a.s.*

Proof: (i) From [14] Proposition 2.6 and the proof of theorem 3.5, there is a bounded martingale $(M_n, \mathcal{A}_n)_{n\geq 1}$ in $\mathcal{L}^1_{ccb(H)}(\mathcal{A})$ such that

$$\sup_{n\geq 1} E|M_n| < +\infty \qquad (3.7.1)$$

and

$$\lim_{n\to\infty} h(X_n, M_n) = 0 \text{ a.s.} \qquad (3.7.2)$$

By (3.7.1) and theorem 3.4, there exists $X \in \mathcal{L}^1_{ccb(H)}(\mathcal{A})$ such that

$$\tau_L - \lim_{n\to\infty} M_n = X \text{ a.s.}$$

From theorem 3.4 (ii) $\lim_{n\to\infty} h(M_n, E^{\mathcal{A}_n} X) = 0$ a.s. and by Lemma 3.3 (a) :

$$\tau_L - \lim_{n\to\infty} E^{\mathcal{A}_n} X = X \text{ a.s.}$$

(ii) $h(X_n, E^{\mathcal{A}_n} X) \leq h(X_n, M_n) + h(M_n, E^{\mathcal{A}_n} X)$.
Then by (3.7.2) and Theorem 3.4 (ii)

$$\lim_{n\to\infty} h(X_n, E^{\mathcal{A}_n} X) = 0 \text{ a.s.}$$

(iii) From (i), (ii) and Lemma 3.3 (a), we get

$$\tau_L - \lim_{n\to\infty} X_n = X \text{ a.s.}$$

□

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