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# Integral Representations of Solutions to Initial–Boundary Value Problems on Mixed Metric Graphs

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ABSTRACT: In this paper, initial-boundary value problems for the heat equation on symmetric metric graphs are investigated. Based on the Fokas unified transform method, global relations are derived and used to establish a correspondence between Dirichlet and Neumann boundary conditions at the vertices of the graph. The problem is reduced to a system of algebraic equations with respect to the unknown values of the solution at the branching points of the graph. As a result, an explicit integral representation of the solution in terms of given initial and boundary data is constructed. The convergence properties of the contour integrals are analyzed, and the conditions ensuring exponential decay of the integrands are justified. The obtained results extend the analytical framework for studying diffusion processes on metric graphs and provide a theoretical basis for modeling heat transfer in complex branched structures.

Keywords: Heat transfer equation, metric graphs, branched structures, Fokas' method, unified transformation, Fourier transformation, initial problem, boundary value problem.

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## 1. Introduction

Partial differential equations on metric graphs have attracted increasing attention due to their relevance in modeling diffusion, wave propagation and transport phenomena in complex network-like structures arising in physics, biology and engineering [1]–[3]. In particular, the heat equation on graphs provides a fundamental framework for studying thermal and diffusive processes in branched systems [7]. Despite significant progress in the theory of PDEs on metric graphs, explicit analytical representations of solutions to initial-boundary value problems remain limited, especially for graphs containing both finite and semi-infinite edges [8].

Usually, for investigation physical properties of quantum graphs used static Schrödinger equation [3]–[7]. In the references [16] was investigated nonlinear Schrödinger equation on two dimensional thin tabular branched domain and proved that the problem on metric graphs for one dimensional nonlinear Schrödinger equation on metric graph, with gluing (Kirchhoff) conditions on the vertex point, can be obtained when width of the branches tends to zero. Similar convergence result in the case of linear Schrödinger equation with different approaches can be found in [6]–[7].

Existing studies mainly focus on purely finite or purely infinite star graphs, while mixed configurations have not been systematically investigated within the unified transform framework. The Fokas unified transform method has proven to be a powerful tool for solving boundary value problems for linear and integrable nonlinear PDEs. However, its extension to metric graphs with Kirchhoff-type vertex conditions presents substantial analytical challenges due to the coupling of boundary data at the graph vertices. In this paper, we develop a unified analytical approach for the heat equation on symmetric star-shaped metric graphs with mixed finite and semi-infinite edges.

Schrödinger equation can be also called to be heat equation with imaginary time. The heat equation on branched structures firstly used in the 50's of the nineteenth century. Thomson (Lord Kelvin) used heat equation (Thomson's cable equation) as mathematical models of signal decay in submarine (under water)

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telegraphic cables (Ch. IV in [9]). Later this method was widely used in neuroscience for theoretical analyzing data collected from intracellular micro electrode recordings and for analyzing the electrical properties of neuronal dendrites (see [10]). Initial and boundary value problems for some other types of PDE on metric graphs and their possible applications can be found in [11], [16]–[17].

We derive global relations, construct explicit integral representations of solutions, and establish a correspondence between Dirichlet and Neumann boundary data at the branching vertex. The obtained results significantly extend the applicability of the Fokas method to complex graph structures and provide a rigorous theoretical basis for diffusion processes on networks.

## 2. Formulation of the Problem and Main Result

We consider symmetric metric graph which obtained by connecting $n$ finite $B_1, ..., B_n$ and $m$ semi infinite $B_{n+1}, B_{n+2}, ..., B_{n+m}$ bonds at one point, called to be vertex of the graph. We correspond the bonds $B_j$, $(j = \overline{1, n})$ to the intervals $(0, L_j)$ and the bonds $B_r$, $(r = \overline{n+1, n+m})$ to intervals $(0, \infty)$ to define coordinates in each bond. Here vertex of the graph corresponds to 0 on each bond (Figure 1).

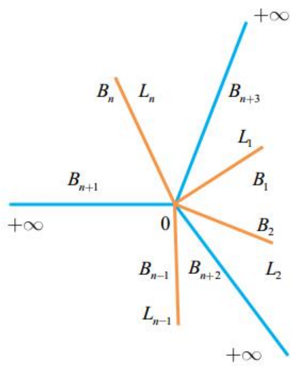


Figure 1. Symmetric metric graph

In each bond of the graph consider the heat transfer equation

$$V_t^{(j)}(x,t) = V_{xx}^{(j)}(x,t) + f^{(j)}(x,t), \; j = \overline{1, n+m}. \tag{2.1}$$

Define initial conditions

$$V^{(j)}(x,0) = V_0^{(j)}(x), \;\; x \in \overline{B_j}, \, j = \overline{1, n+m}. \tag{2.2}$$

Boundary and the asymptotic conditions

$$\left(\alpha_j V^{(j)}(x,t) + \beta_j V_x^{(j)}(x,t)\right)|_{x=L_j-0} = 0, \;\; \alpha_j^2 + \beta_j^2 \neq 0, \; j = \overline{1, n}, \tag{2.3}$$

$$\lim_{x\to\infty} V^{(r)}(x,t) = 0, \, t \geq 0, \, r = \overline{n+1, n+m} \tag{2.4}$$

on finite and semi infinite bonds, respectively.

Moreover, we need to define the following gluing conditions for connectivity of the graph

$$V^{(1)}(+0,t) = V^{(2)}(+0,t) = ... = V^{(n+m)}(+0,t), \tag{2.5}$$

$$\sum_{j=1}^{n+m} \delta_j^2 V_x^{(j)}(+0,t) = 0. \tag{2.6}$$

The last conditions usually called continuity and Kirchhoff conditions on branching point of the graphs. We suppose, that initial data are smooth enough functions and they satisfies the conditions (2.3)–(2.6).

**Theorem 2.1** ***(Cauchy)*** *[18]. Let the function $f(z)$ be holomorphic in a domain $D$ bounded by a finite number of piecewise smooth curves and continuous on $\overline{D}$. Then the integral of $f(z)$ over the oriented boundary of this domain is equal to zero:*

$$\int\limits_{\partial D} f(z)\,dz = 0.$$

**Lemma 2.1** ***(Jordan)*** *[19]. Let $\lambda > 0$ and suppose that the following conditions hold:*

1. *the function $f(z)$ is continuous in the domain $\{\operatorname{Im} z \geq 0,\ |z| \geq R_0 > 0\}$;*
2. *$M(R) = \max_{z\in\gamma_R} |f(z)| \to 0$ as $R \to \infty$, where $\gamma_R = \{|z| = R,\ \operatorname{Im} z \geq 0\}$.*

*Then*

$$\lim_{R\to\infty} \int\limits_{\gamma_R} f(z)e^{i\lambda z}\,dz = 0.$$

Now we consider standard steps in the application of the Fokas method.

**Step 1.** Write the PDE as a one-parameter "local relations" of equations, each of which is in divergence form [14]–[15].

For problem (2.1), a clear choice of an integrating factor to achieve this is $\widetilde{q}(x,t) = e^{-ikx+k^2t}$, leading to the form

$$\left[e^{-ikx+k^2t}V^{(j)}(x,t)\right]_t = \left[e^{-ikx+k^2t}\left(V_x^{(j)}(x,t) + ikV^{(j)}(x,t)\right)\right]_x + \tag{2.7}$$
$$+e^{-ikx+k^2t}f^{(j)}(x,t),\ j = \overline{1, n+m}.$$

**Step 2.** Use the divergence form to determine a global relation coupling the various boundary values. In the case of (2.1), this takes the form

$$e^{wt}\widehat{V}^{(j)}(k,t) - \widehat{V}_0^{(j)}(k) = e^{-ikL_j}\left(h_1^{(j)}(w,t) + ikh_0^{(j)}(w,t)\right) -$$
$$-\left(g_j(w,t) + ikg_0(w,t)\right) + \widehat{F}^{(j)}(k,t),\ j = \overline{1,n}; \tag{2.8}$$
$$e^{wt}\widehat{V}^{(r)}(k,t) - \widehat{V}_0^{(r)}(k) = -g_r(w,t) - ikg_0(w,t) + \widehat{F}^{(r)}(k,t),\ r = \overline{n+1, n+m}. \tag{2.9}$$

where $w = k^2$, $\{k \in \mathbb{C}:\ Imk > 0\}$.

However dispersion relation $w = k^2$ invariant with respect to substitution $k \to -k$ then functions $g_0(w,t),\ g_j(w,t), g_r(w,t),\ h_0^{(j)}(w,t),\ h_1^{(j)}(w,t),\ j = \overline{1,n},\ r = \overline{n+1, n+m}$ also be invariant. Hence from (2.8) and (2.9) we have:

$$e^{wt}\widehat{V}^{(j)}(-k,t) - \widehat{V}_0^{(j)}(-k) = e^{ikL_j}\left(h_1^{(j)}(w,t) - ikh_0^{(j)}(w,t)\right) -$$
$$-\left(g_j(w,t) - ikg_0(w,t)\right) + \widehat{F}^{(j)}(-k,t),\ j = \overline{1,n}; \tag{2.10}$$
$$e^{wt}\widehat{V}^{(r)}(-k,t) - \widehat{V}_0^{(r)}(-k) = -g_r(w,t) + ikg_0(w,t) + \widehat{F}^{(r)}(-k,t), \tag{2.11}$$

where $r = \overline{n+1, n+m}, \{k \in \mathbb{C}:\ Imk < 0\}$.

We can write solution in following form with using inverse Fourier transformation in global relation (2.8) and (2.9) [16]–[17].

$$V^{(j)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\left(\widehat{V}_0^{(j)}(k) + \widehat{F}^{(j)}(k,t)\right)dk-$$
$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-ikL_j-wt}\left(h_1^{(j)}(w,t) + ikh_0^{(j)}(w,t)\right)dk- \tag{2.12}$$

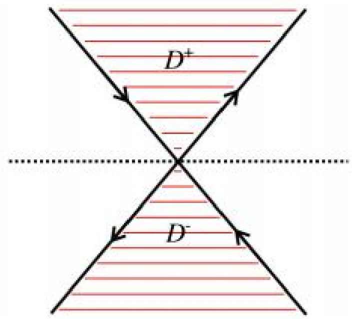


Figure 2. The domains $D^+$ and $D^-$ for the heat transfer equation

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\left(g_j(w,t)+ikg_0(w,t)\right)dk,\ \ j=\overline{1,n},$$

$$V^{(r)}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\left(\widehat{V}_0^{(r)}(k)+\widehat{F}^{(r)}(k,t)\right)dk- \tag{2.13}$$

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\left(g_r(w,t)+ikg_0(w,t)\right)dk,\ \ r=\overline{n+1,n+m}.$$

The integrand of the second integral in (2.12) and (2.13) is entire and decays as $k\to\infty$ for $k\in\mathbb{C}^-\backslash D^-$. The third integral has an integrand that is entire and decays as $k\to\infty$ for $k\in\mathbb{C}^+\backslash D^+$. Using the analyticity of the integrand and applying Jordan's Lemma we can replace the contour of integration of the second integral by - $\int_{\partial D^-}$ and of the third integral by - $\int_{\partial D^+}$ (see [12], [14]–[17]):

$$V^{(j)}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\left(\widehat{V}_0^{(j)}(k)+\widehat{F}^{(j)}(k,t)\right)dk-$$

$$-\frac{1}{2\pi}\int_{\partial D^-} e^{ikx-ikL_j-wt}\left(h_1^{(j)}(w,t)+ikh_0^{(j)}(w,t)\right)dk- \tag{2.14}$$

$$-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-wt}\left(g_j(w,t)+ikg_0(w,t)\right)dk,\ \ j=\overline{1,n},$$

$$V^{(r)}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\left(\widehat{V}_0^{(r)}(k)+\widehat{F}^{(r)}(k,t)\right)dk- \tag{2.15}$$

$$-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-wt}\left(g_r(w,t)+ikg_0(w,t)\right)dk,\ \ r=\overline{n+1,n+m},$$

where $D^\pm=\left\{k\in\mathbb{C}^\pm : Rek^2<0\right\},\ \mathbb{C}^+=\left\{k\in\mathbb{C} : Imk>0\right\},\ \mathbb{C}^-=\left\{k\in\mathbb{C} : Imk<0\right\}.$

We need to find unknowns $g_0(w,t), g_j(w,t), g_r(w,t), h_0^{(j)}(w,t), h_1^{(j)}(w,t),\ j=\overline{1,n}, r=\overline{n+1,n+m}$ representation of the solution.

Now, using vertex conditions we get

$$\begin{cases} e^{wt}\widehat{V}^{(j)}(k,t)-\widehat{V}_0^{(j)}(k)=e^{-ikL_j}h_1^{(j)}(w,t)+ike^{-ikL_j}h_0^{(j)}(w,t)- \\ -g_j(w,t)-ikg_0(w,t)+F^{(j)}(k,t); \\ e^{wt}\widehat{V}^{(j)}(-k,t)-\widehat{V}_0^{(j)}(-k)=e^{ikL_j}h_1^{(j)}(w,t)-ike^{ikL_j}h_0^{(j)}(w,t)- \\ -g_j(w,t)+ikg_0(w,t)+\widehat{F}^{(j)}(-k,t); \\ e^{wt}\widehat{V}^{(r)}(k,t)-\widehat{V}_0^{(r)}(k)=-g_r(w,t)-ikg_0(w,t)+\widehat{F}^{(r)}(k,t); \\ \sum_{j=1}^{6}\delta_j^2 g_j(w,t)=0; \\ \alpha_j h_0^{(j)}(w,t)+\beta_j h_1^{(j)}(w,t)=0. \end{cases} \tag{2.16}$$

Where $j=\overline{1,n}$, $r=\overline{n+1,n+m}$.

Solving this equations for $ikg_0(w,t)$, we have

$$ikg_0(w,t)=\frac{1}{\sum\limits_{j=1}^{n+m}\delta_j^2}\left[\sum_{j=1}^{n}\frac{\delta_j^2}{A_j+i\cdot\frac{\beta_j}{\alpha_j}\cdot k\cdot B_j}\left[e^{ikL_j}\left(1+i\frac{\beta_j}{\alpha_j}\cdot k\right)\widehat{V}_0^{(j)}(k)\ -\right.\right.$$

$$\left.-e^{-ikL_j}\left(1-i\frac{\beta_j}{\alpha_j}\cdot k\right)\widehat{V}_0^{(j)}(-k)\right]+\sum_{r=n+1}^{n+m}\delta_r^2\widehat{V}_0^{(r)}(k)+$$

$$+\sum_{j=1}^{n}\frac{\delta_j^2}{A_j+i\cdot\frac{\beta_j}{\alpha_j}\cdot k\cdot B_j}\left[e^{ikL_j}\left(1+i\frac{\beta_j}{\alpha_j}\cdot k\right)\widehat{F}^{(j)}(k,t)-\right. \tag{2.17}$$

$$\left.-e^{-ikL_j}\left(1-i\frac{\beta_j}{\alpha_j}\cdot k\right)\widehat{F}^{(j)}(-k,t)\right]+\sum_{r=n+1}^{n+m}\delta_r^2\widehat{F}^{(r)}(k,t)-$$

$$-\sum_{j=1}^{n}\frac{\delta_j^2e^{wt}}{A_j+i\cdot\frac{\beta_j}{\alpha_j}\cdot k\cdot B_j}\left[e^{ikL_j}\left(1+i\frac{\beta_j}{\alpha_j}\cdot k\right)\widehat{V}^{(j)}(k,t)-\right.$$

$$\left.\left.-e^{-ikL_j}\left(1-i\frac{\beta_j}{\alpha_j}\cdot k\right)\widehat{V}^{(j)}(-k,t)\right]-\sum_{r=n+1}^{n+m}\delta_r^2e^{wt}\widehat{V}^{(r)}(k,t)\right],$$

where $A_j=e^{ikL_j}-e^{-ikL_j}$, $B_j=e^{ikL_j}+e^{-ikL_j}$, $j=\overline{1,n}$, $\alpha_j\neq 0$.

Now putting $G^{(j)}(k,t)=\widehat{V}_0^{(j)}(k)-ikg_0(w,t)+\widehat{F}^{(j)}(k,t)$, $j=\overline{1,n}$ we can rewrite

$$\begin{cases} e^{wt}V^{(j)}(k,t)=G^{(j)}(k,t)+e^{-ikL_j}\left(1-i\cdot\frac{\beta_j}{\alpha_j}\cdot k\right)h_1^{(j)}(w,t)-g_j(w,t),\\ e^{wt}V^{(j)}(-k,t)=G^{(j)}(-k,t)+e^{ikL_j}\left(1+i\cdot\frac{\beta_j}{\alpha_j}\cdot k\right)h_1^{(j)}(w,t)-g_j(w,t). \end{cases} \tag{2.18}$$

So, we have

$$h_1^{(j)}(w,t)=\frac{1}{A_j+i\cdot\frac{\beta_j}{\alpha_j}\cdot k\cdot B_j}\left[G^{(j)}(k,t)-G^{(j)}(-k,t)-\right.$$

$$\left.-e^{wt}\left(\widehat{V}^{(j)}(k,t)-\widehat{V}^{(j)}(-k,t)\right)\right], \tag{2.19}$$

where $h_0^{(j)}(w,t)=-\frac{\beta_j}{\alpha_j}h_1^{(j)}(w,t),\ \alpha_j\neq 0,\ j=\overline{1,n}$.

$$g_j(w,t)=\frac{1}{A_j+i\cdot\frac{\beta_j}{\alpha_j}\cdot k\cdot B_j}\left[e^{ikL_j}\left(1+i\cdot\frac{\beta_j}{\alpha_j}\cdot k\right)G^{(j)}(k,t)-\right.$$

$$-e^{-ikL_j}\left(1-i\cdot\frac{\beta_j}{\alpha_j}\cdot k\right)G^{(j)}(-k,t)-e^{wt+ikL_j}\left(1+i\cdot\frac{\beta_j}{\alpha_j}\cdot k\right)\widehat{V}^{(j)}(k,t)-$$

$$\left.-e^{wt-ikL_j}\left(1-i\cdot\frac{\beta_j}{\alpha_j}\cdot k\right)\widehat{V}^{(j)}(-k,t)\right], \tag{2.20}$$

this $\alpha_j\neq 0,\ j=\overline{1,n}$.

Solving (2.11) for $g_r(w,t)$ we find

$$g_r(w,t)=\widehat{V}_0^{(r)}(-k)+ikg_0(w,t)+\widehat{F}^{(r)}(-k,t)-e^{wt}\widehat{V}^{(r)}(-k,t). \tag{2.21}$$

Replacing in equation (2.15) $g_r(w,t)$ with the RHS of (2.21) we find [16]–[17]

$$V^{(r)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\left(\widehat{V}_0^{(r)}(k) + \widehat{F}^{(r)}(k,t)\right)dk- \tag{2.22}$$

$$-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-wt}\left(2ikg_0(w,t) + \widehat{V}_0^{(r)}(-k) + \widehat{F}^{(r)}(-k,t)\right)dk, r = \overline{n+1, n+m},$$

The term $e^{wt}\widehat{V}^{(r)}(-k,t)$ gives rise to the term [13]

$$-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx}\widehat{V}^{(r)}(-k,t)dk,\ 0 < x < \infty,\ t > 0,$$

which vanishes, since both $e^{wt}$ and $\widehat{V}^{(r)}(-k,t)$ are bounded and analytic in the upper half of the complex $k$ plane, and furthermore $\widehat{V}^{(r)}(-k,t)$ is of $O\left(\frac{1}{k}\right)$ as $k \to \infty$:

$$\widehat{V}^{(r)}(-k,t) = \int_0^\infty e^{ikx}V^{(r)}(x,t)dx \sim -\frac{V^{(r)}(0,t)}{ik},\ k \to \infty.$$

Thus, Cauchy's theorem supplemented with Jordan Lemma in the domain $D^+$ [12].

We next substitute $g_1^{(j)}(w,t)$ and $h_1^{(j)}(w,t)$ in (2.14). We claim that the terms involving $\widehat{V}^{(j)}(\pm k,t)$ yield a zero contribution. Indeed, since this is a well-posed BVP, the relevant terms are bounded as $k \to \infty$. Let us verify this explicitly: the term in $g_1^{(j)}(w,t)$ involves the following contribution form $\widehat{V}^{(j)}(\pm k,t)$:

$$\frac{e^{ikL_j}\widehat{V}^{(j)}(k,t) - e^{-ikL_j}\widehat{V}^{(j)}(-k,t)}{e^{ikL_j}\left(1 + i\frac{\beta_j}{\alpha_j}k\right) - e^{-ikL_j}\left(1 - i\frac{\beta_j}{\alpha_j}k\right)}.$$

Since $Imk \geq 0$, $e^{-ikL_j}$ grows, and then the above expression, as $k \to \infty$, becomes

$$\frac{-\widehat{V}^{(j)}(-k,t) + e^{ikL_j}\int_0^{L_j} e^{ik(L_j-x)}V^{(j)}(x,t)dx}{i\frac{\beta_j}{\alpha_j}k - 1},$$

which is clearly bounded as $k \to \infty$ with $Imk \geq 0$. Similarly the term in $h_1^{(j)}(w,t)$ involves the following contribution from $\widehat{V}^{(j)}(\pm k,t)$:

$$\frac{\widehat{V}^{(j)}(k,t) - \widehat{V}^{(j)}(-k,t)}{e^{ikL_j}\left(1 + i\frac{\beta_j}{\alpha_j}k\right) - e^{-ikL_j}\left(1 - i\frac{\beta_j}{\alpha_j}k\right)},$$

which as $k \to \infty$, $Imk \leq 0$, simplifies to the expression

$$\frac{\int_0^{L_j} e^{-ik(L_j+x)}V^{(j)}(x,t)dx - e^{-ikL_j}\widehat{V}^{(j)}(-k,t)}{i\frac{\beta_j}{\alpha_j}k - 1},$$

which is clearly bounded as $k \to \infty$, $Imk \leq 0$ [12].

**Theorem 2.2** *Let the functions $V_0^{(j)}(x) \in (\bar{B}_j)$ and $f_j(x,t) \in C(\bar{B}_j \times [0,T])$, where $T > 0$, be absolutely integrable on $\bar{B}_j$ for $j = \overline{1, n+m}$. Then the regular solution of problem (2.1)–(2.6) is given by formulas (2.19), (2.20), (2.21), and (2.22).*

## 3. Conclusion

This study develops a rigorous and unified analytical approach for constructing solutions to initial-boundary value problems on metric graphs, with a particular focus on star-shaped configurations. The proposed framework is not limited to this specific class of graphs but is applicable to a broad range of arbitrary metric graphs, thereby significantly extending the scope of analytical methods available for such problems.

The methodology is built upon the systematic use of the global relation and an auxiliary spectral relation obtained through the transformation of the complex parameter. These relations provide a powerful mechanism for establishing an intrinsic correspondence between Dirichlet and Neumann conditions at graph vertices. As a result, the original initial-boundary value problem is reduced to a finite system of algebraic equations governing the unknown solution values at the branching points of the graph.

A key contribution of this work is the explicit integral representation of the solution in terms of contour integrals of known functions, with integration paths carefully chosen to ensure exponential decay of the integrands at infinity. This property guarantees strong convergence of the resulting integral expressions and is particularly important for the stability and efficiency of numerical implementations.

Overall, the results presented in this paper not only generalize the Fokas unified transform method to the setting of metric graphs but also provide a robust theoretical foundation for the analysis of diffusive processes in complex branched structures. The developed approach opens new directions for further research on partial differential equations on networks and offers promising perspectives for applications in physics, engineering, and interdisciplinary fields where graph-based models play a fundamental role.

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