



Planarity of Permutability Intersection Graph of Subgroups of Finite Groups

A. Esakkiraj and P. Devi

ABSTRACT: Let G be a group. Then the permutability intersection graph of G is a graph and it is denoted by $\text{PI}(G)$ whose vertex set is all the proper subgroups of G and two vertices H, K are adjacent if and only if they permute and $H \cap K \neq \{e\}$. In this paper, we classified all finite groups whose permutability intersection graph is one of planar, bipartite, C_3 -free. Also, we have studied some other properties for the same.

Keywords: Permutability intersection graph, finite groups, planar, bipartite graph.

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1. Introduction

By Connecting an appropriate graph to an algebraic structure and utilizing graph theory tools, one can investigate the attributes of the structure. Algebraic graph theorists have been interested in this area recently and have made major contributions, especially when the algebraic structure is a group see [4,6,7,8,13,14,21,29].

The non-normal permutability graph of a given group G was first presented by Binachi et al. in [9]. In [18,10], they carried on with this effort. In [24], the authors defined a permutability graph of a group is a graph whose vertex set is all proper subgroups of G and two vertices are adjacent if and only if they permute.

The intersection graph for semi group was studied by Bosak in 1964 [12]. In this direction, Csákány and Pollák defined the intersection graph for a group and studied their properties [17]. Let G be a group. A graph that has all of the proper subgroups of G as its vertices is known as the intersection graph of subgroups of G . Two vertices H, K are adjacent if and only if $H \cap K \neq \{e\}$. In [31], Zelinka conducted some research on the intersection graph of finite abelian groups. These have inspired other researchers to define variety of algebraic structures, including rings, vector spaces, etc and produced some intriguing findings see [1,2,3,15,22,28,30]. In [23], Selçuk Kayacan et al examined the planarity of intersection graph of subgroups of finite groups.

we were motivated by these findings to investigate the combination of intersection graphs and permutability graphs. In this research, the investigation is regarding the finite groups whose permutability intersection graph of subgroup is planar and studied some other properties.

The basic definitions and notations of graph theory which are used in this paper are from [20].

The following theorems are used for the proof of classification of finite groups whose permutability intersection graphs are planar.

Let P, Q, R be sylow p, q, r - subgroups of G respectively.

Theorem 1.1 ([25], Theorem 3.1) *Let G be a finite group and p, q be two distinct primes. Then $\Gamma(G)$ is bipartite if and only if G is isomorphic to one of $\mathbb{Z}_{p^3}, \mathbb{Z}_{pq}, \mathbb{Z}_q \rtimes \mathbb{Z}_p$.*

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Theorem 1.2 ([27], Theorem 5.2) *Let G be a finite group and p, q, r be distinct primes. Then $\Gamma(G)$ is bipartite if and only if G is isomorphic to one of \mathbb{Z}_{p^α} ($\alpha = 2, 3$), \mathbb{Z}_{pq} , $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, A_4 or $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^1b^l \rangle$, where $\begin{pmatrix} 0 & -1 \\ 1 & l \end{pmatrix}$ has order q in $GL_2(p)$, where $q \mid (p + 1)$.*

Theorem 1.3 ([24], Theorem 5.1) *Let G be a finite group. Then $\Gamma(G)$ is planar if and only if G is isomorphic to one of the following groups (where p and q are distinct primes):*

1. \mathbb{Z}_{p^α} , $2 \leq \alpha \leq 5$;
2. $\mathbb{Z}_{p^\alpha q}$, $\alpha = 1, 2$;
3. $\mathbb{Z}_p \times \mathbb{Z}_p$, $p = 2, 3$;
4. Q_8 ;
5. $\mathbb{Z}_q \rtimes \mathbb{Z}_p$;
6. A_4 ;
7. $\mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, \text{ord}_q(i) = p^2 \rangle$, with $p^2 \mid (q - 1)$.

In [23] Selçuk Kayacan et al. have classified all the finite groups whose intersection graphs of subgroups are planar, Which is given below.

Theorem 1.4 Let G be a finite group and p, q, r be distinct primes. Then $\Gamma(G)$ is planar if and only if G is one of the following groups:

1. \mathbb{Z}_{p^α} ($\alpha = 2, 3, 4, 5$), $\mathbb{Z}_{p^\alpha q}$ ($\alpha = 1, 2$), \mathbb{Z}_{pqr} , $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_6 \times \mathbb{Z}_2$;
2. Q_8 , M_8 , $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, $\mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2}$, A_4 ;
3. $G_1 \cong \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^1b^l \rangle$, where $\begin{pmatrix} 0 & -1 \\ 1 & l \end{pmatrix}$ has order q in $GL_2(p)$, where $q \mid (p + 1)$;
4. $G_2 \cong \langle a, b, c \mid a^p = b^p = c^{q^2} = 1, ab = ba, cac^{-1} = b^{-1}, cbc^{-1} = a^1b^l \rangle$, where $\begin{pmatrix} 0 & -1 \\ 1 & l \end{pmatrix}$ has order q^2 in $GL_2(p)$, where $q^2 \mid (p + 1)$;
5. $G_3 \cong \langle a, b, c \mid a^p = b^q = c^r = 1, b^{-1}ab = a^\mu, cac^{-1} = a^v, bc = cb \rangle$, q, r are divisors of $(p - 1)$, $\mu, v \neq 1$.

The intersection graph of subgroups of groups listed in Theorem 1.4, are given below for use in the subsequent sections.

$$\Gamma(\mathbb{Z}_{p^\alpha}) \cong K_{\alpha-1}, \alpha = 2, 3, 4, 5. \quad (1.1)$$

$$\Gamma(\mathbb{Z}_{p^\alpha}) \cong \overline{K}_2. \quad (1.2)$$

$$\Gamma(\mathbb{Z}_{p^2q}) \cong K_1 + (K_2 \cup K_2). \quad (1.3)$$

$$\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p) \cong \overline{K}_{p+1}. \quad (1.4)$$

$$\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_2) \cong K_1 + (K_4 \cup \overline{K}_2). \quad (1.5)$$

$$\Gamma(Q_8) \cong K_4. \quad (1.6)$$

$$\Gamma(\mathbb{Z}_q \rtimes \mathbb{Z}_p) \cong \overline{K}_{q+1}. \quad (1.7)$$

$$\Gamma(\mathbb{Z}_p \rtimes_2 \mathbb{Z}_{p^2}) \cong K_1 + (K_1 \cup qK_2). \quad (1.8)$$

$$\Gamma(G_1) \cong (K_{1,p+1} \cup \overline{K}_{p^2}). \quad (1.9)$$

$$\Gamma(A_4) \cong (K_{1,3} \cup \overline{K}_4). \quad (1.10)$$

$$\Gamma(G_2) \cong K_1 + (K_{1,p+1} \cup p^2K_2). \quad (1.11)$$

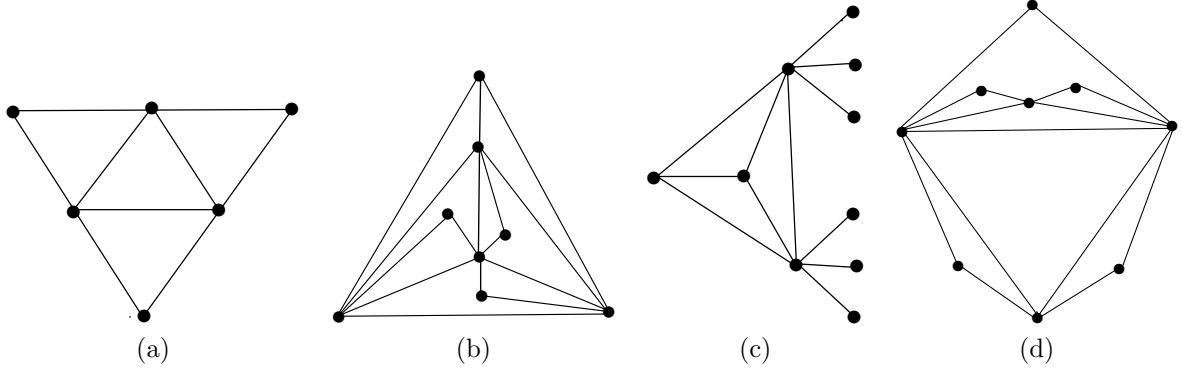


Figure 1: (a) $\Gamma(\mathbb{Z}_{pqr})$, (b) $\Gamma(\mathbb{Z}_6 \times \mathbb{Z}_2)$, (c) $\Gamma(M_8)$, (d) $\Gamma(G_3)$.

2. Finite Abelian Groups

In this section, we are going to classify the finite abelian groups whose permutability intersection graphs of subgroups are planar, C_3 -free, acyclic or bipartite.

Lemma 2.1 *Let G be a finite abelian group. Then $PI(G) \cong I(G)$.*

Proof: Since G is a finite abelian group, so all the proper subgroups of G permutes with each other. But they are need not intersect with each other. Therefore, $PI(G) \cong I(G)$. \square

Proposition 2.1 *Let G be a finite abelian group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_i 's are distinct primes and $\alpha_i \geq 1, k \geq 1$. Then*

1. $PI(G)$ is planar if and only if G is isomorphic to one of \mathbb{Z}_{p^α} ($\alpha = 2, 3, 4, 5$), $\mathbb{Z}_{p^\alpha q}$ ($\alpha = 1, 2$), $\mathbb{Z}_p \times \mathbb{Z}_p$ ($p = 2, 3$), $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ ($p = 2$), $\mathbb{Z}_{pq} \times \mathbb{Z}_p$ ($p = 2$);
2. $PI(G)$ is bipartite if and only if G is isomorphic to \mathbb{Z}_{p^3} .

Proof: By Lemma 2.1 and Theorem 1.4 proof follows. \square

3. Finite Non-Abelian Groups

In this section, we consider finite non-abelian groups.

Proposition 3.1 *Let G be a finite non-abelian group of order p^α , $\alpha \geq 3$ and p is a prime. Then*

1. $PI(G)$ is planar if and only if G is isomorphic to either Q_8 or M_8 ;
2. $PI(G)$ contains C_3 .

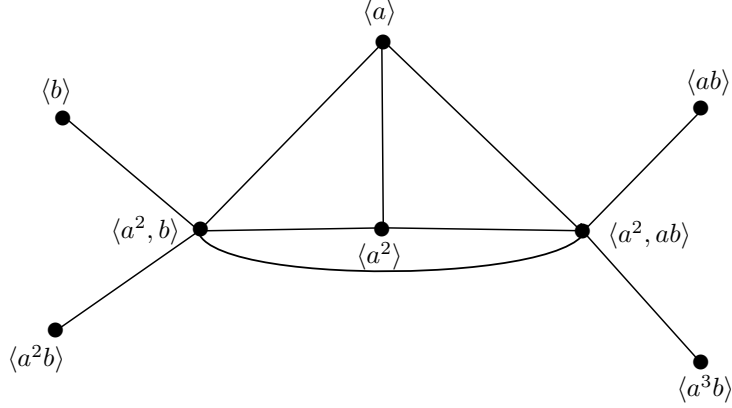
Proof: Let H_1, H_2, H_3 be three subgroups of G of order $p^{\alpha-1}$. Since H_1, H_2, H_3 are normal and has non-trivial intersection. It follows that $PI(G)$ contains C_3 .

Case 1: $\alpha = 3$. If p is even, then $G \cong Q_8$ or M_8 . $G \cong Q_8, M_8$. then by Theorem 1.3 and Theorem 1.4, $PI(Q_8) \cong K_4$ and $PI(M_8)$, which is shown in Figure 2.

If p is odd, then $G \cong M_{p^3}$ and $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.

If $G \cong M_{p^3}$, then every subgroups of G permutes with each other. So $PI(G) \cong I(G)$. By Proposition 4.1 in [27], $PI(G)$ contains K_5 .

suppose $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p = \langle a, b, c \mid a^p = b^p = c^p = 1, ab = ba, ca = ac, cbc^{-1} = cb \rangle$. Here $H_1 = \langle a, b \rangle$, $H_2 = \langle a, c \rangle$, $H_3 = \langle b, c \rangle$, $H_4 = \langle ab, c \rangle$, $H_5 = \langle b, ac \rangle$, $H_6 = \langle ab^2, c \rangle$ are subgroups of G and H_1, H_2, H_3 are normal in G which permutes with every subgroups of G . They are intersect non-trivially with

Figure 2: $PI(M_8)$.

H_4, H_5, H_6 . It follows that $PI(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

Case 2: $\alpha \geq 4$. Let $H_1, H_2, H_3, H_4, H_5, H_6$ be subgroups of G of order $p^{\alpha-1}, p^{\alpha-1}, p^{\alpha-1}, p^{\alpha-2}, p^{\alpha-2}, p^{\alpha-2}$ respectively. Here H_1, H_2, H_3 are normal in G which permutes with every subgroups of G and $H_i \cap H_j \neq e$ for every $i = 1, 2, 3$ and $j = 4, 5, 6$. It follows that $PI(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$. \square

Proposition 3.2 *Let G be a group of order pq , where $p < q$ are distinct primes, then $PI(G)$ is planar and bipartite.*

Proof: By Theorem 1.4 and Proposition 4.2 in [26],

$$PI(G) \cong \overline{K}_{q+1}. \quad (3.1)$$

\square

Proposition 3.3 *Let G be a group of order p^2q , where p and q are distinct primes. Then*

1. $PI(G)$ is planar if and only if G is isomorphic to one of $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}, \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q, (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q, A_4$;
2. $PI(G)$ is bipartite if and only if G is isomorphic to one of $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$.

Proof: Consider $(p, q) \neq (2, 3)$. In this case, we have only six groups.

If $p < q$ and $p \mid (q-1)$, then $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, \text{ord}_q(i) = p \rangle$. Here G has exactly one subgroup of order q , say H ; q subgroups of order p^2 , say $H_1, H_2, H_3, \dots, H_q$; one subgroup of order pq , say N ; one subgroup of order p , say K . H, N, K are normal in G , permutes with every other subgroups of G . No two $H_1, H_2, H_3, \dots, H_q$ permutes; K is a subgroup of $H_1, H_2, H_3, \dots, H_q, N$ and no two remaining subgroups has non-trivial intersection.

$$PI(G) \cong K_1 + (K_1 \cup K_{1,3}). \quad (3.2)$$

Which is planar and contains C_3 .

If $p < q$ and $p \mid (q-1)$, then $G = \langle a, b, c \mid a^q = b^p = c^p = 1, bab^{-1} = a^i, ac = ca, bc = cb, ord_q(i) = p \rangle$. Here $H_1 = \langle a, b \rangle$, $H_2 = \langle a, c \rangle$, $H_3 = \langle b, c \rangle$, $H_4 = \langle a, bc \rangle$, $H_5 = \langle ab, c \rangle$, $H_6 = \langle c \rangle$ are the subgroups of G . H_1, H_2, H_4 are normal in G which permutes with every subgroups of G . H_1 intersect non-trivially with H_2, H_3, H_4, H_5 . H_6 is a subgroup of H_3, H_5 ; H_2, H_3, H_4, H_5 are intersects non-trivially. Therefore, $PI(G)$ contains a subdivision of K_5 and it contain C_3 , which is shown in Figure 3.

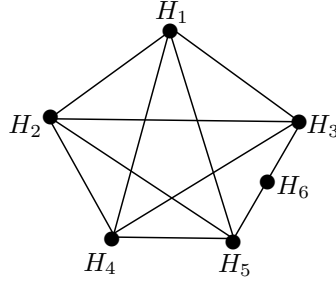


Figure 3: Subdivision of K_5 .

If $p < q$ and $p^2 \mid (q-1)$, then $G \cong \langle a, b \mid a^{p^2} = b^q = 1, bab^{-1} = a^i, ord_q(i) = p^2 \rangle$. By equation (4.5) in [26] and equation (1.8) in Theorem 1.4,

$$PI(G) = K_1 + (K_1 \cup qK_2). \quad (3.3)$$

So, $PI(G)$ is planar and contains C_3 .

If $p > q$ and $q \mid (p-1)$, then $G \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q = \langle a, b \mid a^{p^2} = b^q = 1, bab^{-1} = a^i, ord_{p^2}(i) = q \rangle$. Here $H_1 = \langle a \rangle$, $H_2 = \langle a^3 \rangle$, $H_3 = \langle a^3, b \rangle$, $H_4 = \langle a^3, ab \rangle$, $H_5 = \langle a^3, a^2b \rangle$, $H_6 = \langle b \rangle$, $H_7 = \langle a^3b \rangle$, $H_8 = \langle a^6b \rangle$, $H_9 = \langle ab \rangle$, $H_{10} = \langle a^4b \rangle$, $H_{11} = \langle a^7b \rangle$, $H_{12} = \langle a^2b \rangle$, $H_{13} = \langle a^5b \rangle$, $H_{14} = \langle ba \rangle$ are the proper subgroups. H_3 permutes and intersect non-trivially with H_1, H_2, H_6, H_7 and H_8 . H_4 permutes and intersect non-trivially with H_1, H_2, H_9, H_{10} and H_{11} . H_5 permutes and intersect non-trivially with H_1, H_2, H_{12}, H_{13} and H_{14} . Which is shown in Figure 4.

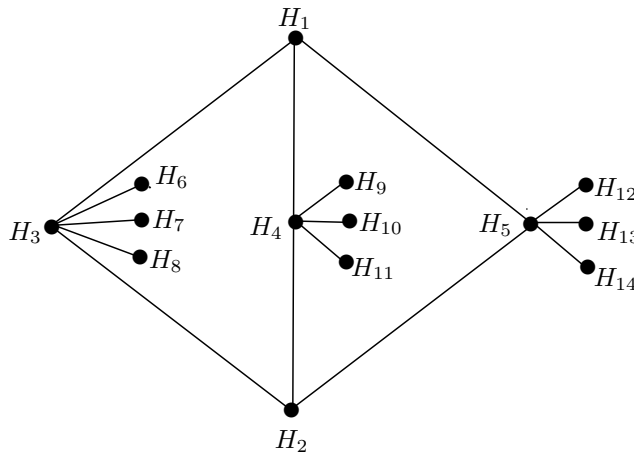


Figure 4: $PI(\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q)$.

If $p > q$ and $q \mid (p-1)$, then $G \cong \langle a, b, c \mid a^p = b^p = c^q = 1, cac^{-1} = a^i, cbc^{-1} = b^{i^t}, ab = ba, ord_p(i) = q \rangle$. There are $(q+3)/2$ isomorphism types in this family one for $t = 0$ and one for each pair $\{x, x^{-1}\}$ in F_p^x . We will refer to all of these groups as $G_{5(t)}$ of order p^2q . Here $H_1 = \langle a, b \rangle$, $H_2 = \langle b, c \rangle$, $H_3 =$

$\langle a, c \rangle, H_4 = \langle ab, c \rangle, H_5 = \langle a^2b, c \rangle$ are proper subgroups of G . $H_i, i = 1, 2, 3, 4, 5$ permutes and intersects non-trivially with each other. So, they forms K_5 as a subgraph.

If $p > q$ and $q \mid (p+1)$, then $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q = \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = a^i b^j, cbc^{-1} = a^k b^l \rangle$, where, $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ has order q in $GL_2(p)$.

By equation (4.7) in [26] and equation (1.9) in Theorem 1.4,

$$PI(G) \cong K_{1,p+1} \cup K_{p^2}. \quad (3.4)$$

So, $PI(G)$ is planar and also it is bipartite.

Note that, $(p, q) = (2, 3)$. Up to isomorphism, there are three non-abelian group of order 12: $\mathbb{Z}_3 \rtimes \mathbb{Z}_4, D_{12}$ and A_4 . But we proved that $PI(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ is planar. In $D_{12}, H_1 = \langle a \rangle, H_2 = \langle a^3 \rangle, H_3 = \langle a^3, b \rangle, H_4 = \langle a^3, ab \rangle, H_5 = \langle a^3, a^2b \rangle$ are proper subgroups of G . $H_i, i = 1, 2, 3, 4, 5$ permutes and intersects non-trivially with each other. So, they forms K_5 as a subgraph of $PI(D_{12})$. For the case of A_4 . By the equation (4.4) in [24] and equation (1.10) in Theorem 1.4,

$$PI(G) \cong K_{1,3} \cup \overline{K}_4. \quad (3.5)$$

So, $PI(A_4)$ is planar and bipartite.

Comparing all the cases, proof follows. \square

Proposition 3.4 Let G be a group of order $p^\alpha q$, when p, q are distinct primes. Then $PI(G)$ is non-planar and contains C_3 .

Proof: Since $\alpha \geq 3$, then G has chain of subgroups of p, p^2, p^3 , say H_1, H_2, H_3 . Here $H_1 \leq H_2 \leq H_3$ and so they are adjacent in $PI(G)$. Therefore, $PI(G)$ contains C_3 .

Let P denote a sylow p -subgroup of G . We shall prove this result by induction on α . Let $\alpha = 3$, If $p > q$, then $n_p(G) = 1$ and $G \cong P \rtimes \mathbb{Z}_q$. If $PI(P)$ is non-planar, then $PI(G)$ is non-planar. By Propositions 2.1, 3.1 and Theorem 1.4, P is isomorphic to one of $\mathbb{Z}_{p^3}, \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ or M_{p^3} .

If $P \cong \mathbb{Z}_{p^3}$, then $G \cong \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_q = \langle a, b \mid a^{p^3} = b^q = 1, bab^{-1} = a^i, \text{ord}_q(i) = q \rangle$. Here $H_1 = \langle a \rangle, H_2 = \langle a^p \rangle, H_3 = \langle a^{p^2} \rangle, H_4 = \langle a^p, b \rangle, H_5 = \langle a^{p^2}, b \rangle, H_6 = \langle a^p, ab \rangle$ are proper subgroups of G . H_1, H_2, H_3 are normal in G ; H_1, H_2, H_3 are intersect non-trivially with H_4, H_5, H_6 . It follows that they form $K_{3,3}$ as a subgraph of $PI(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $P \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, then $G \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$. Here $H_1 = \langle a^p \rangle, H_2 = \langle a, c \rangle, H_3 = \langle a \rangle, H_4 = \langle a^p, c \rangle, H_5 = \langle a^p, b, c \rangle, H_6 = \langle a^p, b \rangle$ are the proper subgroups of G . H_1, H_2, H_3 are permutes and intersect non-trivially with H_4, H_5, H_6 . It follows that they form $K_{3,3}$ as a subgraph of $PI(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $P \cong M_{p^3}$, then $G \cong M_{p^3} \rtimes \mathbb{Z}_q$. let $\mathbb{Z}_q = \langle c \rangle$ Here $H_1 = \langle a, c \rangle, H_2 = \langle a^2, c \rangle, H_3 = \langle ab \rangle, H_4 = \langle ab, c \rangle, H_5 = \langle a^p, b \rangle, H_6 = \langle a, b \rangle$ are the proper subgroups of G . where $\langle a, b \rangle = P$. H_1, H_2, H_3 are permutes and intersect non-trivially with H_4, H_5, H_6 . It follows that they form $K_{3,3}$ as a subgraph of $PI(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

Now, let us consider the case $p < q$ and $(p, q) \neq (2, 3)$. we have $G \cong \mathbb{Z}_q \rtimes P$. By Propositions 2.1, 3.1 and Theorem 1.4, P is isomorphic to one of $\mathbb{Z}_{p^3}, \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p, Q_8, M_8$ or M_{p^3} .

If $P \cong \mathbb{Z}_{p^3}$, then $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^3} = \langle a, b \mid a^q = b^3 = 1, bab^{-1} = a^i, \text{ord}_q(i) = p \rangle$. Here $H_1 = \langle a, b^{p^2} \rangle, H_2 = \langle a, b^p \rangle, H_3 = \langle b^{p^2} \rangle, H_4 = \langle a^3b \rangle, H_5 = \langle ab \rangle, H_6 = \langle a^2b \rangle$ are proper subgroups of G . Here H_1 is normal in G and intersect non-trivially with H_2, H_4, H_5, H_6 . H_1 is permutes and intersect non-trivially with H_4, H_5, H_6 . It follows that they form $K_{3,3}$ as a subgraph of $PI(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $P \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, then $G \cong \mathbb{Z}_q \rtimes (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$. Here $H_1 = \langle a^p \rangle, H_2 = \langle a, c \rangle, H_3 = \langle a \rangle, H_4 = \langle a^p, b, c \rangle, H_5 = \langle a^p, b \rangle, H_6 = \langle a^p, c \rangle$ are the proper subgroups of G . H_1, H_2, H_3 are permutes and intersect non-trivially with H_4, H_5, H_6 . It follows that they form $K_{3,3}$ as a subgraph of $PI(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $P \cong M_{p^3}, G \cong \mathbb{Z}_q \rtimes M_{p^3}$. It has the same subgroup lattices of the above. By the above argument $PI(G)$ is non-planar.

If $P \cong Q_8$, $G \cong \mathbb{Z}_q \times Q_8$. Here $H_1 = \langle a \rangle$, $H_2 = \langle a, b \rangle$, $H_3 = \langle a, c \rangle$, $H_4 = \langle a^2 \rangle$, $H_5 = \langle a^2, c \rangle$, $H_6 = \langle ab, c \rangle$ are the proper subgroups of G ; Here H_1, H_2, H_3 are normal in G . H_1, H_2, H_3 are intersect non-trivially with H_4, H_5, H_6 . It follows that they form $K_{3,3}$ as a subgraph of $\text{PI}(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $P \cong M_8$, then $G \cong \mathbb{Z}_q \times M_8$. Here $H_1 = \langle a \rangle$, $H_2 = \langle a^2 \rangle$, $H_3 = \langle a, c \rangle$, $H_4 = \langle a^2, c \rangle$, $H_5 = \langle a^2, b \rangle$, $H_6 = \langle a^2, b, c \rangle$ are the proper subgroups of G ; Here H_1, H_2, H_3 permutes and intersect non-trivially with H_4, H_5, H_6 . It follows that they form $K_{3,3}$ as a subgraph of $\text{PI}(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $(p, q) = (2, 3)$, then $G \cong S_4$. In this Case G has atleast two copies of D_8 . By Proposition 4.1, $\text{PI}(G)$ is non-planar.

Assume that the result is true for $\alpha > 3$. Now $|G| = p^\alpha q$, $\alpha \geq 4$. If $n_p(G) = 1$, then $G \cong \mathbb{Z}_{p^4} \times \mathbb{Z}_q$ and G has a subgroup isomorphic to $\mathbb{Z}_{p^3} \times \mathbb{Z}_q$, by the above argument $\text{PI}(\mathbb{Z}_{p^3} \times \mathbb{Z}_q)$ is non-planar and contains C_3 . Otherwise G has a normal subgroup, say N with prime index, $|N| = p^{\alpha-1}q$. Now by induction hypothesis, $\text{PI}(N)$ is non-planar and contains C_3 . So, Proof follows. \square

Proposition 3.5 Let G be a group of order p^2q^2 , where p, q are distinct primes.

1. $\text{PI}(G)$ planar if and only if G is isomorphic to $(\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_{q^2}$
2. $\text{PI}(G)$ contains C_3 .

Proof: G has chain of subgroups of p, p^2, p^2q , say H_1, H_2, H_3 . Here $H_1 \leq H_2 \leq H_3$ and so they are adjacent in $\text{PI}(G)$. Therefore, $\text{PI}(G)$ contains C_3 .

Without loss of generality $p > q$.

Let P and Q be a sylow p, q subgroups of G respectively.

Case 1: $(p, q) \neq (3, 2)$, then $n_p(G) = 1$ and $G \cong P \times Q$.

If $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} = \langle a, b \mid a^{p^2} = b^{q^2} = 1, bab^{-1} = a^i, i^{q^2} \equiv 1 \pmod{p^2} \rangle$, then we have $H = \langle a^p, b \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{q^2}$, so by Proposition 3.3, H together with its subgroups form K_5 as a subgroup of $\text{PI}(G)$.

If $G \cong \mathbb{Z}_{p^2} \times (\mathbb{Z}_q \times \mathbb{Z}_q)$ then we have proper subgroups $H_1 = \langle a \rangle$, $H_2 = \langle a, b \rangle$, $H_3 = \langle a, c \rangle$, $H_4 = \langle a^p \rangle$, $H_5 = \langle a^p, b, c \rangle$ in which H_1, H_2, H_3 are normal in G ; H_4 and H_5 permutes; $H_i, i = 1, 2, 3, 4, 5$ are intersects non-trivially with each other. It follows that form K_5 as a subgroup of $\text{PI}(G)$.

If $(\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_{q^2} = \langle a, b, c \mid a^p = b^p = c^{q^2} = 1, ab = ba, cac^{-1} = a^i b^j, cbc^{-1} = a^k b^l \rangle$, where $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ has order q^2 in $GL_2(p)$, $q^2 \mid (p+1)$. By equation (1.11) in Theorem 1.4 and Proposition 4.4 in [26],

$$\text{PI}(G) \cong K_1 + (K_{1,p+1} \cup p^2 K_2) \quad (3.6)$$

So, $\text{PI}(G)$ is planar. It contains C_3 .

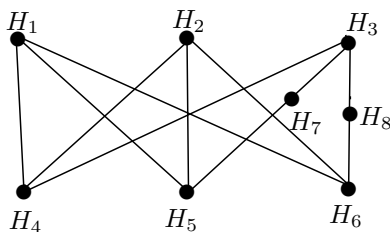
Case 2: $(p, q) = (3, 2)$ upto isomorphism, there are nine groups of order 36.

If $G \cong D_{18}$. Here $H_1 = \langle a \rangle$, $H_2 = \langle a^2 \rangle$, $H_3 = \langle a^3 \rangle$, $H_4 = \langle a^2, b \rangle$, $H_5 = \langle a^2, ab \rangle$ are the proper subgroups of G . Here $H_i = 1, 2, 3, 4, 5$ intersect with each other and all normal in G . H_4 and H_5 permutes. It follows that $\text{PI}(G)$ contains K_5 as a subgroup.

If $G \cong S_3 \times S_3$. Here $H_1 = S_3 \times \{e\}$, $H_2 = \{e\} \times S_3$, $H_3 = \langle a \rangle \times \{e\}$, $H_4 = \langle a \rangle \times \langle a \rangle$, $H_5 = \langle b \rangle \times \langle b \rangle$, $H_6 = \langle ab \rangle \times \langle ab \rangle$, $H_7 = \langle a \rangle \times \langle b \rangle$, $H_8 = \langle a \rangle \times \langle ab \rangle$ are the proper subgroups of G . H_1, H_2, H_3 permutes with H_4, H_5, H_6 ; H_1, H_2 are intersect non-trivially with H_4, H_5, H_6 ; H_3 intersect non-trivially with H_4 ; H_3 intersect non-trivially with H_7 ; H_7 intersect non-trivially with H_5 ; H_3 intersect non-trivially with H_8 ; H_8 intersect non-trivially with H_6 . It follows that $\text{PI}(G)$ contains a subdivision of $K_{3,3}$ as a subgraph which is shown in Figure 5.

If $G \cong \mathbb{Z}_3 \times A_4$. Here $H_1 = \{e\} \times A_4$, $H_2 = \{e\} \times \langle (12)(34), (13)(24) \rangle$, $H_3 = \mathbb{Z}_3 \times \langle (12)(34), (13)(24) \rangle$, $H_4 = \mathbb{Z}_3 \times \langle (12)(34) \rangle$, $H_5 = \{e\} \times \langle (13)(24) \rangle$, $H_6 = \{e\} \times \langle (12)(34) \rangle$ are the proper subgroups of G . H_1, H_2, H_3 permutes with H_4, H_5, H_6 and H_1, H_2, H_3 intersect nontrivially with H_4, H_5, H_6 . It follows that $K_{3,3}$ is a subgroup of $\text{PI}(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $G \cong \mathbb{Z}_6 \times S_3$. Here $H_1 = \{e\} \times S_3$, $H_2 = \mathbb{Z}_2 \times S_3$, $H_3 = \mathbb{Z}_3 \times S_3$, $H_4 = \mathbb{Z}_2 \times \langle (12) \rangle$, $H_5 = \mathbb{Z}_3 \times \langle (12) \rangle$, $H_6 = \{e\} \times \langle (12) \rangle$, are the proper subgroups of G . H_1, H_2, H_3 permutes with H_4, H_5, H_6 and H_1, H_2, H_3 intersect non-trivially with H_4, H_5, H_6 . It follows that $K_{3,3}$ is a subgroup of $\text{PI}(G)$ with bipartition

Figure 5: Subdivision of $K_{3,3}$.

$X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $G \cong Z_9 \rtimes Z_4 = \langle a, b \mid a^9 = b^4 = 1, bab^{-1} = a^i, i^4 \equiv 1 \pmod{9} \rangle$. Then $H_1 = \langle a \rangle$, $H_2 = \langle a^3 \rangle$, $H_3 = \langle a, b^2 \rangle$, $H_4 = \langle a^3, b \rangle$, $H_5 = \langle a^3, b^2 \rangle$ are proper subgroups of G . H_1, H_2, H_3 are normal in G ; H_4 permutes with H_5 ; they intersect non-trivially. So they form K_5 as a subgraph of $\text{PI}(G)$.

If $Z_3 \times (Z_3 \times Z_4) = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, cbc^{-1} = b^i, \text{ord}_2(i) = 3 \rangle$, then $H_1 = \langle ab^2 \rangle$, $H_2 = \langle b^2 \rangle$, $H_3 = \langle a, b^2 \rangle$, $H_4 = \langle b \rangle$, $H_5 = \langle ab \rangle$, $H_6 = \langle a^2b \rangle$ are proper subgroups of G . H_1, H_2, H_3 and intersect non-trivially with H_4, H_5, H_6 . It follows that $K_{3,3}$ as a subgroup of $\text{PI}(G)$ with bipartition $X = \{H_1, H_2, H_3\}$ and $Y = \{H_4, H_5, H_6\}$.

If $(Z_3 \times Z_3) \rtimes Z_4 = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, cac^{-1} = a^i b^j, cbc^{-1} = a^k b^l \rangle$, where $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ has order 4 in $GL_2(3)$. we already discussed the permutability intersection graph of subgroups of this group in the above case 1. Which is planar.

If $Z_2 \times ((Z_3 \times Z_3) \times Z_2)$. Then $Z_2 \times (Z_3 \times Z_3)$ is a proper subgroup of G . Therefore, $Z_6 \times Z_3$ is a subgroup. By Proposition 2.1, $\text{PI}(Z_6 \times Z_3)$ is non-planar.

If $(Z_2 \times Z_2) \rtimes Z_9$. Then $H_1 = \langle a, c \rangle$, $H_2 = \langle b, c \rangle$, $H_3 = \langle a, b \rangle$, $H_4 = \langle a, b, c^3 \rangle$, $H_5 = \langle a, b, c^2 \rangle$, H_6 are the proper subgroups of G . H_1 and H_2 are normal in G . $H_i, i = 1, 2, 3, 4, 5$ permutes and intersects non-trivially with each other. It follows that K_5 as a subgroup of $\text{PI}(G)$. \square

Proposition 3.6 Let G be a group of order $p^\alpha q^\beta$, where p, q are distinct primes and $\alpha + \beta \geq 5$, then $\text{PI}(G)$ is non-planar and contains C_3

Proof: since G is solvable, it has a normal subgroup, say N with prime index. If $\text{PI}(N)$ is non-planar and contains C_3 , then $\text{PI}(G)$ also the same. So, by Propositions 2.1, 3.4 and 3.5, $N \cong (Z_p \times Z_p) \rtimes Z_{q^2}$, $Z_{p^2q} \times Z_p$.

If $[G:N] = p$, then $|N| = p^2q^2$. If $N \cong (Z_p \times Z_p) \rtimes Z_{q^2}$, then by Proposition 3.5, $\text{PI}(N) \cong K_1 + (K_{1,p+1} \cup p^2K_2)$. N together with its proper subgroups forms $K_{3,3}$. Therefore, $\text{PI}(G)$ is non-planar.

If $[G:N] = q$, then $|N| = p^3q$. If $N \cong Z_{p^2q} \times Z_p$. $H_i, i = 1, 2, 3, 4, 5$ be subgroup of N of order p, p^2, p^3, pq, p^2q respectively, then they intersect non-trivially and permutes with each other. It follows that K_5 as a subgraph of $\text{PI}(N)$. Therefore, $\text{PI}(G)$ is non-planar. \square

Proposition 3.7 Let G be a non-abelian solvable group of order pqr , where p, q, r are distinct primes, $p > q > r$. then,

1. $\text{PI}(G)$ is planar if and only if $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, b^{-1}ab = a^u, c^{-1}ac = a^v, bc = cb \rangle$, $u, v \neq 1$;
2. $\text{PI}(G)$ contains C_3 .

Proof: If $r \mid (p-1)$ and $r \mid (q-1)$ and $q \nmid (p-1)$, then $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, ac = ca, c^{-1}bc = b^v \rangle$, $v \neq 1$. Here $H_1 = \langle a \rangle$, $H_2 = \langle a, b \rangle$, $H_3 = \langle a, c \rangle$, $H_4 = \langle a, bc \rangle$, $H_5 = \langle a, b^2c \rangle$. $H_i, i = 1, 2, 3, 4, 5$ intersect non-trivially and permutes with each other. It follows that K_5 as a subgraph of $\text{PI}(G)$ and it contains C_3 .

If $r \mid (p-1)$, $r \nmid (q-1)$ and $q \nmid (p-1)$, then the group is different from the group given in the above argument only in the exchange the roles of a , b . So $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, bc = cb, c^{-1}ac = a^u \rangle$, $u \neq 1$. Here $H_1 = \langle b \rangle$, $H_2 = \langle a, b \rangle$, $H_3 = \langle b, ac \rangle$, $H_4 = \langle b, a^2c \rangle$, $H_5 = \langle b, a^3c \rangle$. H_i , $i = 1, 2, 3, 4, 5$ are intersect non-trivially and permutes with each other. It follows that K_5 as a subgraph of $PI(G)$ and it contains C_3 .

If $r \mid (p-1)$ and $r \mid (q-1)$ and $q \nmid (p-1)$, then If $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, bc = cb, c^{-1}ac = a^u \rangle$, $u \neq 1$. Here $H_1 = \langle a \rangle$, $H_2 = \langle a, b \rangle$, $H_3 = \langle a, c \rangle$, $H_4 = \langle a, bc \rangle$, $H_5 = \langle a, b^2c \rangle$. H_i , $i = 1, 2, 3, 4, 5$ are intersect non-trivially and permutes with each other. It follows that K_5 as a subgraph of $PI(G)$ and it contains C_3 .

If $q \mid (p-1)$, then we have the group is of essentially the same form as the first group, second group and third group in Proposition 3.2. The planarity of the Permutability intersection graphs of subgroups of this groups are as described in Proposition 3.2 .

If $r \mid (p-1)$ and $q \mid (p-1)$, then $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, b^{-1}ab = a^u, c^{-1}ac = a^v, bc = cb \rangle$, $v, u \neq 1$. By Theorem 1.4, Figure 1d, $PI(G)$ is planar and it contains C_3 . □

Proposition 3.8 *Let G be a group of order p^2qr , where p, q, r are distinct primes, then $PI(G)$ is non-planar and contains C_3 .*

Proof: Let $PQ \leq G$. By Proposition 2.1 and proposition 3.3, $PQ \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q$. By Proposition 3.3, $PI(PQ)$ is isomorphic to a graph which is shown in Figure 4. PQ together with its subgroups form $K_{3,3}$ and it contains C_3 . □

Proposition 3.9 *Let G be a group of order p^3qr , where p, q, r are distinct primes, then $PI(G)$ is non-planar and contains C_3 .*

Proof: Let $|N| \cong p^3q$. By Proposition 2.1 and Proposition 3.4, $PI(N)$ is non-planar and it contains C_3 . □

Proposition 3.10 *Let G be a group of order p^2q^2r , where p, q, r are distinct primes, then $PI(G)$ is non-planar and contains C_3 .*

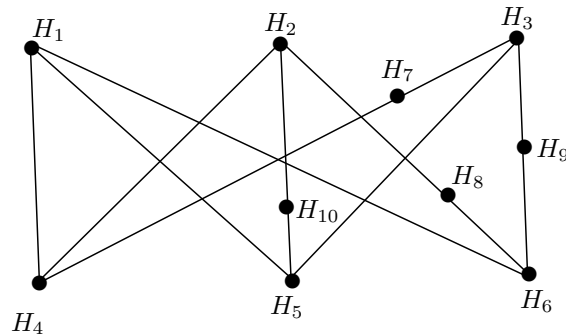
Proof: Let $PQ \leq G$. By Proposition 2.1 and Proposition 3.5, $PQ \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_{q^2}$. By Proposition 3.5, $PI(PQ) \cong K_1 + (K_{1,p+1} \cup p^2K_2)$. PQ together its subgroups form $K_{3,3}$ and it contains C_3 . □

Proposition 3.11 *Let G be a group of order $p^\alpha q^\beta r^\gamma$, where p, q, r are distinct primes and $\alpha + \beta + \gamma \geq 6$, then $PI(G)$ is non-planar and contains C_3 .*

Proof: Let $\alpha \geq \beta \geq \gamma$ and $PQ \leq G$. By Proposition 2.1 and Proposition 3.6, $PI(PQ)$ is non-planar and it contains C_3 . □

Proposition 3.12 *Let G be a finite solvable group of order n , $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $k \geq 4$, $\alpha_i \geq 1$, p_i 's are distinct primes, then $PI(G)$ is non-planar and contains C_3 .*

Proof: Let P_1, P_2, P_3, P_4 be the sylow p_1, p_2, p_3, P_4 subgroups of G . Since G is solvable, so $P_1, P_2, P_3, \dots, P_k$ is the sylow basis of G . Here $H_i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ and 10 are subgroups of G of order $p_1, p_2, p_3, p_1p_2, p_1p_3, p_1p_4, p_2p_3, p_2p_4, p_3p_4$ and $p_1p_2p_3$ respectively. H_1, H_2, H_3 permutes with H_4, H_5, H_6 . H_1 intersect H_4, H_5, H_6 ; H_2 intersect H_4, H_8, H_{10} ; H_3 intersect H_5, H_7, H_9 ; H_4 intersects H_7 ; H_5 intersects H_{10} ; H_6 intersects H_8, H_9 . It follows that $PI(G)$ contains a subdivision of $K_{3,3}$ as a subgraph. which is shown in Figure 6. H_1, H_4, H_5 are permutes and intersects non-trivially with each other. So, they forms C_3 . □

Figure 6: Subdivision of $K_{3,3}$.

Proposition 3.13 *If G is a non-solvable group then $PI(G)$ contains K_5 and C_3 .*

Proof: Since every non-solvable group contains simple group as a subquotient; every simple group contains minimal simple group as a subquotient.

Case 1: $G \cong L_2(q^p)$, where $q = 2, 3$ and p is prime. If $p = 2$, then the only non-solvable group is $L_2(4)$. Also $L_2(4) \cong A_5$ [5]. Here $H_1 = \{(12, 34)\}$, $H_2 = \{(12, 34), (13, 34)\}$, $H_3 = \{(12, 34), (12354)\}$, $H_4 = \{(12, 34), (12453)\}$, $H_5 = \{(12, 34), (345)\}$ are the proper subgroups of G . H_i , $i = 1, 2, 3, 4, 5$ are intersect non-trivially and permutes with each other. It follows that K_5 as a subgraph of $PI(A_5)$.

If $p > 2$, then $L_2(q^p)$ contains a subgroup isomorphic to $(\mathbb{Z}_q)^p$, namely the subgroup of matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with F_{q^p} . By Proposition 3.1, $PI((\mathbb{Z}_q)^p)$ is non-planar.

Case 2: $G \cong L_3(3)$. In $SL_3(3)$ the only matrix in the subgroup H is the identity matrix so $L_3(3) \cong SL_3(3)$. Let us consider the subgroup consisting of matrices of the form $\begin{pmatrix} 1 & c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ with $a, b, c \in F_3$. This subgroup to the group $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ with $p = 3$. By Proposition 3.1 $PI((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)$ contains $K_{3,3}$ and contains C_3 .

Case 3: $G \cong L_2(p)$, where p is any prime exceeding 3 such that, $p^2 + 1 \equiv 0 \pmod{5}$. We have to consider two subcases.

Subcase 3a: If $p \equiv 1 \pmod{4}$, then $L_2(p)$ has a subgroup isomorphic to D_{p-1} [11]. Now we consider the group D_{4n} . Here $H_1 = \{a\}$, $H_2 = \{a^n\}$, $H_3 = \{a^n, b\}$, $H_4 = \{a^n, ba\}$, $H_5 = \{a^n, ba^2\}$ are the proper subgroups of D_{4n} . H_i , $i = 1, 2, 3, 4, 5$ are intersect non-trivially and permutes with each other. It follows that K_5 as a subgraph of $PI(D_{4n})$. By this argument $PI(D_{p-1})$ contains K_5 .

If $p = 5$, then $L_2(5) \cong A_5 \cong L_2(4)$ [5]. By case 1, $PI(G)$ is non-planar and it contains C_3 .

Subcase 3b: If $p \equiv 3 \pmod{4}$, then $L_2(p)$ has a subgroup isomorphic to D_{p+1} [11]. by the above argument, $PI(D_{p+1})$ contains K_5 and it contains C_3 .

If $p = 7$, then S_4 is a maximal subgroup of $L_2(7)$ [5]. $PI(S_4)$ is non-planar and it contains C_3 .

Case 4: If $G \cong S_z(2^q)$, where q is any prime, then $S_z(2^q)$ has a subgroup isomorphic to $(\mathbb{Z}_2)^q$ [19], $q \geq 3$. By Proposition 3.1, $PI((\mathbb{Z}_2)^q)$ is non-planar, $q \geq 3$. Therefore, $PI(G)$ is non-planar and it contains C_3 . \square

4. Main Results

By combining all the results obtained in Sections 2 and 3 above, we have the following general main result, which classifies the finite groups whose permutability intersection graphs of subgroups are planar and bipartite.

Theorem 4.1 *Let G be a finite group and p, q, r be distinct primes. Then*

1. $PI(G)$ is planar if and only if G is isomorphic to one of the following groups:

(a) \mathbb{Z}_{p^α} ($\alpha = 2, 3, 4, 5$);

- (b) $\mathbb{Z}_{p^\alpha q}$ ($\alpha = 1, 2$), $\mathbb{Z}_p \times \mathbb{Z}_p$ ($p = 2, 3$), $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ ($p = 2$), $\mathbb{Z}_{pq} \times \mathbb{Z}_p$ ($p = 2$);
- (c) Q_8 and M_8 ;
- (d) $\mathbb{Z}_q \rtimes \mathbb{Z}_p$;
- (e) $\mathbb{Z}_q \times \mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$, $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$, A_4 ;
- (f) $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$;
- (g) $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, b^{-1}ab = a^u, c^{-1}ac = a^v, bc = cb \rangle$, $u, v \neq 1$.

2. $PI(G)$ is bipartite if and only if G is isomorphic to one of the following groups:

- (a) \mathbb{Z}_{p^3} ;
- (b) $\mathbb{Z}_q \rtimes \mathbb{Z}_p$;
- (c) $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$.

Corollary 4.1 Let G be a finite group and p, q, r are distinct primes. Then

1. The following are equivalent:

- (a) $G \cong \mathbb{Z}_{p^3}$;
- (b) $PI(G)$ is tree;
- (c) $PI(G)$ is star graph;
- (d) $PI(G)$ is complete bipartite;
- (e) $PI(G)$ is P_n ($n = 2$).

2. $PI(G)$ is C_n if and only if $n = 3$ and $G \cong \mathbb{Z}_{p^4}$.

5. Conclusion

In this paper, we have studied about whose permutability intersection graph of subgroups are planar, bipartite, C_3 -free. Also, we have studied about whose permutability intersection graph of subgroups are one of tree, path, cycle, acyclic, star, complete bipartite or totally disconnected graph.

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A. Esakkiraj,

Research Scholar, Reg. No.: 241112312030,

Department of Mathematics,

Sri Paramakalyani College, Alwarkurichi - 627 412, India.

(Affiliated to Manonmaniam Sundaranar University, Tirunelveli - 627 012, India.)

E-mail address: esakkiraj1008@gmail.com

and

P. Devi,

Assistant Professor,

Department of Mathematics,

Sri Paramakalyani College, Alwarkurichi - 627 412, Tamilnadu, India

E-mail address: pdevispkc@gmail.com