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Constrained Problem of Fractional-Order Itô-Integral Equation Subject to a Nonlocal Fractional Order Stochastic-Integral Problem of Itô-Differential Equation

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ABSTRACT: In this paper, an issue of fractional-order Itô-integral equation constrained by a nonlocal fractional order stochastic-integral problem of Itô-differential equation. Firstly, the existence of solutions and the sufficient conditions for its uniqueness of the constraint will be proved. The dependency of the solution will be studied and the Hyers-Ulam stability of the constraint will be discussed. Secondly, we prove that for every solution of the constraint there exists a unique solution for the constrained problem of fractional-order Itô-integral equation. Also, for the unique solution that we get, we will discuss some dependency properties and the Hyers-Ulam stability for the constrained problem it-self will be proved. Finally some examples will be given.

Keywords: Stochastic integral equation, stochastic Itô-integral of fractional-order, Constrained problem, Schauder's fixed point theorem, existence of solutions, continuous dependence, Hyers-Ulam stability, nonlinear equations, fractional derivatives.

Contents

1. Introduction

Stochastic problems have evolved into an essential instrument for modeling intricate systems across various disciplines, encompassing several areas. Fractional stochastic systems have attracted many researchers in various fields due to their ability to model complex and unpredictable real-world phenomena, (see [2], [4], [6], [9]- [16], [27], [30] and [31]).

In our work, we let $(\Lambda, \mathcal{F}, \mu)$ be a complete probability space [7], μ is a probability measure occurring during the time interval $[0,T]$ and $\mathcal{F}$ is a $\sigma-$algebra defined on a sample space Λ.

Let $\mathcal{I} = [0,T]$ and $X(t; w) = X(t),\ t \in \mathcal{I},\ w \in \Lambda$, considering the space of all second order stochastic processes $\mathsf{L}_2(\Lambda)$ which is identified with the norm

$$\|X(t)\|_2 \ = \ \sqrt{E(X^2(t))}.$$

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Let $\mathbb{C}([0,T],\mathsf{L}_2(\Lambda))$ be the space of all continuous second order stochastic processes on $\mathcal{I}=[0,T]$. The norm of X is given by

$$\|X\|_{\mathbb{C}} = \sup_{t\in[0,\tau]} \|X(t)\|_2.$$

Let $\mathsf{L}_2^*(\mathcal{I},\mathsf{L}_2(\Lambda))$ be the space of all second order integrable stochastic processes on $\mathcal{I}=[0,T]$ with norm

$$\|X\|_2^* = \sqrt{\int_0^T \|X(t)\|_2^2 dt}.$$

Here, we follow the work in [17], authors in that work introduced the stochastic Itô-integral of fractional-order of the function $f \in \mathbb{C}([0,T],\mathsf{L}_2(\Lambda))$, $\beta\in(\frac{1}{2},1)$ with respect to Brownian motion $\mathcal{W}(t)$, denoted by (F_β).

Definition 1.1 *[17] Let $f \in \mathbb{C}([0,T],\mathsf{L}_2(\Lambda))$ be a given second order mean square continuous process. the stochastic Itô-integral of fractional-order of the function f with respect to Brownian motion $\mathcal{W}(t)$ is given by*

$$F_\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)d\mathcal{W}(s), \quad \beta\in(\frac{1}{2},1).$$

Authors in [17] presented some important properties for stochastic Itô-integral of fractional-order and solved some minor problems.

Here, we will continue their work and add to their study. For that, we consider the following fractional-order Itô integral equation

$$\begin{aligned} X(t) &= \mathcal{B}t^{\alpha-1} + \lambda\, F_\beta H(t,X(t),\mathcal{U}(t)), \quad t\in\mathcal{I}, \quad \alpha\in(0,1), \beta\in(\frac{1}{2},1) \\ &= \mathcal{B}t^{\alpha-1} + \lambda\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} H(s,X(s),\mathcal{U}(s))d\mathcal{W}(s), \end{aligned} \tag{1.1}$$

subject to the constraint

$$\begin{cases} d\mathcal{U}(t) = -\sigma\mathcal{U}(t)dt + \mathcal{K}(t,\mathcal{U}(t))d\mathcal{W}_1(t), \quad t\in\mathcal{I}, \\ \mathcal{U}(0) = \mathcal{U}_0 + \int_0^T \frac{(T-t)^{\gamma-1}}{\Gamma(\gamma)} G(t,\mathcal{U}(t))d\mathcal{W}_2(t), \quad \gamma\in(\frac{1}{2},1). \end{cases} \tag{1.2}$$

where $\mathcal{B},\mathcal{U}_0$ are second order stochastic processes, λ,σ are nonnegative parameters and $\mathcal{W}(t)$, $\mathcal{W}_1(t)$, $\mathcal{W}_2(t)$ are three independent standard Brownian motions on the complete probability space $(\Lambda,\mathcal{F},\mu)$.

Here, Firstly, we prove the existence and the uniqueness of the solution $\mathcal{U}\in\mathbb{C}(\mathcal{I},\mathsf{L}_2(\Lambda))$ of the constrained problem (1.2) and discuss some continuous dependencies of the solution $\mathcal{U}$ and the stability of the solution $\mathcal{U}$. Secondly, we prove for every solution of constraind problem (1.2), there exists a unique solution $X\in\mathsf{L}_2^*(\mathcal{I},\mathsf{L}_2(\Lambda))$ of the problem (1.1), then, we study the continuous dependence of X on some parameters and random functions and Hyers-Ulam satability.

2. Solution of the Constraint

Consider the following assumptions:

(A1) The functions $\mathcal{K},G:\mathcal{I}\times\mathsf{L}_2(\Lambda))\to\mathsf{L}_2(\Lambda)$ are continuous in $t\in\mathcal{I}$ $\forall\mathcal{U}\in\mathsf{L}_2(\Lambda)$, continuous in $\mathcal{U}\in\mathsf{L}_2(\Lambda)$ $\forall t\in\mathcal{I}$. and there exists constants a_1,a_2 and second order stochastic processes $\gamma_1(t),\gamma_2(t)\in\mathbb{C}(\mathcal{I},\mathsf{L}_2(\Lambda))$, such that

$$\|\mathcal{K}(t,\mathcal{U}(t))\|_2 \le ||\gamma_1(t)||_2 + a_1 \|\mathcal{U}(t)\|_2,$$

and

$$\|G(t,\mathcal{U}(t))\|_2 \le ||\gamma_2(t)||_2 + a_2 \|\mathcal{U}(t)\|_2 .$$

$(A2)$ Let $a = \max\{a_1, a_2\}$ be such that $a(\aleph + \sqrt{T}) < 1$, where $\aleph = \frac{T^{\gamma-\frac{1}{2}}}{\sqrt{2\gamma-1}\ \Gamma(\gamma)}$.

Lemma 2.1 *The problem (1.2) is equivalent to Itô−stochastic integral equation*

$$\mathcal{U}(t) = e^{-\sigma t}\mathcal{U}_0 + e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} G(s,\mathcal{U}(s))d\mathcal{W}_2(s) + \int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s). \tag{2.1}$$

Proof: Using Itô formula [28], [30], we conclude that the problem (1.2) is equivalent to the stochastic integral equation

$$\begin{aligned}
\mathcal{U}(t) &= e^{-\sigma t}\,\mathcal{U}(0) + e^{-\sigma t}\int_0^t e^{\sigma s}\,\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s) \\
&= e^{-\sigma t}\,\mathcal{U}_0 + e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} G(s,\mathcal{U}(s))d\mathcal{W}_2(s) + e^{-\sigma t}\int_0^t e^{\sigma s}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s) \\
&= e^{-\sigma t}\mathcal{U}_0 + e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} G(s,\mathcal{U}(s))d\mathcal{W}_2(s) + \int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s).
\end{aligned}$$

□

2.1. The existence of the solution $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$

To achieve our first main objective, we study the existence of at least one solution $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$ of the stochastic integral equation (2.1), for that, define the nonempty convex set Q_r by

$$Q_r = \{\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda)) : \|\mathcal{U}(t)\|_2 \le r\} \subset \mathbb{C}([0,r], \mathsf{L}_2(\Lambda)),$$

such that

$$r = \frac{||\mathcal{U}_0||_2 + b(\aleph + \sqrt{T})}{1 - a(\aleph + \sqrt{T})}, \quad \text{where} \quad b = \max\{\sup_{t\in\mathcal{I}} ||\gamma_1(t)||_2, \sup_{t\in\mathcal{I}} ||\gamma_2(t)||_2\}.$$

Theorem 2.1 *Let the assumptions* $(A1) - (A2)$ *be satisfied, then there exists at least one solution* $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$ *of the stochastic integral equation (2.1) involving two independent Brownian motions.*

Proof: Let the mapping $\mathcal{A}$ be defined as

$$\mathcal{A}\,\mathcal{U}(t) = e^{-\sigma t}\mathcal{U}_0 + e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} G(s,\mathcal{U}(s))d\mathcal{W}_2(s) + \int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s). \tag{2.2}$$

Let $\mathcal{U} \in Q_r$, then we have

$$\begin{aligned}
||\mathcal{A}\,\mathcal{U}(t)||_2 &\leq e^{-\sigma t}\left(||\mathcal{U}_0||_2 + \sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)}||G(s,\mathcal{U}(s))||_2^2 ds}\right) \\
&+ \sqrt{\int_0^t \left|\left|e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}(s))\right|\right|_2^2 ds} \\
&\leq e^{-\sigma t}\left(||\mathcal{U}_0||_2 + \sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)}\left(||\gamma_2(s)||_2 + a_2 \ \| \mathcal{U}(s) \|_2\right)^2 ds}\right) \\
&+ \sqrt{\int_0^t e^{-2\sigma(t-s)}\left(||\gamma_1(s)||_2 + a_1 \ \| \mathcal{U}(s) \|_2\right)^2 ds},
\end{aligned}$$

by following

$$\begin{aligned}
||\mathcal{A}\,\mathcal{U}||_{\mathbb{C}} &\leq e^{-\sigma t}||\mathcal{U}_0||_2 + e^{-\sigma t}\left(b + a \ \| \mathcal{U} \|_{\mathbb{C}}\right)\sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)}ds} \\
&+ \left(b + a \ \| \mathcal{U} \|_{\mathbb{C}}\right)\sqrt{\int_0^t e^{-2\sigma(t-s)}ds} \\
&\leq ||\mathcal{U}_0||_2 + \left(b + a \ \| \mathcal{U} \|_{\mathbb{C}}\right)\sqrt{\frac{T^{2\gamma-1}}{(2\gamma-1)\Gamma^2(\gamma)}} + \left(b + a \ \| \mathcal{U} \|_{\mathbb{C}}\right)\sqrt{T} \\
&\leq ||\mathcal{U}_0||_2 + \left(b + a \ \| \mathcal{U} \|_{\mathbb{C}}\right)\frac{T^{\gamma-\frac{1}{2}}}{\sqrt{2\gamma-1} \ \ \Gamma(\gamma)} + \left(b + a \ \| \mathcal{U} \|_{\mathbb{C}}\right)\sqrt{T} \\
&\leq ||\mathcal{U}_0||_2 + \aleph\,(b + a\ r) + (b + a\ r))\,\sqrt{T} = r,
\end{aligned}$$

where $\aleph = \frac{T^{\gamma-\frac{1}{2}}}{\sqrt{2\gamma-1} \ \ \Gamma(\gamma)}$. Thus, we obtain that

$$r \leq \frac{||\mathcal{U}_0||_2 + b(\aleph + \sqrt{T})}{1 - a(\aleph + \sqrt{T})}.$$

This gives that $\mathcal{A} : Q_r \to Q_r$ and the class $\{\mathcal{A}Q_r\}$ is uniformly bounded on Q_r.
Now, let $t_1, t_2 \in [0,T]$ be such that, $|t_2 - t_1| < \varepsilon$, then

$$\begin{aligned}
||\mathcal{A}\,\mathcal{U}(t_2) - \mathcal{A}\,\mathcal{U}(t_1)||_2 \;\leq\;& \left|e^{-\sigma t_2} - e^{-\sigma t_1}\right|\; ||\mathcal{U}_0||_2 + \left|e^{-\sigma t_2} - e^{-\sigma t_1}\right| \sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)} ||G(s,\mathcal{U}(s))||_2^2 ds} \\
&+ \; ||\int_0^{t_2} e^{-\sigma(t_2-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s) - \int_0^{t_1} e^{-\sigma(t_2-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s)||_2 \\
&+ \; ||\int_0^{t_1} e^{-\sigma(t_2-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s) - \int_0^{t_1} e^{-\sigma(t_1-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s)||_2 \\
\leq \;& \left|e^{-\sigma t_2} - e^{-\sigma t_1}\right|\; ||\mathcal{U}_0||_2 \\
&+ \; \left|e^{-\sigma t_2} - e^{-\sigma t_1}\right| \sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)} \left(||\gamma_2(s)||_2 + a_2 \;\; \|\,\mathcal{U}(s)\,\|_2\right)^2 ds} \\
&+ \; ||\int_{t_1}^{t_2} e^{-\sigma(t_2-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s)||_2 \\
&+ \; ||\int_0^{t_1} \left[e^{-\sigma(t_2-s)} - e^{-\sigma(t_1-s)}\right] \mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s)||_2,
\end{aligned}$$

and following by,

$$\begin{aligned}
||\mathcal{A}\,\mathcal{U}(t_2) - \mathcal{A}\,\mathcal{U}(t_1)||_2 \;\leq\;& \left|e^{-\sigma t_2} - e^{-\sigma t_1}\right|\; ||\mathcal{U}_0||_2 \\
&+ \; \left|e^{-\sigma t_2} - e^{-\sigma t_1}\right| \sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)} \left(||\gamma_2(s)||_2 + a_2 \;\; \|\,\mathcal{U}(s)\,\|_2\right)^2 ds} \\
&+ \; \sqrt{\int_{t_1}^{t_2} e^{-2\sigma(t_2-s)} \left(||\gamma_1(s)||_2 + a_1 \;\; \|\,\mathcal{U}(s)\,\|_2\right)^2 ds} \\
&+ \; \sqrt{\int_0^{t_1} [e^{-\sigma(t_2-s)} - e^{-\sigma(t_1-s)}]^2 \left(||\gamma_1(s)||_2 + a_1 \;\; \|\,\mathcal{U}(s)\,\|_2\right)^2 ds.}
\end{aligned}$$

This proves the equi-continuity of the class $\{\mathcal{A}Q_r\}$ on Q_r.
Now, let $\mathcal{U}_n \in Q_r, \;\; \mathcal{U}_n \to \mathcal{U} \;\; w.p.1$, this implies that

$$\|\mathcal{A}\mathcal{U}(t_2) - \mathcal{A}\mathcal{U}(t_1)\|_2 \to 0 \;\; \text{as } \varepsilon \to 0,$$

and the class of function $\{\mathcal{A}\mathcal{U}\}$ is equicontinuous. Therefore the closure of $\{\mathcal{A}\mathcal{U}\}$ is a compact subset of $\mathbb{C}$ (Arzelà-Ascoli theorem [8]) and implies that the operator $\mathcal{A}$ is compact.
Now, consider $\mathcal{U}_n \in Q_r$ being such that $\underset{n\to\infty}{L.i.m}\; \mathcal{U}_n = \mathcal{U} \quad w.p.1$, where $L.i.m$ denotes the limit of the continuous second order process in the mean square sense ([20], [28]), when we apply Lebesgue dominated

theorem [21], we can obtain

$$\begin{aligned}
\underset{n\to\infty}{L.i.m}\ \mathcal{A}\mathcal{U}_n &= \underset{n\to\infty}{L.i.m}\left[e^{-\sigma t}\mathcal{U}_0 + e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\mathcal{U}_n(s))d\mathcal{W}_2(s) + \int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}_n(s))d\mathcal{W}_1(s)\right] \\
&= e^{-\sigma t}\mathcal{U}_0 + e^{-\sigma t}\ \underset{n\to\infty}{L.i.m}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\mathcal{U}_n(s))d\mathcal{W}_2(s) + \underset{n\to\infty}{L.i.m}\int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}_n(s))d\mathcal{W}_1(s) \\
&= e^{-\sigma t}\mathcal{U}_0 + e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\underset{n\to\infty}{L.i.m}\,\mathcal{U}_n(s))d\mathcal{W}_2(s) + \int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\underset{n\to\infty}{L.i.m}\,\mathcal{U}_n(s))d\mathcal{W}_1(s) \\
&= \mathcal{A}\mathcal{U}(t).
\end{aligned}$$

This implies that the operator $\mathcal{A} : Q_r \to Q_r$ is continuous, here applying Schauder fixed point theorem [8] and [21], there exists at least one solution $\mathcal{U} \in \mathbb{C}$ of the stochastic integral equation (2.1). □

2.2. Uniqueness of the solution $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$

To discuss the uniqueness of the solution of the problem (1.2), let us replace hypotheses ($A2$) by

($A2^*$) $\mathcal{K}, G : [0,T] \times \mathsf{L}_2(\Lambda) \to \mathsf{L}_2(\Lambda)$ are continuous in $t \in [0,T]$, $\forall\ \mathcal{U} \in \mathsf{L}_2(\Lambda)$ and satisfy Lipschitz condition

$$\|\mathcal{K}(t,\mathcal{U}_1(t)) - \mathcal{K}(t,\mathcal{U}_2(t))\|_2 \le a_1\|\mathcal{U}_1(t) - \mathcal{U}_2(t)\|_2,$$

and

$$\|G(t,\mathcal{U}_1(t)) - G(t,\mathcal{U}_2(t))\|_2 \le a_2\|\mathcal{U}_1(t) - \mathcal{U}_2(t)\|_2.$$

Now, we have the following theorem.

Theorem 2.2 *Let the assumptions of Theorem (2.1) be satisfied, then the solution $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$ of the problem (1.2) is unique.*

Proof: From the assumption ($A2^*$), we can deduce that

$$||\mathcal{K}(t,\mathcal{U}(t))||_2 - ||\mathcal{K}(t,0)||_2 \le ||\mathcal{K}(t,\mathcal{U}(t)) - \mathcal{K}(t,0)||_2 \le a_1||\mathcal{U}(t)||_2.$$

Thus

$$||\mathcal{K}(t,\mathcal{U}(t))||_2 \le b_1 + a_1||\mathcal{U}(t)||_2, \quad \text{where} \ \ b_1 = ||\mathcal{K}(t,0)||_2,$$

similarly

$$||G(t,\mathcal{U}(t))||_2 \le b_2 + a_2||\mathcal{U}(t)||_2, \quad \text{where} \ \ b_2 = ||G(t,0)||_2.$$

Then, the assumptions of Theorem (2.1) are satisfied and the problem (1.2) has at least one solution. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two solutions of (1.2), then

$$\begin{aligned}
||\mathcal{U}_1(t) - \mathcal{U}_2(t)||_2 &\le ||e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}[G(s,\mathcal{U}_1(s)) - G(s,\mathcal{U}_2(s))]d\mathcal{W}_2(s)||_2 \\
&+ ||\int_0^t e^{-\sigma(t-s)}[\mathcal{K}(s,\mathcal{U}_1(s)) - \mathcal{K}(s,\mathcal{U}_2(s))]d\mathcal{W}_1(s)||_2 \\
&\le |e^{-\sigma t}|\sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)}||G(s,\mathcal{U}_1(s)) - G(s,\mathcal{U}_2(s))||_2^2 ds} \\
&+ \sqrt{\int_0^t e^{-2\sigma(t-s)}||\mathcal{K}(s,\mathcal{U}_1(s)) - \mathcal{K}(s,\mathcal{U}_2(s))||_2^2 ds}.
\end{aligned}$$

So, we get

$$\begin{aligned} ||\mathcal{U}_1 - \mathcal{U}_2||_{\mathbb{C}} &\leq a\, ||\mathcal{U}_1 - \mathcal{U}_2||_{\mathbb{C}} \left(\sqrt{\int\limits_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)}} + \sqrt{\int\limits_0^t e^{-2\sigma(t-s)} ds} \right) \\ &\leq a(\aleph + \sqrt{T})\, ||\mathcal{U}_1 - \mathcal{U}_2||_{\mathbb{C}}. \end{aligned}$$

Then

$$\left(1 - a(\aleph + \sqrt{T})\right) ||\mathcal{U}_1 - \mathcal{U}_2||_{\mathbb{C}} \leq 0.$$

This implies that

$$||\mathcal{U}_1 - \mathcal{U}_2||_{\mathbb{C}} = 0 \quad \Rightarrow \quad \mathcal{U}_1 = \mathcal{U}_2,$$

which proves the uniqueness of the solution to the problem (1.2). □

2.3. Continuous dependence of $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$

In this part, we prove that the solution of the problem (2.1) dependents continuously on the random process $\mathcal{U}_0$ and on the two functions of the problem $\mathcal{K}, G$.

Definition 2.1 *The solution $\mathcal{U} \in \mathbb{C}(\mathcal{I}, L_2(\Lambda))$ of the problem (2.1) depends continuously on the random process $\mathcal{U}_0$, if for all $\varepsilon_1 > 0$, $\exists\ \delta_1 > 0$ such that*

$$\|\mathcal{U}_0 - \mathcal{U}_0^*\|_2 \leq \delta_1 \quad \textit{implies that} \ \| \mathcal{U} - \mathcal{U}^* \|_{\mathbb{C}} \leq \varepsilon_1,$$

where $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t) = e^{-\sigma t}\mathcal{U}_0^* + e^{-\sigma t} \int\limits_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} G(s, \mathcal{U}^*(s)) d\mathcal{W}_2(s) + \int\limits_0^t e^{-\sigma(t-s)} \mathcal{K}(s, \mathcal{U}^*(s)) d\mathcal{W}_1(s).$$

Theorem 2.3 *Let the assumptions of Theorem (2.2) be satisfied, then the solution of (2.1) depends continuously on the random process $\mathcal{U}_0$.*

Proof: Let $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t) = e^{-\sigma t}\mathcal{U}_0^* + e^{-\sigma t} \int\limits_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} G(s, \mathcal{U}^*(s)) d\mathcal{W}_2(s) + \int\limits_0^t e^{-\sigma(t-s)} \mathcal{K}(s, \mathcal{U}^*(s)) d\mathcal{W}_1(s)$$

such that $\|\mathcal{U}_0 - \mathcal{U}_0^*\|_2 \leq \delta_1$. Then we have

$$\begin{aligned} \mathcal{U}(t) - \mathcal{U}^*(t) &= e^{-\sigma t}[\mathcal{U}_0 - \mathcal{U}_0^*] + e^{-\sigma t} \int\limits_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} [G(s, \mathcal{U}(s)) - G(s, \mathcal{U}^*(s))] d\mathcal{W}_2(s) \\ &+ \int\limits_0^t e^{-\sigma(t-s)} [\mathcal{K}(s, \mathcal{U}(s)) - \mathcal{K}(s, \mathcal{U}^*(s))] d\mathcal{W}_1(s) \end{aligned}$$

and this gives

$$\begin{aligned} \|\mathcal{U}(t) - \mathcal{U}^*(t)\|_2 &\leq e^{-\sigma t}\|\mathcal{U}_0 - \mathcal{U}_0^*\|_2 + \sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)} (a_2\|\mathcal{U}(s) - \mathcal{U}^*(s)\|_2)^2 ds} \\ &+ \sqrt{\int\limits_0^t e^{-2\sigma(t-s)} (a_1\|\mathcal{U}(s) - \mathcal{U}^*(s)\|_2)^2 ds}. \end{aligned}$$

So, we can obtain that
$$\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \leq \delta_1 + a(\aleph+\sqrt{T})\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}},$$
thus
$$\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \leq \frac{\delta_1}{1-a(\aleph+\sqrt{T})} = \varepsilon_1 \to 0 \ \text{ whenever } \ \delta_1 \to 0.$$
Which completes the proof. □

Definition 2.2 *The solution $\mathcal{U} \in \mathbb{C}(\mathcal{I}, L_2(\Lambda))$ of the problem (2.1) depends continuously on the random function $\mathcal{K}$, if for all $\varepsilon_2 > 0$, $\exists\ \delta_2 > 0$ such that*
$$\|\mathcal{K}(t,\mathcal{U}(t)) - \mathcal{K}^*(t,\mathcal{U}(t))\|_2 \leq \delta_2 \quad \textit{implies that} \ \| \mathcal{U}-\mathcal{U}^* \|_{\mathbb{C}} \leq \varepsilon_2,$$
where $\mathcal{U}^*$ be the solution of
$$\mathcal{U}^*(t) = e^{-\sigma t}\mathcal{U}_0 + e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} G(s,\mathcal{U}^*(s))d\mathcal{W}_2(s) + \int_0^t e^{-\sigma(t-s)}\mathcal{K}^*(s,\mathcal{U}^*(s))d\mathcal{W}_1(s).$$

Theorem 2.4 *Let the assumptions of Theorem (2.2) be satisfied, then the solution of (2.1) depends continuously on the random function $\mathcal{K}$.*

Proof: Let $\mathcal{U}^*$ be the solution of
$$\mathcal{U}^*(t) = e^{-\sigma t}\mathcal{U}_0 + e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} G(s,\mathcal{U}^*(s))d\mathcal{W}_2(s) + \int_0^t e^{-\sigma(t-s)}\mathcal{K}^*(s,\mathcal{U}^*(s))d\mathcal{W}_1(s)$$
such that $\|\mathcal{K}(t,\mathcal{U}(t)) - \mathcal{K}^*(t,\mathcal{U}(t))\|_2 \leq \delta_2$. Then we have
$$\begin{aligned}
\mathcal{U}(t)-\mathcal{U}^*(t) &= e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}[G(s,\mathcal{U}(s)) - G(s,\mathcal{U}^*(s))]d\mathcal{W}_2(s)\\
&+ \int_0^t e^{-\sigma(t-s)}[\mathcal{K}(s,\mathcal{U}(s)) - \mathcal{K}^*(s,\mathcal{U}^*(s))]d\mathcal{W}_1(s)\\
&= e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}[G(s,\mathcal{U}(s)) - G(s,\mathcal{U}^*(s))]d\mathcal{W}_2(s)\\
&+ \int_0^t e^{-\sigma(t-s)}[\mathcal{K}(s,\mathcal{U}(s)) - \mathcal{K}(s,\mathcal{U}^*(s))]d\mathcal{W}_1(s)\\
&+ \int_0^t e^{-\sigma(t-s)}[\mathcal{K}(s,\mathcal{U}^*(s)) - \mathcal{K}^*(s,\mathcal{U}^*(s))]d\mathcal{W}_1(s),
\end{aligned}$$
and this gives
$$\begin{aligned}
\|\mathcal{U}(t)-\mathcal{U}^*(t)\|_2 &\leq \sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)}(a_2\|\mathcal{U}(s)-\mathcal{U}^*(s)\|_2)^2 ds}\\
&+ \sqrt{\int_0^t e^{-2\sigma(t-s)}(a_1\|\mathcal{U}(s)-\mathcal{U}^*(s)\|_2)^2 ds} + \sqrt{\int_0^t e^{-2\sigma(t-s)}\delta_2^2 ds}.
\end{aligned}$$

So, we can obtain that

$$\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \leq a(\aleph+\sqrt{T})\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}}+\delta_2\sqrt{T},$$

thus

$$\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \leq \frac{\delta_2\sqrt{T}}{1-a(\aleph+\sqrt{T})}=\varepsilon_2 \to 0 \ \text{ whenever } \ \delta_2 \to 0.$$

Which completes the proof. □

Definition 2.3 *The unique solution $\mathcal{U} \in \mathbb{C}(\mathcal{I}, L_2(\Lambda))$ of the problem (2.1) depends continuously on the function G, if for all $\varepsilon_3>0$, $\exists\ \delta_3>0$ such that*

$$|G(t,\mathcal{U}(t))-G^*(t,\mathcal{U}(t))| \leq \delta_3 \quad \textit{implies that} \ \| \mathcal{U}-\mathcal{U}^* \|_{\mathbb{C}} \leq \varepsilon_3,$$

where $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t)=e^{-\sigma t}\mathcal{U}_0+e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G^*(s,\mathcal{U}^*(s))d\mathcal{W}_2(s)+\int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}^*(s))d\mathcal{W}_1(s).$$

Theorem 2.5 *Let the assumptions of Theorem (2.2) be satisfied, then the solution of (2.1) depends continuously on the random function G.*

Proof: Let $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t)=e^{-\sigma t}\mathcal{U}_0+e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G^*(s,\mathcal{U}^*(s))d\mathcal{W}_2(s)+\int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}^*(s))d\mathcal{W}_1(s).$$

Let $\|G(t,\mathcal{U}(t))-G^*(t,\mathcal{U}(t))\|_2 \leq \delta_3$, and

$$\begin{aligned}
\mathcal{U}(t)-\mathcal{U}^*(t) &= e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}[G(s,\mathcal{U}(s))-G^*(s,\mathcal{U}^*(s))]d\mathcal{W}_2(s)\\
&+ \int_0^t e^{-\sigma(t-s)}[\mathcal{K}(s,\mathcal{U}(s))-\mathcal{K}(s,\mathcal{U}^*(s))]d\mathcal{W}_1(s)\\
&= e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}[G(s,\mathcal{U}(s))-G(s,\mathcal{U}^*(s))]d\mathcal{W}_2(s)\\
&+ e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}[G(s,\mathcal{U}^*(s))-G^*(s,\mathcal{U}^*(s))]d\mathcal{W}_2(s)\\
&+ \int_0^t e^{-\sigma(t-s)}[\mathcal{K}(s,\mathcal{U}(s))-\mathcal{K}(s,\mathcal{U}^*(s))]d\mathcal{W}_1(s),
\end{aligned}$$

then

$$\begin{aligned}
\|\mathcal{U}(t)-\mathcal{U}^*(t)\|_2 &\leq \sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)}(a_2\|\mathcal{U}(s)-\mathcal{U}^*(s)\|_2)^2ds}+\delta_3\sqrt{\int_0^T \frac{(T-s)^{2\gamma-2}}{\Gamma^2(\gamma)}ds}\\
&+ \sqrt{\int_0^t e^{-2\sigma(t-s)}(a_1\|\mathcal{U}(s)-\mathcal{U}^*(s)\|_2)^2ds},
\end{aligned}$$

following by

$$\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}}(1-a(\aleph+\sqrt{T}))\leq \aleph\delta_3.$$

Then

$$\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}}\leq \frac{\aleph\delta_3}{1-a(\aleph+\sqrt{T})}=\varepsilon_3\to 0 \ \text{ whenever } \ \delta_3\to 0.$$

Thus completes the proof. □

2.4. Hyers-Ulam stability

Hyers-Ulam stability is one of the essential properties to measure the behavior of the solution of any problem. (This concept can be found in [1], [3], [5], [13], [19], [22]- [26]).
Now, consider the following definition.

Definition 2.4 *Let the solution $\mathcal{U}\in\mathbb{C}([0,\sigma],L_2(\Lambda))$ of the problem (2.1) be exists uniquely, then the problem (2.1) is Hyers-Ulam stable, if for every $\varepsilon_4>0$, $\exists\ \delta_4(\varepsilon_4)$ such that for any $\delta-$approximate solution $\mathcal{U}_p$ of the problem (2.1) satisfies,*

$$\left\|\mathcal{U}_p-e^{-\sigma t}\mathcal{U}_0-e^{-\sigma t}\int_0^T\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\mathcal{U}_p(s))d\mathcal{W}_2(s)-\int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}_p(s))d\mathcal{W}_1(s)\right\|_2\leq\delta_4 \tag{2.3}$$

implies $\|\mathcal{U}_p-\mathcal{U}\|_{\mathbb{C}}<\varepsilon_4.$

Theorem 2.6 *Let the hypotheses of Theorem (2.2) be satisfied. Then the problem (2.1) is Hyers-Ulam stable.*

Proof: Let $\mathcal{U}$ and $\mathcal{U}_p$ be the exact and approximate solutions of the problem (2.1) respectively, then

$$\begin{aligned}
\mathcal{U}_p(t)-\mathcal{U}(t) &= \mathcal{U}_p(t)-e^{-\sigma t}\mathcal{U}_0-e^{-\sigma t}\int_0^T\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\mathcal{U}(s))d\mathcal{W}_2(s)-\int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s))\\
&= \mathcal{U}_p-e^{-\sigma t}\mathcal{U}_0-e^{-\sigma t}\int_0^T\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\mathcal{U}_p(s))d\mathcal{W}_2(s)\\
&+ e^{-\sigma t}\int_0^T\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\mathcal{U}_p(s))d\mathcal{W}_2(s)-e^{-\sigma t}\int_0^T\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\mathcal{U}(s))d\mathcal{W}_2(s)\\
&- \int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}_p(s))d\mathcal{W}_1(s))+\int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}_p(s))d\mathcal{W}_1(s))\\
&- \int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}(s))d\mathcal{W}_1(s))\\
&= \mathcal{U}_p-e^{-\sigma t}\mathcal{U}_0-e^{-\sigma t}\int_0^T\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}G(s,\mathcal{U}_p(s))d\mathcal{W}_2(s)-\int_0^t e^{-\sigma(t-s)}\mathcal{K}(s,\mathcal{U}_p(s))d\mathcal{W}_1(s))\\
&+ e^{-\sigma t}\int_0^T\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}[G(s,\mathcal{U}_p(s))-G(s,\mathcal{U}(s))]d\mathcal{W}_2(s)\\
&+ \int_0^t e^{-\sigma(t-s)}[\mathcal{K}(s,\mathcal{U}_p(s))-\mathcal{K}(s,\mathcal{U}(s))]d\mathcal{W}_1(s)).
\end{aligned}$$

Then

$$\begin{aligned}\|\mathcal{U}_p(t)-\mathcal{U}(t)\|_2 &\leq \delta_4+\|e^{-\sigma t}\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}[G(s,\mathcal{U}_p(s))-G(s,\mathcal{U}_(s))]d\mathcal{W}_2(s)\|_2 \\ &+ \|\int_0^t e^{-\sigma(t-s)}[\mathcal{K}(s,\mathcal{U}_p(s))-\mathcal{K}(s,\mathcal{U}(s))]d\mathcal{W}_1(s))\|_2,\end{aligned}$$

that gives

$$[1-a(\aleph+\sqrt{T})]\|\mathcal{U}_p-\mathcal{U}\|_{\mathbb{C}} \leq \delta_4.$$

Then

$$\|\mathcal{U}_p-\mathcal{U}\|_{\mathbb{C}} \leq \frac{\delta_4}{1-a(\aleph+\sqrt{T})}=\varepsilon_4,$$

and the problem (2.1) is Hyers-Ulam stable. □

From the equivalent between the problem (1.2) and the integral equation (2.1), we have the following corollary.

Corollary 2.1
Let the assumptions of Theorem (2.6) be satisfied then the problem (1.2) is Hyers-Ulam stable.

3. Fractional-Order Itô-Integral Equation

In this section, we finally have reached the study of the problem that related to the stochastic Itô-integral of fractional-order F_β, $\beta \in (\frac{1}{2},1)$ with respect to Brownian motion $\mathcal{W}(t)$, the study of the solution $X \in L_2^*(\mathcal{I},\mathsf{L}_2(\lambda))$ of the fractional-order Itô integral equation (1.1) and its behaviour will give to the interested readers and researchers in this topic a clear understanding of the difference between the stochastic Itô-integral of fractional-order (F_β) and the random fractional operator I^β.
We study the existence of a unique solution $X \in \mathsf{L}_2^*(\mathcal{I},\mathsf{L}_2(\Lambda))$ to the fractional-order Itô integral equation (1.1), we discuss the continuous dependence of this solution X on some parameters and random functions, we also explained the Hyers-Ulam stability for X.
Consider the following assumptions:

(M1) The function $H:\mathcal{I}\times\mathsf{L}_2(\Lambda))\times\mathsf{L}_2(\Lambda))\to\mathsf{L}_2(\Lambda)$ is measurable in $t\in\mathcal{I}$ $\forall\mathcal{U}\in\mathsf{L}_2$ (Λ), continuous in $\mathcal{U}\in\mathsf{L}_2(\Lambda)$ $\forall t\in\mathcal{I}$ and satisfies Lipschitz condition

$$\| H(t,X(t),\mathcal{U}(t))-H(t,\bar{X}(t),\bar{U}(t))\|_2 \leq \hbar\{\| X(t)-\bar{X}(t)\|_2+\|\mathcal{U}(t)-\bar{\mathcal{U}}(t)\|_2\},$$

where $\hbar$ is a constant.

$(M2)$ Let $4\lambda^2$ $\hbar^2 T^*<1$, where $T^*=\max\{\frac{T^{2\alpha-1}}{2\alpha-1},\frac{T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)}\}$.

Lemma 3.1 *The assumption* $(M1)$ *implies that*

$$\| H(t,X(t),\mathcal{U}(t))\|_2 \leq \hbar\ \{\| X(t)\|_2+\|\mathcal{U}(t)\|_2\}+\| H(t,0,0)\|_2\ .$$

Proof:

$$\begin{aligned}\| H(t,X(t),\mathcal{U}(t))\|_2 &= |\,\| H(t,X(t),\mathcal{U}(t))\|_2-\| H(t,0,0)\|_2+\| H(t,0,0)\|_2\,| \\ &\leq |\,\| H(t,X(t),\mathcal{U}(t))-H(t,0,0)\|_2\,|+\| H(t,0,0)\|_2 \\ &\leq \hbar\ \{\| X(t)\|_2+\|\mathcal{U}(t)\|_2\}+\| H(t,0,0)\|_2\ .\end{aligned}$$

□

Now, we define the set Q_ρ by

$$Q_\rho = \{X \in \mathsf{L}_2^*(\mathcal{I}, \mathsf{L}_2(\Lambda)), \|X\|_2^* \leq \rho\} \subset \mathsf{L}_2^*(\mathcal{I}, \mathsf{L}_2(\Lambda)),$$

where

$$\rho = \sqrt{\frac{T^* \left[2\|B\|_{\mathbb{C}}^2 + 8\lambda^2 T\hbar^2 r^2 + 8\lambda^2 A^2 T\right]}{1 - 4\lambda^2 \ \hbar^2 T^*}}.$$

3.1. Existence of solution $X \in \mathsf{L}_2^*(\mathcal{I}, \mathsf{L}_2(\Lambda))$

We have the following theorem for the existence of solution of the fractional-order Itô integral equation (1.1).

Theorem 3.1 *Let the hypotheses $(A1), (M1) - (M2)$ be satisfied, then for every solution $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$ of the constraint (1.2), there exists a unique solution $X \in \mathsf{L}_2^*(\mathcal{I}, \mathsf{L}_2(\Lambda))$ of (1.1).*

Proof: Let $X \in Q_\rho$, and define the mapping

$$\mathcal{T}X(t) = \mathcal{B}t^{\alpha-1} + \lambda F_\beta H(t, X(t), \mathcal{U}(t)),$$

Since $(a+b)^2 \leq 2a^2 + 2b^2$ and $(a+b+c)^2 \leq 2a^2 + 2(b+c)^2 \leq 2a^2 + 4b^2 + 4c^2$, we can estimate that

$$\begin{aligned}
\|\mathcal{T}X(t)\|_2^2 &= \|B\ t^{\alpha-1} + \lambda F_\beta H(t, X(t), \mathcal{U}(t))\|_2^2 \\
&\leq 2\|B\|_2^2\ t^{2\alpha-2} + 2\lambda^2 \|F_\beta H(t, X(t), \mathcal{U}(t))\|_2^2 \\
&\leq 2\|B\|_2^2\ t^{2\alpha-2} + 2\lambda^2 \|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} H(s, X(s), \mathcal{U}(s)) d\mathcal{W}(s)\|_2^2 \\
&\leq 2\|B\|_2^2\ t^{2\alpha-2} + 2\lambda^2 \int_0^t |\frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)}| \|H(s, X(s), \mathcal{U}(s))\|_2^2 ds \\
&\leq 2\|B\|_2^2\ t^{2\alpha-2} + 2\lambda^2 \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} [\hbar\ \parallel X(s) \parallel_2 + \hbar\ r + A]^2\, ds \\
&\leq 2\|\mathcal{B}\|_2^2\ t^{2\alpha-2} + 2\lambda^2 \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \left[2\hbar^2 \parallel X(s) \parallel_2^2 + 4\hbar^2\ r^2 + 4A^2\right] ds,
\end{aligned}$$

where $A = \parallel H(t,0,0) \parallel_2$. Following that,

$$\begin{aligned}
(\|\mathcal{T}X\|_2^*)^2 &= \int_0^T \|\mathcal{T}X(t)\|_2^2 dt \\
&\leq 2\|\mathcal{B}\|_{\mathbb{C}}^2 \int_0^T t^{2\alpha-2} dt + 2|\lambda|^2\ \Im \\
&= 2\|\mathcal{B}\|_{\mathbb{C}}^2 \frac{T^{2\alpha-1}}{2\alpha-1} + 2\lambda^2\ \Im,
\end{aligned}$$

where,

$$\begin{aligned}
\Im &= \int_0^T \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \left[2\hbar^2\ \parallel X(s) \parallel_2^2 + 4\hbar^2\ r^2 + 4A^2\right] ds dt \\
&= \int_0^T \left[2\hbar^2\ \parallel X(s) \parallel_2^2 + 4\hbar^2\ r^2 + 4A^2\right] \int_s^T \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} dt ds \\
&\leq \frac{T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)} \int_0^T \left[2\hbar^2\ \parallel X(s) \parallel_2^2 + 4\hbar^2\ r^2 + 4A^2\right] ds \\
&\leq \frac{T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)} \left[2\hbar^2 (\|X\|_2^*)^2 + 4\hbar^2 \int_0^T r^2 ds + 4 \int_0^T A^2 ds\right].
\end{aligned}$$

This gives

$$\Im \leq \frac{T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)} \left[2\hbar^2(\|X\|_2^*)^2 + 4\hbar^2 r^2 T + 4A^2 T\right].$$

Hence, we obtain

$$(\|\mathcal{T}X\|_2^*)^2 \leq 2\|\mathcal{B}\|_{\mathbb{C}}\, \frac{T^{2\alpha-1}}{2\alpha-1} + 2\lambda^2 \; \frac{T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)} \left[2\hbar^2\rho^2 + 4\hbar^2 r^2 T + 4A^2 T\right] = \rho^2$$

and $\|\mathcal{T}X\|_2^* \leq \rho,$ where

$$(1 - 4\lambda^2\ \hbar^2 T^*)\rho^2 = T^* \left[2\|\mathcal{B}\|_{\mathbb{C}}^2 + 8\lambda^2 T\hbar^2 r^2 + 8\lambda^2 A^2 T\right],$$

$$\rho = \sqrt{\frac{T^*\left[2\|B\|_{\mathbb{C}}^2 + 8\lambda^2 T\hbar^2 r^2 + 8\lambda^2 A^2 T\right]}{1 - 4\lambda^2\ \hbar^2 T^*}} \quad \text{and} \quad T^* = \max\left\{\frac{T^{2\alpha-1}}{2\alpha-1}, \frac{T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)}\right\}.$$

Let $X, \bar{X} \in Q_\rho$ be two solutions of (1.1), then we have

$$\begin{aligned}
\|\mathcal{T}X(t) - \mathcal{T}\bar{X}(t)\|_2^2 &= \|\lambda[F_\beta H(t, X(t), \mathcal{U}(t)) - F_\beta H(t, \bar{X}(t), \mathcal{U}(t))]\|_2^2 \\
&\leq \lambda^2\|F_\beta[H(t, X(t), \mathcal{U}(t)) - H(t, \bar{X}(t), \mathcal{U}(t))]\|_2^2 \\
&\leq \lambda^2\|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}[H(s, X(s), \mathcal{U}(s)) - H(s, \bar{X}(s), \mathcal{U}(s))]d\mathcal{W}(s)\|_2^2 \\
&\leq \lambda^2 \int_0^t |\frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)}|\|H(s, X(s), \mathcal{U}(s)) - H(s, \bar{X}(s), \mathcal{U}(s))\|_2^2 ds \\
&\leq \lambda^2 \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \left[\hbar\ \ \| X(s) - \bar{X}(s) \|_2\right]^2 ds \\
&\leq \lambda^2 \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \left[\hbar^2\ \| X(s) - \bar{X}(s) \|_2^2\right] ds \\
&\leq \frac{\lambda^2\hbar^2 T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)}\|X(t) - \bar{X}(t)\|_2^2.
\end{aligned}$$

For every solution $\mathcal{U}$ of the constrained problem (1.2), we get

$$(\|\mathcal{T}X - \mathcal{T}\bar{X}\|_2^*)^2 \leq \lambda^2\hbar^2 T^*(||X - \bar{X}||_2^*)^2.$$

Which proves that T is contraction on Q_ρ [8] and the fractional order stochastic integral equation (1.1) has a unique solution $X \in Q_\rho \subset \mathsf{L}_2^*(\mathcal{I}, \mathsf{L}_2(\Lambda))$. □

3.2. Continuous dependence of $X \in \mathsf{L}_2^*(\mathcal{I}, \mathsf{L}_2(\Lambda))$

Firstly, discussing the continuous dependence of the solution $X \in \mathsf{L}_2^*([0,T], \mathsf{L}_2(\Lambda))$ of the stochastic integral equation (1.1) on λ and $\mathcal{B}$, after that we study the continuous dependence on $H(t, X(t), \mathcal{U}(t))$ for every unique solution $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$ of the constraint (1.2).

Definition 3.1 *The solution $X \in \mathsf{L}_2^*([0,T], \mathsf{L}_2(\Lambda))$ of the fractional order stochastic integral equation (1.1) depends continuously on the parameter λ and the second-order stochastic process $\mathcal{B}$, for every unique solution $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$ of the constrained (1.2), if for all $\varepsilon_5 > 0$, there exists $\delta_5 > 0$ such that $\max\{|\lambda - \lambda^*|, \|\mathcal{B} - \mathcal{B}^*\|_2\} \leq \delta_5$ implies that $||X - X^*||_2^* \leq \varepsilon_5$.*

Theorem 3.2 *Let the assumptions of Theorem (3.1) be satisfied, then the solution $X \in \mathsf{L}_2^*([0,T], \mathsf{L}_2(\Lambda))$ of the integral equation (1.1) depends continuously on the parameter λ and the stochastic process B.*

Proof: Let X^* be the solution of the equation

$$X^*(t) = \mathcal{B}^* t^{\alpha-1} + \lambda^* F_\beta H(t, X^*(t), \mathcal{U}(t)), \quad t \in (0, T]. \tag{3.1}$$

Here

$$\begin{aligned} X(t) - X^*(t) &= (\mathcal{B} - \mathcal{B}^*)t^{\alpha-1} + \lambda F_\beta[H(t, X(t), \mathcal{U}(t)) - H(t, X^*(t), \mathcal{U}(t))] \\ &+ (\lambda - \lambda^*)F_\beta H(t, X^*(t), \mathcal{U}(t)). \end{aligned}$$

It follows that

$$\begin{aligned} \|X(t) - X^*(t)\|_2^2 &\leq 2\|\mathcal{B} - \mathcal{B}^*\|_2^2 t^{2\alpha-2} + 4\lambda^2 \|F_\beta[H(t, X(t), \mathcal{U}(t)) - H(t, X^*(t), \mathcal{U}(t))]\|_2^2 \\ &+ 4(\lambda - \lambda^*)^2 \ \|F_\beta H(t, X^*(t), \mathcal{U}(t))\|_2^2 \\ &\leq 2\delta_5^2 \ t^{2\alpha-2} + 4\lambda^2 \| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} [H(s, X(s), \mathcal{U}(s)) - H(s, X^*(s), \mathcal{U}(s))] d\mathcal{W}(s)\|_2^2 \\ &+ 4(\lambda - \lambda^*)^2 \ \| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} H(s, X^*(s), \mathcal{U}(s)) d\mathcal{W}(s)\|_2^2 \\ &\leq 2\delta_5^2 \ t^{2\alpha-2} + 4\lambda^2 \int_0^t |\frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)}| \|H(s, X(s), U(s)) - H(s, X^*(s), \mathcal{U}(s))\|_2^2 ds \\ &+ 4(\lambda - \lambda^*)^2 \ \int_0^t |\frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)}| \|H(s, X^*(s), \mathcal{U}(s))\|_2^2 ds \\ &\leq 2\delta_5^2 \ t^{2\alpha-2} + 4\lambda^2 \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \ [\hbar \ \| \ X(s) - X^*(s) \ \|_2]^2 \ ds \\ &+ 4(\lambda - \lambda^*)^2 \ \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \ [\hbar \ \| \ X^*(s) \ \|_2 + \hbar \ r + A]^2 \ ds \\ &\leq 2\delta_5^2 \ t^{2\alpha-2} + 4\lambda^2 \hbar^2 \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \ \| \ X(s) - X^*(s) \ \|_2^2 \ ds \\ &+ 4(\lambda - \lambda^*)^2 \ \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \ \left[2\hbar^2 \ \| \ X^*(s) \ \|_2^2 + 4\hbar^2 \ r^2 + 4A^2\right] ds \end{aligned}$$

For every unique solution $\mathcal{U} \in \mathbb{C}([0, T], \mathsf{L}_2(\Lambda))$ of the constraint (1.2), we get

$$\begin{aligned} (\|X - X^*\|_2^*)^2 &\leq \frac{2\delta_5^2 T^{2\alpha-1}}{(2\alpha - 1)} + \frac{4\lambda^2 \hbar^2 T^{2\beta-1}}{(2\beta - 1)\Gamma^2(\beta)} (\|X - X^*\|_2^*)^2 \\ &+ 4(\lambda - \lambda^*)^2 \ \frac{T^{2\beta-1}}{(2\beta - 1)\Gamma^2(\beta)} \left[2\hbar^2 \ (\| \ X^* \ \|_2^*)^2 + 4\hbar^2 \ r^2 T + 4A^2 T\right] \\ &\leq 2\delta_5^2 \ (T^*) + 4T^* \lambda^2 \hbar^2 (\|X - X^*\|_2^*)^2 \\ &+ 4(\lambda - \lambda^*)^2 \ T^* \left[2\hbar^2 \ r + 4\hbar^2 \ r^2 T + 4A^2 T\right], \end{aligned}$$

this implies to

$$(\|X - X^*\|_2^*)^2 \leq \frac{2\delta_5^2 \ T^* + 4(\lambda - \lambda^*)^2 \ T^* \left[2\hbar^2 \ \rho + 4\hbar^2 \ r^2 T + 4A^2 T\right]}{1 - 4T^* \lambda^2 \hbar^2} = \varepsilon_5^2.$$

Which completes the proof. □

Definition 3.2 *The solution $X \in L_2^*([0, T], \mathsf{L}_2(\Lambda))$ of the fractional order stochastic integral equation (1.1) depends continuously on the random function $H(t, X(t), \mathcal{U}(t))$, for every unique solution $\mathcal{U} \in \mathbb{C}([0, T], \mathsf{L}_2(\Lambda))$ of a constraint (1.2) and for all $\varepsilon_6 > 0$, there exists $\delta_6 > 0$ such that $\|H(t, X(t), \mathcal{U}(t)) - H^*(t, X(t), \mathcal{U}(t))\|_2 \leq \delta_6$ implies that $||X - X^*||_2^* \leq \varepsilon_6$.*

Theorem 3.3 *Let the assumptions of theorem (3.1) be satisfied, then the solution $X \in \mathsf{L}_2^*([0,T], \mathsf{L}_2(\Lambda))$ of the integral equation (1.1) depends continuously on the main random function $H(t, X(t), \mathcal{U}(t))$.*

Proof: Let X^* be the solution of the equation

$$X^*(t) = \mathcal{B}t^{\alpha-1} + \lambda F_\beta H^*(t, X^*(t), \mathcal{U}(t)), \quad t \in (0,T]. \tag{3.2}$$

Here,

$$\begin{aligned} X(t) - X^*(t) &= \lambda F_\beta\{H(t, X(t), \mathcal{U}(t)) - H^*(t, X^*(t), \mathcal{U}(t))\} \\ &= \lambda F_\beta\{H(t, X(t), \mathcal{U}(t)) - H(t, X^*(t), \mathcal{U}(t))\} \\ &+ \lambda F_\beta\{H(t, X^*(t), \mathcal{U}(t)) - H^*(t, X^*(t), \mathcal{U}(t))\}. \end{aligned}$$

It follows that,

$$\begin{aligned} \|X(t) - X^*(t)\|_2^2 &\le 2\lambda^2 \|F_\beta\{H(t, X(t), \mathcal{U}(t)) - H(t, X^*(t), \mathcal{U}(t))\|_2^2 + 2\lambda^2\delta_6^2 \\ &\le 2\lambda^2 \| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} [H(s, X(s), \mathcal{U}(s)) - H(s, X^*(s), \mathcal{U}(s))] dW(s)\|_2^2 + 2\lambda_8^2\delta_6^2 \\ &\le 2\lambda^2 \int_0^t |\frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)}| \|H(s, X(s), \mathcal{U}(s)) - H(s, X^*(s), \mathcal{U}(s))\|_2^2 ds + 2\lambda^2\delta_6^2 \\ &\le 2\lambda^2 \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} [\hbar \ \| X(s) - X^*(s) \|_2]^2 \, ds + 2\lambda^2\delta_6^2 \\ &\le 2\lambda^2\hbar^2 \int_0^t \frac{(t-s)^{2\beta-2}}{\Gamma^2(\beta)} \| X(s) - X^*(s) \|_2^2 \, ds + 2\lambda^2\delta_6^2 \end{aligned}$$

For every unique solution $\mathcal{U} \in \mathbb{C}([0,T], \mathsf{L}_2(\Lambda))$ of the constraint (1.2), we get

$$\begin{aligned} (\|X - X^*\|_2^*)^2 &\le \frac{2\lambda^2\hbar^2 T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)} (\|X - X^*\|_2^*)^2 + 2\lambda^2\delta_6^2 \\ &\le 2T^*\lambda^2\hbar^2 \|X - X^*\|_2^* + 2\lambda^2\delta_6^2, \end{aligned}$$

this implies to

$$(\|X - X^*\|_2^*)^2 \le \frac{2\lambda^2\delta_6^2}{1 - 2T^*\lambda^2\hbar^2} = \varepsilon_6^2.$$

Which completes the proof. □

3.3. Hyers - Ulam stability of (1.1)

For this section, The main purpose is to explore the Hyers-Ulam stability of the solution $X_p \in \mathsf{L}_2^*(\mathcal{I}, \mathsf{L}_2(\Lambda))$.To get the desired result, we have the following definition.

Definition 3.3 *Let the solution $X \in \mathsf{L}_2^*(\mathcal{I}, \mathsf{L}_2(\Lambda))$ of the problem (1.1) be exists uniquely, then the problem (1.1) is Hyers-Ulam stable, if for every $\varepsilon_7 > 0$, $\exists\ \delta_7(\varepsilon_7)$ and for any $\delta-$approximate solution X_p of the problem (1.1) satisfies,*

$$\|X_p(t) - \mathcal{B}t^{\alpha-1} - \lambda I^\beta H(t, X_p(t), \mathcal{U}_p(t))\|_2 \le \delta_7, \qquad t \in \mathcal{I}, \tag{3.3}$$

implies $\|X_p - X\|_2^* \le \varepsilon_7$, $\ t \in \mathcal{I}$, where $\mathcal{U}_p$ is the approximate solution of (2.1).

Theorem 3.4 *Let the assumptions of Theorems (2.2) and (3.1) be satisfied. Then the problem (1.1) is Hyers-Ulam stable.*

Proof: Let X be the solution of (1.1), for $\delta_7 > 0$ we can get

$$\begin{aligned} ||X_p(t) - X(t)||_2^2 &= ||X_p(t) - Bt^{\alpha-1} - \lambda F_\beta H(t, X(t), \mathcal{U}(t))||_2^2 \\ &\leq 2||X_p(t) - \mathcal{B}t^{\alpha-1} - \lambda F_\beta H(t, X_p(t), \mathcal{U}_p(t))\|_2^2 \\ &+ 2\lambda^2 \|F_\beta[H(t, X_p(t), \mathcal{U}_p(t)) - H(t, X(t), \mathcal{U}(t))]\|_2^2 \\ &\leq 2\delta_7^2 T + \frac{2\lambda^2 \hbar^2 T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)} \left(\int_0^T [2\|X_p(s) - X(s)\|_2^2 + 2\|\mathcal{U}_p(s) - \mathcal{U}(s)\|_2^2] ds \right) \\ &\leq 2\delta_7^2 T + \frac{2\lambda^2 \hbar^2 T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)} \left(\int_0^T [2\|X_p(s) - X(s)\|_2^2 + 2\varepsilon_4^2] ds \right), \end{aligned}$$

then

$$(\|X_p - X\|_2^*)^2 \quad \leq \quad 2\delta_7^2 T + \frac{2\lambda^2 \hbar^2 T^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)} \left(2(\|X_p - X\|_2^*)^2 + 2\varepsilon_4^2 T\right).$$

Then

$$(\|X_p - X\|_2^*)^2 \quad \leq \quad \frac{2\delta_7^2 T + 4\lambda^2 \hbar^2 T^* T \ \varepsilon_4^2}{1 - 4\lambda^2 \hbar^2 T^*}.$$

Finally,

$$\|X_p - X\|_2^* \leq \varepsilon_7, \qquad \text{where} \qquad \varepsilon_7 = \sqrt{\frac{2\delta_7^2 T + 4\lambda^2 \hbar^2 T^* T \ \varepsilon_4^2}{1 - 4\lambda^2 \hbar^2 T^*}}.$$

Which completes the proof. □

4. Examples

Here, we have the following examples for some known mathematical models as constraints:

(I) Consider the fractional-order Itô integral equation

$$X(t) = \mathcal{B}t^{\alpha-1} + \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} H(s, X(s), \mathcal{U}(s)) d\mathcal{W}(s), \tag{4.1}$$

subject to the constraint which describes a mathematical model of the motion of a Brownian particle model frictional forces as follow for the one dimensional case

$$\begin{cases} d\mathcal{U}(t) = -b\, \mathcal{U} dt + \pounds\ d\mathcal{W}_1(t), \quad t \in (0, T], \\ \\ \mathcal{U}(0) = \mathcal{U}_0 + \int_0^T \frac{(1-t)^{\gamma-1}}{\Gamma(\gamma)} G(t, \mathcal{U}(t)) d\mathcal{W}_2(t), \quad \gamma \in (\frac{1}{2}, 1). \end{cases} \tag{4.2}$$

where $b > 0$ is a coefficient of friction, and $\pounds$ is a diffusion coefficient, $\mathcal{U}(.)$ is the velocity of the Brownian particle subject to nonlocal fractional order stochastic-integral condition, we get the solution of (1.2) as following

$$\mathcal{U}(t) = e^{-bt}\ (\mathcal{U}_0 + \int_0^T \frac{(1-t)^{\gamma-1}}{\Gamma(\gamma)} G(t, \mathcal{U}(t)) d\mathcal{W}_2(t)) + \pounds \int_0^t e^{-b(t-s)} d\mathcal{W}_1(s), \quad t \geq 0 \tag{4.3}$$

Note that, this solution in Equation (4.3) is called Langevin's process (see [18])
This function $\mathcal{U}$ satisfies our conditions for the function in (2.1), where $\mathcal{K} = \pounds$.

(II) Consider the fractional-order Itô integral equation (4.1) subject to the following one-dimensional equation

$$d\mathcal{U}(t) = \frac{b-\mathcal{U}(t)}{1-t}dt + d\mathcal{W}(t) \quad 0 \le t < 1 \quad \mathcal{U}_0 = a \quad a, b \in R$$

The solution of this equation is called Brownian bridge process (see [29]) and it gives

$$\mathcal{U}(t) = a(1-t) + bt + (1-t)\int_0^t \frac{d\mathcal{W}(s)}{(1-s)}, \quad 0 \le t < 1 \tag{4.4}$$

This function $\mathcal{U}$ in Equation (4.4) satisfies our conditions for the function in (2.1), where $\mathcal{K} = 1, \quad G = 0$.

(III) Consider the fractional-order Itô integral equation (4.1) subject to the constraint which describes a mathematical model of the motion of a Brownian particle model frictional forces as follow for the one dimensional case

$$d\mathcal{U}(t) = a(m - \mathcal{U}(t))dt + \pounds d\mathcal{W}(t), \quad \mathcal{U}(0) = \mathcal{U}_0$$

where $a, \pounds > 0$ and m is a real number, the solution will be

$$\mathcal{U}(t) = m + (\mathcal{U}_0 - m)e^{-at}\mathcal{U}_0 + \pounds\int_0^t e^{-a(t-s)}d\mathcal{W}(s), \quad t \ge 0. \tag{4.5}$$

Note that, this solution in Equation (4.5) is called Ornstein-Uhlenbeck process (see [25]) This function $\mathcal{U}$ satisfies our conditions for the function in (2.1), where $\mathcal{K} = \pounds, \quad G = 0$.

5. Conclusions

In this study, we investigated the fractional-order Itô-integral equation (1.1), that constrained by a nonlocal fractional order stochastic-integral problem of Itô-differential equation (1.2). The existence and the sufficient conditions for uniqueness of the solution $\mathcal{U} \in C(\mathcal{I}, \mathsf{L}_2(\Lambda))$ of the constraint are given in Theorems (2.1) and (2.2), we proved that the solution $\mathcal{U}$ is continuously dependent on the random process $\mathcal{U}_0$ and on the two random functions $\mathcal{K}, G$ in Theorems (2.3), (2.4) and (2.5). We proved also in theorem (2.6) and corollary (2.1) that the integral equation (2.1) and the constraint (1.2) are Hyers-Ulam stable. After that, we proved that, for every unique solution $\mathcal{U}$ of the equation (1.2), there exists a unique solution $X \in L_2^*(\mathcal{I}, \mathsf{L}_2(\lambda))$ equation (1.1) which depends continuously on the parameter λ and the stochastic process B and Hyers-Ulam stable, all that can seen in Theorems (3.2), (3.3) and (3.4).

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