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Exploring the Rectangular Inequality in Bicomplex Valued Metric Spaces

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ABSTRACT: In this paper, we present a new space that extends the concept of bicomplex valued metric spaces. We used rectangular inequality instead of triangular inequality. As a result, we have obtained some new results regarding the complete bicomplex valued rectangular metric spaces. We employed the well-known Banach contraction to investigate fixed points in bicomplex valued rectangular metric spaces. Moreover, we give enough parameters so that two contractive mappings in bicomplex valued rectangular metric spaces have a common fixed point. We also present several non-trivial cases to support the accuracy of our established findings.

Key Words: Contraction mapping, fixed point, common fixed point, rectangular metric space.

Contents

1. Introduction

Segre [25] endeavored to create special algebras in a novel way. He proposed bicomplex numbers, tricomplex numbers, and other commutative generalization of complex numbers as components of an endless collection of algebras. Subsequently, during the 1930s, further academics made significant contributions to this field (see [7]- [12]). Unfortunately, no advancements were achieved in this domain during the subsequent half-century. Bicomplex algebra and function theory were later pioneered by Price [21]. This topic holds significant relevance in various mathematical science domains, as well as other domains within the realm of science and technology, which have recently garnered a resurgence of interest. A noteworthy investigation into the fundamental functions using bicomplex numbers was carried out by Luna-Elizarraras et. al. [20]. Beg et al. [7] looked into whether maps on cone metric spaces portrayed in topological vector spaces have common fixed points. Azam et al. [3] extended the scope of the study to encompass complex valued metric space. Additionally, they demonstrated a few of the common fixed point theorems (FPT in shortly) for two self-contracting maps. In addition to proving a well-known common FPT that fulfills a rational inequality in complex valued metric spaces, Rouzkard and Imdad [24] also expanded on the results of Azam et al. [1].

Research on the principle of Banach contraction [6] is ongoing and is a well-liked and useful tool for resolving existing problems in many areas of mathematical analysis. Rectangular metric space (RMS) was first conceptualized by Branciari [12], who performed an analogy to the Banach Contraction Principle in such space. He achieved this by substituting a three-term formula and the sum on the right side of the triangle inequality for the definition of a metric space. Several FPT have been developed for various contractions on rectangular metric space.

Choudhury et al. [15] showed some fixed point solutions for rational type expressions in partly ordered complex valued metric spaces. The fixed point in complex valued metric spaces of a mapping satisfying

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rational inequality was demonstrated by Bhat et al. [11]. The attempts are also visible in [6,11]. Choi et al. [13] proved a few standard FPTs about two mappings in bicomplex valued metric space ($BVMS$ in shortly) that are weakly compatible. Jebril et al. [19] established a number of classic FPT for one pair of mappings in bicomplex valued metric space under rational contractions.

Many researcher have made their significant contribution on this field. Azam et al. [2] worked on "Banach contraction principle on cone rectangular metric spaces". Ahmad et al. [4] studied "Common fixed point results for contractive mappings in complex valued metric spaces". In the year 2013 Arshad et al. [5] study "Some common fixed point results in rectangular metric spaces". Singularities of functions of one and several bicomplex variables studied by Colombo et al. [14] in 2010. Datta et al. [16,17], Dragoni and Scorza [18] have made many significant work in bicomplex fixed point theory. Ran and Reurings [22] have studied A FPT in partially ordered sets and some applications to matrix equations. Roshan et al. [23] worked on ihe topic "new fixed point results in b-rectangular metric spaces". Spampinato [26] studied Singularities of functions of one and several bicomplex variables. Spampinato [27] worked on the topic "On the representation of totally differentiable functions of a bicomplex variable". Recently, contraction principles in bicomplex valued G-metric and multiplicative metric spaces have been studied by Bhattacharjee *et al.* [9,10].

In this work, we first study the bicomplex valued rectangular metric space ($BVRMS$ in shortly), and then we give some FPT for two contractive type mappings satisfying a rational inequality. Furthermore, we resolve a few unique fixed point (UFP in shortly) theorems within this new generalized $BVRMS$.

After that, we outline basic ideas and icons for further reference. The notations $\mathbb{C}_0, \mathbb{C}_1$, and $\mathbb{C}_2$ represent the sets of real, complex, and bicomplex numbers respectively.

2. Preliminaries

2.1. Bicomplex Number

The bicomplex number, as stated by Segre [25], is:

$$\kappa = t_1 + t_2 i_1 + t_3 i_2 + t_4 i_1 i_2,$$

where $t_1, t_2, t_3, t_4 \in \mathbb{C}_0$, and the standalone units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we denote the collection of bicomplex numbers $\mathbb{C}_2$ as follows:

$$\mathbb{C}_2 = \{\kappa : \kappa = t_1 + t_2 i_1 + t_3 i_2 + t_4 i_1 i_2, t_1, t_2, t_3, t_4 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_2 = \{\kappa : \kappa = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = t_1 + t_2 i_1 \in \mathbb{C}_1$ and $z_2 = t_3 + t_4 i_1 \in \mathbb{C}_1$.
In $\mathbb{C}_2$, there are four idempotent elements: 0, 1, $\delta_1 = \frac{1+i_1 i_2}{2}$, and $\delta_2 = \frac{1-i_1 i_2}{2}$ out of which the nontrivial components δ_1 and δ_2, such that $\delta_1 + \delta_2 = 1$ and $\delta_1 \delta_2 = 0$. One unique way to express every bicomplex number $z_1 + i_2 z_2$ is as the sum of δ_1 and δ_2, specifically

$$\kappa = z_1 + i_2 z_2 = (z_1 - i_1 z_2)\,\delta_1 + (z_1 + i_1 z_2)\,\delta_2.$$

The idempotent illustration of bicomplex numbers and the complex coefficients $\kappa_1 = (z_1 - i_1 z_2)$ and $\kappa_2 = (z_1 + i_1 z_2)$ are the names given to this representation of κ. The bicomplex numbers κ have idempotent components, denoted by $\kappa_1 = (z_1 - i_1 z_2)$ and $\kappa_2 = (z_1 + i_1 z_2)$.
The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a non-negative real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined in [13], by

$$\begin{aligned}\|\kappa\| = \|z_1 + i_2 z_2\| &= \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} = \left(t_1^2 + t_2^2 + t_3^2 + t_4^2\right)^{\frac{1}{2}},\end{aligned}$$

where $\kappa = t_1 + t_2 i_1 + t_3 i_2 + t_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

Since $\mathbb{C}_2$ is complete and the linear space $\mathbb{C}_2$ is a norm linear space with respect to the specified norm, $\mathbb{C}_2$ is the Banach space.
The definition of $\precsim_{i_2}$, the partial order relation on $\mathbb{C}_2$, is given by Choi et al. [13], as follows:
If $\mathbb{C}_2$ is the collection of bicomplex numbers, and $\kappa = \kappa_1 + i_2\kappa_2, \aleph = \aleph_1 + i_2\aleph_2 \in \mathbb{C}_2$, then $\kappa \precsim_{i_2} \aleph$ iff $\kappa_1 \precsim \aleph_1$ and $\kappa_2 \precsim \aleph_2$,
that is, $\kappa \precsim_{i_2} \aleph$, if any of the subsequent circumstances holds true:

(1) $\kappa_1 = \aleph_1, \kappa_2 = \aleph_2$,
(2) $\kappa_1 \prec \aleph_1, \kappa_2 = \aleph_2$,
(3) $\kappa_1 = \aleph_1, \kappa_2 \prec \aleph_2$, and
(4) $\kappa_1 \prec \aleph_1, \kappa_2 \prec \aleph_2$.

Specifically, if $\kappa \precsim_{i_2} \aleph$ and $\kappa \neq \aleph$, that is, if one of (2), (3), and (4) is satisfied, it is possible to write $\kappa \lnsim_{i_2} \aleph$; if only (4) is satisfied, it is possible to write $\kappa \prec_{i_2} \aleph$.
According to the Choi et al. [13], for any two bicomplex numbers $\kappa, \aleph \in \mathbb{C}_2$ the following characteristics of the norm on the bicomplex algebra must be observed:

(i) $\kappa \precsim_{i_2} \aleph \Rightarrow \|\kappa\| \leq \|\aleph\|$,
(ii) $\|\kappa + \aleph\| \leq \|\kappa\| + \|\aleph\|$,
(iii) $\|a\kappa\| = a\|\kappa\|$, where $a > 0$ is any real number,
(iv) $\|\kappa\aleph\| \leq \sqrt{2}\|\kappa\|\|\aleph\|$, and the equality is only true in the case that $\aleph$ and κ are at least partially degenerated,
(v) $\left\|\kappa^{-1}\right\| = \|\kappa\|^{-1}$ with $0 \prec \kappa$, where κ is a degenerated bicomplex number,
(vi) $\left\|\frac{\kappa}{\aleph}\right\| = \frac{\|\kappa\|}{\|\aleph\|}$, if the bicomplex number $\aleph$ is degenerated.

2.2. Bicomplex valued metric space

The $BVMS$ were defined as follows by Choi et al. [13]:

Definition 2.1 [13] Assuming a set $\bar{P} \neq \emptyset$, and $\wp : \bar{P}^2 \to \mathbb{C}_2$ be any mapping. Then the mapping $\wp$ is referred to as a bicomplex valued metric if that fulfills the requirements listed below:

1. $0 \prec_{i_2} \wp(\Theta_1, \Theta_2)$ for all distinct $\Theta_1, \Theta_2 \in \bar{P}$ (positivity),
2. $\wp(\Theta_1, \Theta_2) = 0 \iff \Theta_1 = \Theta_2$,
3. $\wp(\Theta_1, \Theta_2) = \wp(\Theta_2, \Theta_1)$ for all $\Theta_1, \Theta_2 \in \bar{P}$ (symmetry), and
4. $\wp(\Theta_1, \Theta_2) \precsim_{i_2} \wp(\Theta_1, \bar{z}) + \wp(\bar{z}, \Theta_2)$ for all $\Theta_1, \Theta_2, \bar{z} \in \bar{P}$ (triangle inequility).

And the pair $(\bar{P}, \wp)$ is called the $BVMS$.

2.3. Research Insufficiency

The idea of $BVRMS$ is relatively novel and hasn't been explored in the literature yet, let's specify it precisely and examine its characteristics. This will serve as the starting point for the research and emphasize its uniqueness.

Bi-complex valued rectangular metric spaces provide fertile ground for investigation, with various possible research gaps. These gaps provide potential for important contributions to theoretical mathematics as well as practical applications in a variety of scientific and technical fields.

In this work, we are interested in studying some of its topological features, such as compactness, connectedness, and completeness in bi-complex valued rectangular metric spaces.

3. Main Results

This part begins with the introduction of the concept of bicomplex valued rectangular metric space (BVRMS in short), which is a generalization of the concept of $BVMS$ in this paper. A $BVRMS$ is defined as follows:

Definition 3.1 Let $\bar{P}(\neq 0)$ be any set and $\wp : \bar{P} \times \bar{P} \to \mathbb{C}_2$ be a mapping. Then the mapping $\wp$ is referred to as a bicomplex valued rectangular metric if that meets the requirements listed below:

1. $\emptyset \precsim_{i_2} \wp(\Theta_1,\Theta_2)$ for all $\Theta_1,\Theta_2 \in \bar{P}$ (positivity),
2. $\wp(\Theta_1,\Theta_2)=0$ iff $\Theta_1=\Theta_2$,
3. $\wp(\Theta_1,\Theta_2)=\wp(\Theta_2,\Theta_1)$ for all $\Theta_1,\Theta_2 \in \bar{P}$ (symmetry), and
4. $\wp(\Theta_1,\Theta_2) \precsim_{i_2} \wp(\Theta_1,x)+\wp(x,y)+\wp(y,\Theta_2)\ \forall\ \Theta_1,\Theta_2 \in \bar{P}$ and all nonidentical points $\Theta_1,\Theta_2 \in \bar{P}\setminus\{x,y\}$ (rectangular inequility).

The pair $(\bar{P},\wp)$ is called the BVRMS.

Example 3.1 Consider $\bar{P}=\{0,\frac{1}{10},\frac{1}{4},\frac{1}{2},2\}$, define a bicomplex valued rectangular metric $\wp:\bar{P}\times\bar{P}\to\mathbb{C}_2$ by $\wp(\Theta_1,\Theta_2)=(1+i_2)\,|\Theta_1-\Theta_2|, \forall\Theta_1,\Theta_2\in\bar{P}$.
It is simple to ensure that $(\bar{P},\wp)$ is a BVRMS using the Definition 2.1 given above.

Example 3.2 Let $\bar{P}=\mathbb{N}$, $\alpha>0$ and define a symmetrical metric $\wp:\bar{P}\times\bar{P}\to\mathbb{C}_2$ by

$$\wp(\Theta_1,\Theta_2)=\begin{cases}0 & \iff \Theta_1=\Theta_2,\\ 3\alpha i_2 & \text{if } \Theta_1\neq\Theta_2 \text{ and } \Theta_1,\Theta_2\in\{1,2\},\\ \alpha i_2 & \text{if } \Theta_1\neq\Theta_2 \text{ and } \Theta_1 \textit{ or } \Theta_2 \textit{ or both} \notin\{1,2\}.\end{cases}$$

By utilizing the Definition 2.1 provided above, it is easy to ensure that $(\bar{P},\wp)$ is a BVRMS.

Definition 3.2 Let $(\bar{P},\wp)$ be a BVRMS. Then,

1. A sequence $\{\Theta_n\}$ in $\bar{P}$ is convergent to a point Θ if for any $0\prec_{i_2}\kappa\in\mathbb{C}_2$, there is a number $n_0\in\mathbb{W}\setminus\{0\}$ such that $\wp(\Theta_n,\Theta)\prec_{i_2}\kappa\quad\forall\ n>n_0$ and we express it mathematically as $\lim\limits_{n\to\infty}\Theta_n=\Theta$ or $\Theta_n\to\Theta$ as $n\to\infty$.
2. A sequence $\{\Theta_n\}$ in $\bar{P}$ is called a Cauchy sequence in $(\bar{P},\wp)$ if for any $0\prec_{i_2}\kappa\in\mathbb{C}_2$, there is a number $n_0\in\mathbb{W}\setminus\{0\}$ such that $\wp(\Theta_n,\Theta_{n+m})\prec_{i_2}\kappa$ for any $m,n\in\mathbb{W}\setminus\{0\}$ and $n,m>n_0$.
3. A BVRMS $(\bar{P},\wp)$ is complete if each Cauchy sequence is convergent in $\bar{P}$.

Lemma 3.1 *Let $(\bar{P},\wp)$ be a BVRMS and $\{\Theta_n\}$ be a sequence in $\bar{P}$. Then $\{\Theta_n\}$ is a convergent sequence and converges to a point Θ iff $\lim\limits_{n\to\infty}\|\wp(\Theta_n,\Theta)\|=0$.*

Lemma 3.2 *Let $(\bar{P},\wp)$ be a BVRMS and $\{\Theta_n\}$ be a sequence in $\bar{P}$. Then $\{\Theta_n\}$ is a Cauchy sequence in $\bar{P}$ iff $\lim\limits_{n,m\to\infty}\|\wp(\Theta_n,\Theta_{n+m})\|=0$.*

Theorem 3.1 *Let $(\bar{P},\wp)$ be a complete BVRMS and $\Gamma:\bar{P}\to\bar{P}$ be any mapping satisfying:*

$$\wp(\Gamma\Theta,\Gamma\bar{y})\precsim_{i_2}\beta\wp(\Theta,\bar{y}) \tag{3.1}$$

for all $\Theta,\bar{y}\in\bar{P}$, where $\beta\in[0,1)$. Then Γ has a UFP.

Proof: *Let Γ satisfy Equation (3.1). Asuming $\Theta_0\in\bar{P}$ denotes a random point, and we define the sequence $\{\Theta_n\}$ by $\Theta_n=\Gamma^n\Theta_0$. From Equation (3.1), we get*

$$\begin{aligned}\wp(\Theta_n,\Theta_{n+1}) &= \wp(\Gamma\Theta_{n-1},\Gamma\Theta_n)\\ &\precsim_{i_2}\beta\wp(\Theta_{n-1},\Theta_n).\end{aligned} \tag{3.2}$$

Using again Equation (3.1), we have

$$\wp(\Theta_{n-1},\Theta_n)\precsim_{i_2}\beta\wp(\Theta_{n-2},\Theta_{n-1}),$$

and by Equation (3.2), we get

$$\wp(\Theta_n,\Theta_{n+1})\precsim_{i_2}\beta^2\wp(\Theta_{n-2},\Theta_{n-1}).$$

If we carry out this procedure further, we get

$$\wp(\Theta_n, \Theta_{n+1}) \precsim_{i_2} \beta^n \wp(\Theta_0, \Theta_1). \tag{3.3}$$

By using rectangular inequality and Equation (3.3), for any two numbers $n, m \in \mathbb{W} \setminus \{0\}$ with $n < m$, we have

$$\begin{aligned}
\wp(\Theta_n, \Theta_m) &\precsim_{i_2} \wp(\Theta_n, \Theta_{n+1}) + \wp(\Theta_{n+1}, \Theta_{n+2}) + \wp(\Theta_{n+2}, \Theta_m) \\
&\vdots \\
&\precsim_{i_2} \wp(\Theta_n, \Theta_{n+1}) + \wp(\Theta_{n+1}, \Theta_{n+2}) + \wp(\Theta_{n+2}, \Theta_{n+3}) + \cdots + \wp(\Theta_{m-1}, \Theta_m) \\
&\precsim_{i_2} (\beta^n + \beta^{n+1} + \beta^{n+2} + \cdots + \beta^{m-1})\wp(\Theta_0, \Theta_1) \\
&\precsim_{i_2} \beta^n[1 + \beta + (\beta)^2 + (\beta)^3 + \cdots + (\beta)^{m-n-1}]\wp(\Theta_0, \Theta_1) \\
&\precsim_{i_2} \frac{\beta^n}{1-\beta}\wp(\Theta_0, \Theta_1).
\end{aligned}$$

Thus, we have

$$\|\wp(\Theta_n, \Theta_m)\| \leq \frac{\beta^n}{1-\beta} \|\wp(\Theta_0, \Theta_1)\|.$$

Since $\beta \in [0, 1)$, taking limits as $n \to \infty$, we obtain

$$\frac{\beta^n}{1-\beta} \|\wp(\Theta_0, \Theta_1)\| \to 0.$$

This means that

$$\|\wp(\Theta_n, \Theta_m)\| \to 0.$$

Then $\{\Theta_n\}$ is a Cauchy sequence as stated by Lemma 2.6. Since $(\bar{P}, \wp)$ is complete, consequently $\{\Theta_n\}$ is a convergent sequence and converges to $v \in \bar{P}$.
Next, we aim to demonstrate that $\bar{w}$ is a fixed point under the mapping Γ.
i.e., $\Gamma\bar{w} = \bar{w}$. For any $n \in \mathbb{N}$, we get

$$\begin{aligned}
\wp(\bar{w}, \Gamma\bar{w}) &\precsim_{i_2} [\wp(\bar{w}, \Theta_n) + \wp(\Theta_n, \Theta_{n+1}) + \wp(\Theta_{n+1}, \Gamma\bar{w})] \\
&= [\wp(\bar{w}, \Theta_n) + \wp(\Theta_n, \Theta_{n+1}) + \wp(\Gamma\Theta_n, \Gamma\bar{w})] \\
&\precsim_{i_2} [\wp(\bar{w}, \Theta_n) + \wp(\Theta_n, \Theta_{n+1}) + \beta\wp(\Theta_n, \bar{w})].
\end{aligned} \tag{3.4}$$

Since Θ_n converges to $\bar{w}$ as $n \to \infty$, so from Equation (3.4), it is evident that $\wp(\bar{w}, \Gamma\bar{w}) = 0$, i.e., $\Gamma\bar{w} = \bar{w}$.
Lastly, we demonstrate that the fixed point of Γ remains unique. Assume that Γ lacks a single fixed point and that $w \neq \bar{w}$ is one of the additional fixed points of Γ. Thus $\Gamma w = w$. Now using Equation (3.1), we have

$$\wp(\bar{w}, w) = \wp(\Gamma\bar{w}, \Gamma w) \precsim_{i_2} \beta\wp(\bar{w}, w)$$

and

$$\|\wp(\bar{w}, w)\| \leq \beta \|\wp(\bar{w}, w)\|.$$

Since $\beta < 1$, we have, $\|\wp(\bar{w}, w)\| \leq 0$ which is contradictory. Thus, we conclude that $\bar{w} = w$ and so $\bar{w}$ is a UFP of Γ. □

Example 3.3 In Example (3.1), we consider the mapping $\Gamma : \bar{P} \to \bar{P}$ defined by

$$\Gamma(\Theta) = \begin{cases} 0 & \text{if } \Theta = 0, \\ 0 & \text{if } \Theta = \frac{1}{10}, \\ \frac{1}{10} & \text{if } \Theta = \frac{1}{4}, \\ 0 & \text{if } \Theta = \frac{1}{2}, \\ \frac{1}{2} & \text{if } \Theta = 2. \end{cases}$$

Let $a = \frac{1}{2}$, then clearly $a < 1$. Also, the condition (3.1) of the Theorem 2.7 is satisfied. Hence $\Theta = 0$ is the UFP of Γ.

Theorem 3.2 *Let $(\bar{P}, \wp)$ be a complete BVRMS and $\Gamma : \bar{P} \to \bar{P}$ consist of a continuous mapping such that, for a given function $\phi : \bar{P} \to \mathbb{C}_0$. If for each $\Theta \in \bar{P}$ the following condition holds:*

$$\wp(\Theta, \Gamma(\Theta)) \precsim_{i_2} \phi(\Theta) - \phi(\Gamma(\Theta)), \tag{3.5}$$

then $\{\Gamma^n(\Theta)\}$ converges to a fixed point of Γ for all $\Theta \in \bar{P}$.

Proof: *For a random point $\Theta \in \bar{P}$ which is fixed, let $\Theta_n = \Gamma^n(\Theta), n \in \mathbb{N}$. From Equation (3.5), we obtain*

$$0 \precsim_{i_2} \phi(\Theta) - \phi(\Gamma(\Theta)) \quad \Leftrightarrow \quad \phi(\Gamma(\Theta)) \precsim_{i_2} \phi(\Theta)$$

for all $\Theta \in \bar{P}$ and so we have

$$\phi(\Theta_{n+1}) = \phi\left(\Gamma^{n+1}(\Theta)\right) = \phi\left(\Gamma\left(\Gamma^n(\Theta)\right)\right) = \phi\left(\Gamma\left(\Theta_n\right)\right) \precsim_{i_2} \phi\left(\Theta_n\right).$$

Here, $\{\phi(\Gamma^n(\Theta))\}$ is monotonically decreasing and bounded below, we have $\lim_{n\to\infty} \phi(\Gamma^n(\Theta)) = a \geq 0$. If m, n are two natural number with $m > n$, then by using rectangular inequality and Equation(3.5), we have

$$\begin{aligned}
& \wp(\Theta_n, \Theta_m) \\
\precsim_{i_2} \; & \wp(\Theta_n, \Theta_{n+1}) + \wp(\Theta_{n+1}, \Theta_{n+2}) + \wp(\Theta_{n+2}, \Theta_m) \\
= \; & \wp(\Theta_n, \Theta_{n+1}) + \wp(\Theta_{n+1}, \Theta_{n+2}) + \wp(\Theta_{n+2}, \Theta_m) \\
\precsim_{i_2} \; & \wp(\Theta_n, \Theta_{n+1}) + \wp(\Theta_{n+1}, \Theta_{n+2}) + \wp(\Theta_{n+2}, \Theta_{n+3}) + \wp(\Theta_{n+3}, \Theta_{n+4}) \\
& + \wp(\Theta_{n+4}, \Theta_m) \\
& \vdots \\
\precsim_{i_2} \; & \wp(\Theta_n, \Theta_{n+1}) + \wp(\Theta_{n+1}, \Theta_{n+2}) + \cdots + \wp(\Theta_{m-3}, \Theta_{m-2}) + \wp(\Theta_{m-2}, \Theta_{m-1}) \\
& + \wp(\Theta_{m-1}, \Theta_m) \\
\precsim_{i_2} \; & \phi(\Theta_n) - \phi(\Theta_{n+1}) + \phi(\Theta_{n+1}) - \phi(\Theta_{n+2}) + \cdots + \phi(\Theta_{m-2}) - \phi(\Theta_{m-1}) \\
& + \phi(\Theta_{m-1}) - \phi(\Theta_m) \\
= \; & \phi(\Theta_n) - \phi(\Theta_m).
\end{aligned}$$

Using the fact that $\lim_{n\to\infty} \phi(\Theta_n) = a$, we have,

$$\begin{aligned}
\wp(\Theta_n, \Theta_m) & \precsim_{i_2} \phi(\Theta_n) - \phi(\Theta_m). \\
\Longrightarrow \wp(\Theta_n, \Theta_m) & \precsim_{i_2} a - a \; [\textit{taking limits as } n, m \to \infty] \\
& = 0.
\end{aligned}$$

Thus, we have

$$\|\wp(\Theta_n, \Theta_m)\| \leq 0 \text{ as } n, m \to \infty.$$

$$\Longrightarrow \|\wp(\Theta_n, \Theta_m)\| \to 0 \text{ as } n, m \to \infty.$$

Thus $\lim_{m,n\to\infty} \wp(\Theta_n, \Theta_m) = 0$ *and* $\{\Gamma^n(\Theta)\}$ *is a Cauchy sequence in* $\bar{P}$ *by Lemma* 2.6. *Since* $\bar{P}$ *is complete BVRMS. So there is a point* $u \in \bar{P}$ *such that* $\lim_{n\to\infty} \Gamma^n(\Theta) = u$ *and from continuity of* $\Gamma, u = \Gamma(u)$. □

Example 3.4 Let $\dot{P} = \mathbb{R}$ and $\wp(p,q) = |p-q| + |p-q|\, j$. Then $(\dot{P}, \wp)$ is a complete BVRMS. Define $I : \dot{P} \to \dot{P}$ by $I(p) = \frac{p+1}{2}$. For all $p, q \in \dot{P}$,

$$\wp(Ip, Iq) \preceq \tfrac{1}{2}\, \wp(p,q),$$

so condition (2.5) of Theorem 2.9 holds with $\varphi(t) = \frac{1}{2}t$. Hence $I^n(p) \to 1$, where 1 is the unique fixed point of I.

Theorem 3.3 *Let* $(\bar{P}, \wp)$ *be a complete BVRMS with degenerated* $1 + \wp(p,q)$ *and* $\|1 + \wp(p,q)\| > 0$ *for all* $p, q \in \bar{P}$ *and let* $\sigma, \Gamma : \bar{P} \to \bar{P}$ *be mappings satisfying the condition*

$$\wp(\sigma p, \Gamma q) \precsim_{i_2} \lambda_0 \wp(p,q) + \frac{\mu_0 \wp(p, \sigma p) \wp(q, \Gamma q)}{1 + \wp(p,q)} \tag{3.6}$$

for all $p, q \in \bar{P}$*, where* $\lambda_0, \mu_0 \in \mathbb{R}^+ \cup \{0\}$ *with the condition* $\lambda_0 + \sqrt{2}\mu_0 < 1$. *Then* S, Γ *has a common UFP.*

Proof: *Assume that any random point in* $\bar{P}$ *is* Θ_0. *We take a sequence in* $\bar{P}$*, say,* $\{\Theta_n\}$ *such that*

$$\Theta_{2k+1} = \sigma \Theta_{2k}, \quad \Theta_{2k+2} = T\Theta_{2k+1}, \quad k = 0, 1, 2, \ldots$$

Then we have

$$\begin{aligned} \wp(\Theta_{2k+1}, \Theta_{2k+2}) &= \wp(\sigma\Theta_{2k}, \Gamma\Theta_{2k+1}) \\ &\precsim_{i_2} \lambda_0 \wp(\Theta_{2k}, \Theta_{2k+1}) + \frac{\mu_0 \wp(\Theta_{2k}, \sigma\Theta_{2k}) \wp(\Theta_{2k+1}, \Gamma\Theta_{2k+1})}{1 + \wp(\Theta_{2k}, \Theta_{2k+1})} \\ &\precsim_{i_2} \lambda_0 \wp(\Theta_{2k}, \Theta_{2k+1}) + \frac{\mu_0 \wp(\Theta_{2k}, \Theta_{2k+1}) \wp(\Theta_{2k+1}, \Theta_{2k+2})}{1 + \wp(\Theta_{2k}, \Theta_{2k+1})}. \end{aligned}$$

Therefore

$$\begin{aligned} \|\wp(\Theta_{2k+1}, \Theta_{2k+2})\| \leq & \lambda_0 \|\wp(\Theta_{2k}, \Theta_{2k+1})\| \\ & + \sqrt{2}\mu_0 \frac{\|\wp(\Theta_{2k}, \Theta_{2k+1})\|}{\|1 + \wp(\Theta_{2k}, \Theta_{2k+1})\|} \|\wp(\Theta_{2k+1}, \Theta_{2k+2})\|. \end{aligned}$$

Also, $\|\wp(\Theta_{2k}, \Theta_{2k+1})\| \leq \|1 + \wp(\Theta_{2k}, \Theta_{2k+1})\|$.
Thus,

$$\begin{aligned} \|\wp(\Theta_{2k+1}, \Theta_{2k+2})\| &\leq \lambda_0 \|\wp(\Theta_{2k}, \Theta_{2k+1})\| + \sqrt{2}\mu_0 \|\wp(\Theta_{2k+1}, \Theta_{2k+2})\|. \\ \text{i.e., } (1 - \sqrt{2}\mu_0) \|\wp(\Theta_{2k+1}, \Theta_{2k+2})\| &\leq \lambda_0 \|\wp(\Theta_{2k}, \Theta_{2k+1})\|. \\ \text{i.e., } \|\wp(\Theta_{2k+1}, \Theta_{2k+2})\| &\leq \frac{\lambda_0}{(1 - \sqrt{2}\mu_0)} \|\wp(\Theta_{2k}, \Theta_{2k+1})\|. \end{aligned}$$

Similarly,

$$\begin{aligned} \wp(\Theta_{2k+2}, \Theta_{2k+3}) &= \wp(\Gamma\Theta_{2k+1}, \sigma\Theta_{2k+2}) = \wp(\sigma\Theta_{2k+2}, \Gamma\Theta_{2k+1}) \\ &\precsim_{i_2} \lambda_0 \wp(\Theta_{2k+2}, \Theta_{2k+1}) + \frac{\mu_0 \wp(\Theta_{2k+2}, \sigma\Theta_{2k+2}) \wp(\Theta_{2k+1}, \Gamma\Theta_{2k+1})}{1 + \wp(\Theta_{2k+2}, \Theta_{2k+1})} \\ &\precsim_{i_2} \lambda_0 \wp(\Theta_{2k+2}, \Theta_{2k+1}) + \frac{\mu_0 \wp(\Theta_{2k+2}, \Theta_{2k+3}) \wp(\Theta_{2k+1}, \Theta_{2k+2})}{1 + \wp(\Theta_{2k+2}, \Theta_{2k+1})}. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\|\wp(\Theta_{2k+2},\Theta_{2k+3})\| &\leq \lambda_0 \|\wp(\Theta_{2k+2},\Theta_{2k+1})\| \\ &\quad +\sqrt{2}\mu_0 \frac{\|\wp(\Theta_{2k+1},\Theta_{2k+2})\|}{\|1+\wp(\Theta_{2k+1},\Theta_{2k+2})\|}\|\wp(\Theta_{2k+2},\Theta_{2k+3})\| \\ i.e.,\ \|\wp(\Theta_{2k+2},\Theta_{2k+3})\| &\leq \lambda_0 \|\wp(\Theta_{2k+2},\Theta_{2k+1})\| + \sqrt{2}\mu_0 \|\wp(\Theta_{2k+2},\Theta_{2k+3})\|, \\ as\ \|\wp(\Theta_{2k+1},\Theta_{2k+2})\| &\leq \|1+\wp(\Theta_{2k+1},\Theta_{2k+2})\| \\ i.e.,\ (1-\sqrt{2}\mu_0)\|\wp(\Theta_{2k+2},\Theta_{2k+3})\| &\leq \lambda_0 \|\wp(\Theta_{2k+2},\Theta_{2k+3})\| \\ i.e.,\ \|\wp(\Theta_{2k+2},\Theta_{2k+3})\| &\leq \frac{\lambda_0}{(1-\sqrt{2}\mu_0)}\|\wp(\Theta_{2k+2},\Theta_{2k+3})\|. \qquad (3.7)\end{aligned}$$

Suppose that $\beta = \frac{\lambda_0}{1-\sqrt{2}\mu_0}$. *So clearly we claim that* $\beta < 1$ *as* $\lambda_0 + \sqrt{2}\mu_0 < 1$. *Now letting* $2k+1 = n$ *and from Equation (3.7) it follows that*

$$\begin{aligned}\|\wp(\Theta_{n+1},\Theta_{n+2})\| &\leq \beta\|\wp(\Theta_n,\Theta_{n+1})\| \\ &\leq \beta^2\|\wp(\Theta_{n-1},\Theta_n)\| \leq \cdots \leq \beta^{n+1}\|\wp(\Theta_0,\Theta_1)\|.\end{aligned}$$

Also for any natural numbers m, n *with* $n < m$, *we get*

$$\wp(\Theta_n,\Theta_m) \precsim_{i_2} \wp(\Theta_n,\Theta_{n+1}) + \wp(\Theta_{n+1},\Theta_{n+2}) + \cdots + \wp(\Theta_{m-1},\Theta_m).$$

Therefore

$$\begin{aligned}\|\wp(\Theta_n,\Theta_m)\| &\leq \|\wp(\Theta_n,\Theta_{n+1})\| + \|\wp(\Theta_{n+1},\Theta_{n+2})\| + \cdots + \|\wp(\Theta_{m-1},\Theta_m)\| \\ &\leq \left[\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}\right]\|\wp(\Theta_0,\Theta_1)\|. \\ i.e.,\ \|d(\Theta_n,\Theta_m)\| &\leq \beta^n\left[1+\beta+\beta^2+\cdots+\beta^{m-n-1}\right]\|\wp(\Theta_0,\Theta_1)\|.\end{aligned}$$

Since, $0 \leq \beta < 1$, *then* $1+\beta+\beta^2+\cdots+\beta^{m-n-1} \leq \frac{1}{1-\beta}$.
Hence,

$$\|\wp(\Theta_n,\Theta_m)\| \leq \frac{\beta^n}{1-\beta}\|\wp(\Theta_0,\Theta_1)\|.$$

Again $\frac{\beta^n}{1-\beta} \longrightarrow 0$ *as* $n \longrightarrow \infty$. *Hence* $\{\Theta_n\}$ *is a Cauchy sequence in* $\bar{P}$ *by Lemma 2.6. Also,* $\bar{P}$ *is a complete BVRMS. Then there exists* $\bar{w} \in \bar{P}$ *such that* $\lim_{n\longrightarrow\infty}\Theta_n = \bar{w}$.
We now demonstrate that $\bar{w} = \sigma\bar{w}$. *If not, then there exists* $0 \prec_{i_2} \Theta \in \mathbb{C}_2$ *such that* $\wp(\bar{w},\sigma\bar{w}) = \Theta$. *Therefore we have*

$$\begin{aligned}\Theta &= \wp(\bar{w},\sigma\bar{w}) \\ &\precsim_{i_2} \wp(\bar{w},\Theta_{2k+1}) + \wp(\Theta_{2k+1},\Theta_{2k+2}) + \wp(\Theta_{2k+2},\sigma\bar{w}) \\ &\precsim_{i_2} \wp(\bar{w},\Theta_{2k+1}) + \wp(\Theta_{2k+1},\Theta_{2k+2}) + \wp(\Gamma\Theta_{2k+1},\sigma\bar{w}) \\ &\precsim_{i_2} \wp(\bar{w},\Theta_{2k+1}) + \wp(\Theta_{2k+1},\Theta_{2k+2}) + \lambda_0\wp(\Theta_{2k+1},\bar{w}) + \frac{\mu_0\wp(\Theta_{2k+1},\Gamma\Theta_{2k+1})\wp(\bar{w},\sigma\bar{w})}{1+\wp(\Theta_{2k+1},\bar{w})} \\ i.e., \Theta &\precsim_{i_2} \wp(\bar{w},\Theta_{2k+1}) + \wp(\Theta_{2k+1},\Theta_{2k+2}) + \lambda_0\wp(\Theta_{2k+1},\bar{w}) + \frac{\mu_0\wp(\Theta_{2k+1},\Theta_{2k+2})\Theta}{1+\wp(\Theta_{2k+1},\bar{w})}.\end{aligned}$$

Hence

$$\|\Theta\| \leq \|\wp(\bar{w},\Theta_{2k+1})\| + \|\wp(\Theta_{2k+1},\Theta_{2k+2})\| + \lambda_0\|\wp(\Theta_{2k+1},\bar{w})\| + \sqrt{2}\mu_0\frac{\|\wp(\Theta_{2k+1},\Theta_{2k+2})\|\,\|\Theta\|}{\|1+\wp(\Theta_{2k+1},\bar{w})\|}.$$

When we take the limit on both sides as $n \longrightarrow \infty$, *we obtain* $\|\Theta\| \leq 0$, *which is contradictory. As a result,* $\|\Theta\| = 0 \Rightarrow \|\wp(\bar{w},\sigma\bar{w})\| = 0 \Rightarrow \bar{w} = \sigma\bar{w}$. *In a similar vein,* $\bar{w} = \Gamma\bar{w}$ *can be demonstrated. As a*

consequence, σ and Γ have a fixed point. We now demonstrate the common UFP shared by σ and Γ. Let $\mu^* \in \bar{P}$ be an additional common fixed point of σ and Γ, if at all possible.
Then we have

$$\wp(\bar{w}, \mu^*) = \wp(\sigma\bar{w}, \Gamma\mu^*) \precsim_{i_2} \lambda_0 \wp(\bar{w}, \mu^*) + \frac{\mu_0 \wp(\bar{w}, \sigma\bar{w}) \wp(\mu^*, \Gamma\mu^*)}{1 + \wp(\bar{w}, \mu^*)}.$$

$$\begin{aligned} &i.e., \ \|\wp(\bar{w}, \mu^*)\| \leq \lambda_0 \|\wp(\bar{w}, \mu^*)\| + \sqrt{2}\mu_0 \frac{\|\wp(\bar{w}, \sigma\bar{w})\| \, \|\wp(\mu^*, \Gamma\mu^*)\|}{\|1 + \wp(\bar{w}, \mu^*)\|}. \\ &i.e., \ \|\wp(\bar{w}, \mu^*)\| \leq \lambda_0 \|\wp(\bar{w}, \mu^*)\|. \\ &i.e., \ \|\wp(\bar{w}, \mu^*)\| = 0. \\ &\qquad i.e., \ \bar{w} = \mu^*. \end{aligned}$$

This consummates the theorem's proof. □

Example 3.5 Consider $\bar{P} = A \cup B \cup C$, where $A = \{1, 2, 3\}$, $B = \left\{\frac{1}{4}, \frac{1}{9}\right\}$, and $A = \left\{\frac{1}{8}, \frac{1}{27}\right\}$. Let us define a bicomplex valued rectangular metric $\wp : \bar{P} \times \bar{P} \to \mathbb{C}_2$ by

$$\wp(\Theta_1, \Theta_2) = \begin{cases} 0 & \text{if } \Theta_1 = \Theta_2, \\ 3i_2 & \text{if } \Theta_1 \neq \Theta_2 \text{ and } \Theta_1, \Theta_2 \in \{1, 2\}, \\ |\Theta_1 - \Theta_2| i_2 & \text{if } \Theta_1 \neq \Theta_2 \text{ and } \Theta_1 \textit{ or } \Theta_2 \textit{ or both} \notin \{1, 2\}. \end{cases}$$

It is simple to ensure that $(\bar{P}, \wp)$ is a BVRMS using the Definition 2.1 given above.

Now we consider the self mappings Γ and σ on $\bar{P}$ such as

$$\Gamma(\Theta) = \begin{cases} \frac{1}{\Theta^2} & \text{if } \Theta \in A, \\ 1 & \text{if } \Theta \in B \cup C, \end{cases}$$

and

$$\sigma(\Theta) = \begin{cases} \frac{1}{\Theta^3} & \text{if } \Theta \in A, \\ 1 & \text{if } \Theta \in B \cup C. \end{cases}$$

Let us pick any value for $\lambda_0 \geq 0.49$ and $\mu_0 \geq 0.09$ so that $\lambda_0 + \sqrt{2}\mu_0 < 1$. Consequently, the condition (3.6) of Theorem 2.11 is met. Thus, it is evident that $\Theta = 1$ is the common UFP of Γ and σ.

Corollary 3.1 *Let $(\bar{P}, \wp)$ be a complete BVRMS with degenerated $1 + \wp(p, q)$ and $\|1 + \wp(p, q)\| > 0$ for all $p, q \in \bar{P}$ and take any mapping $\sigma : \bar{P} \to \bar{P}$ that satisfies the condition:*

$$\wp(\sigma p, \sigma q) \precsim_{i_2} \lambda_0 \wp(p, q) + \frac{\mu_0 \wp(p, \sigma p) \wp(q, \sigma q)}{1 + \wp(p, q)} \tag{3.8}$$

for all $p, q \in \bar{P}$, where $\lambda_0, \mu_0 \in \mathbb{R}^+ \cup \{0\}$ with the condition $\lambda_0 + \sqrt{2}\mu_0 < 1$. Then σ has a UFP.

Proof: *The result can be conveniently demonstrated by utilizing the Theorem 2.11 and taking $\Gamma = \sigma$.* □

Example 3.6 In Example (3.3), we take the mapping $\Gamma : \bar{P} \to \bar{P}$ defined by

$$\Gamma(\Theta) = \begin{cases} 0 & \text{if } \Theta = 0, \\ 0 & \text{if } \Theta = \frac{1}{10}, \\ \frac{1}{10} & \text{if } \Theta = \frac{1}{4}, \\ 0 & \text{if } \Theta = \frac{1}{2}, \\ \frac{1}{2} & \text{if } \Theta = 2. \end{cases}$$

And let $\lambda_0 = \frac{1}{2}$ and $\mu_0 = \frac{1}{8}$, then $\lambda_0 + \sqrt{2}\mu_0 = \frac{1}{2} + \sqrt{2}\frac{1}{8} = 0.67675 < 1$. Also, the Equation (3.8) of the Corollary 2.13 is satisfied. So clearly, $\Theta = 0$ is the UFP of Γ.

Corollary 3.2 *Let $(\bar{P}, \wp)$ be a complete BVRMS with degenerated $1+\wp(p,q)$ and $\|1+\wp(p,q)\|>0$ for all $p,q \in \bar{P}$ and take any mapping $\sigma : \bar{P} \to \bar{P}$ that satisfies the condition*

$$\wp\left(\sigma^n p, \sigma^n q\right) \precsim_{i_2} \lambda_0 \wp(p,q) + \frac{\mu_0 \wp\left(p, \sigma^n p\right) \wp\left(q, \sigma^n q\right)}{1+\wp(p,q)} \tag{3.9}$$

for all $p,q \in \bar{P}$, where $\lambda_0, \mu_0 \in \mathbb{R}^+ \cup \{0\}$ such that $\lambda_0 + \sqrt{2}\mu_0 < 1$. Then σ has a UFP.

Proof: *According to Corollary 2.13, there is a UFP of σ, say $\Theta \in \bar{P}$. Then, we get*

$$\begin{aligned} \sigma\Theta &= \Theta. \\ i.e., \ \sigma^2\Theta &= \sigma\Theta = \Theta. \\ &\vdots \\ i.e., \ \sigma^n\Theta &= \Theta. \end{aligned}$$

Therefore, by using Equation (3.9), we have

$$\begin{aligned} &\wp(\sigma\Theta, \Theta) = \wp\left(\sigma\sigma^n\Theta, \sigma^n\Theta\right) = \wp\left(\sigma^n\sigma\Theta, \sigma^n\Theta\right) \precsim_{i_2} \lambda_0\wp(\sigma\Theta,\Theta) + \frac{\mu_0 \wp\left(\sigma\Theta, \sigma^n\sigma\Theta\right)\wp\left(\Theta, \sigma^n\Theta\right)}{1+\wp(\sigma\Theta,\Theta)} \\ &\implies \wp(\sigma\Theta,\Theta) \precsim_{i_2} \lambda_0\wp(\sigma\Theta,\Theta) + \frac{\mu_0 \wp\left(\sigma\Theta, \sigma^n\sigma\Theta\right)\wp(\Theta,\Theta)}{1+\wp(\sigma\Theta,\Theta)} \\ &\implies \wp(\sigma\Theta,\Theta) \precsim_{i_2} \lambda_0\wp(\sigma\Theta,\Theta) \\ &i.e., \ \|\wp(\sigma\Theta,\Theta)\| \leq \lambda_0\|\wp(\sigma\Theta,\Theta)\| \\ &\implies \|\wp(\sigma\Theta,\Theta)\| = 0 \\ &\implies \sigma\Theta = \Theta. \end{aligned}$$

Hence the corollary. □

Theorem 3.4 *Let $(\bar{P}, \wp)$ be a complete BVRMS and assume the mappings $\sigma, \Gamma : \bar{P} \to \bar{P}$ satisfy the condition*

$$\wp(\sigma p, \Gamma q) \precsim_{i_2} \frac{\beta[\wp(p,\sigma p)\wp(p,\Gamma q) + \wp(q,\Gamma q)\wp(q,\sigma p)]}{\wp(p,\Gamma q)+\wp(q,\sigma p)} \tag{3.10}$$

for all $p,q \in \bar{P}$ and if $\|\wp(p,\Gamma q)+\wp(q,\sigma p)\| \neq 0$ and $\wp(p,\Gamma q)+\wp(q,\sigma p)$ is degenerated, where β be any number such that $0 \leq \beta < 1$. Then there exists a common unique point that is fixed under the mappings σ and Γ.

Proof: *Assume that any random point in $\bar{P}$ is Θ_0. We take a sequence in $\bar{P}$, say $\{\Theta_n\}$, such that*

$$\Theta_{2n+1} = \sigma\Theta_{2n}, \ \text{and} \ \Theta_{2n+2} = \Gamma\Theta_{2n+1} \ \text{for all} \ n = 0,1,2,\ldots$$

Then, by using Equation (3.10), we have,

$$\begin{aligned} &\wp\left(\Theta_{2n+1}, \Theta_{2n+2}\right) \\ &= \wp\left(\sigma\Theta_{2n}, \Gamma\Theta_{2n+1}\right) \\ &\precsim_{i_2} \frac{\beta\left[\wp\left(\Theta_{2n}, \sigma\Theta_{2n}\right)\wp\left(\Theta_{2n}, \Gamma\Theta_{2n+1}\right) + \wp\left(\Theta_{2n+1}, \Gamma\Theta_{2n+1}\right)\wp\left(\Theta_{2n+1}, \sigma\Theta_{2n}\right)\right]}{\wp\left(\Theta_{2n}, \Gamma\Theta_{2n+1}\right) + \wp\left(\Theta_{2n+1}, S\Theta_{2n}\right)} \\ &\precsim_{i_2} \frac{\beta\left[\wp\left(\Theta_{2n}, \Theta_{2n+1}\right)\wp\left(\Theta_{2n}, \Theta_{2n+2}\right) + \wp\left(\Theta_{2n+1}, \Theta_{2n+2}\right)\wp\left(\Theta_{2n+1}, \Theta_{2n+1}\right)\right]}{\wp\left(\Theta_{2n}, \Theta_{2n+2}\right) + \wp\left(\Theta_{2n+1}, \Theta_{2n+1}\right)} \\ &\precsim_{i_2} \frac{\beta\wp\left(\Theta_{2n}, \Theta_{2n+1}\right)\wp\left(\Theta_{2n}, \Theta_{2n+2}\right)}{\wp\left(\Theta_{2n}, \Theta_{2n+2}\right)}, \quad \text{since} \ \wp\left(\Theta_{2n+1}, \Theta_{2n+1}\right) = 0 \\ &\precsim_{i_2} \beta\wp\left(\Theta_{2n}, \Theta_{2n+1}\right) \\ \implies &\wp\left(\Theta_{2n+1}, \Theta_{2n+2}\right) \precsim_{i_2} \beta\wp\left(\Theta_{2n}, \Theta_{2n+1}\right). \end{aligned} \tag{3.11}$$

So from Equation (3.11), it follows that:

$$\wp(\Theta_{n+1},\Theta_{n+2}) \precsim_{i_2} \beta\wp(\Theta_n,\Theta_{n+1}).$$

Therefore for any $n \in \mathbb{N}\cup\{0\}$ we get

$$\wp(\Theta_{n+1},\Theta_{n+2}) \precsim_{i_2} \beta\wp(\Theta_n,\Theta_{n+1}) \precsim_{i_2} \beta^2\wp(\Theta_{n-1},\Theta_n) \precsim_{i_2} \cdots \precsim_{i_2} \beta^{n+1}\wp(\Theta_0,\Theta_1).$$

Thus for any two natural numbers m, n with $n < m$ we have

$$\begin{aligned}
\wp(\Theta_n,\Theta_m) &\precsim_{i_2} \wp(\Theta_n,\Theta_{n+1}) + \wp(\Theta_{n+1},\Theta_{n+2}) + \cdots + \wp(\Theta_{m-1},\Theta_m)\\
&\precsim_{i_2} \beta^n\wp(\Theta_0,\Theta_1) + \beta^{n+1}\wp(\Theta_0,\Theta_1) + \cdots + \beta^{m-1}\wp(\Theta_0,\Theta_1)\\
&\precsim_{i_2} \left[\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}\right]\wp(\Theta_0,\Theta_1)\\
&\precsim_{i_2} \beta^n\left[1 + \beta + \beta^2 + \cdots + \beta^{m-n-1}\right]\wp(\Theta_0,\Theta_1)\\
&\precsim_{i_2} \frac{\beta^n}{1-\beta}\wp(\Theta_0,\Theta_1).
\end{aligned}$$

Since $0 \le \beta < 1$, then $1 + \beta + \beta^2 + \cdots + \beta^{m-n-1} \le \frac{1}{1-\beta}$. Hence

$$\|\wp(\Theta_n,\Theta_m)\| \le \frac{\beta^n}{1-\beta}\|\wp(\Theta_0,\Theta_1)\|.$$

Again $\frac{\beta^n}{1-\beta} \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $\{\Theta_n\}$ is a Cauchy sequence in $\bar{P}$ by Lemma 2.6. Also, $\bar{P}$ is a complete BVRMS.

Therefore, there exists $\bar{w} \in \bar{P}$ such that $\lim_{n\to\infty}\Theta_n = \bar{w}$. Now we will show that $\bar{w} = \sigma\bar{w}$ by contradiction. If we consider $\bar{w} \neq \sigma\bar{w}$ then $\wp(\bar{w},\sigma\bar{w}) = \Theta$ exists for each $0 \prec_{i_2} \Theta \in \mathbb{C}_2$. Therefore we have

$$\begin{aligned}
&\Theta = \wp(\bar{w},\sigma\bar{w})\\
&\precsim_{i_2} \wp(\bar{w},\Theta_{n+1}) + \wp(\Theta_{n+1},\Theta_{n+2}) + \wp(\Theta_{n+2},\sigma\bar{w})\\
&\precsim_{i_2} \wp(\bar{w},\Theta_{n+1}) + \wp(\Theta_{n+1},\Theta_{n+2}) + \wp(\sigma\bar{w},\Gamma\Theta_{n+1})\\
&\precsim_{i_2} \wp(\bar{w},\Theta_{n+1}) + \wp(\Theta_{n+1},\Theta_{n+2}) + \frac{\beta\left[\wp(\bar{w},\sigma\bar{w})\wp(\bar{w},\Gamma\Theta_{n+1}) + \wp(\Theta_{n+1},\Gamma\Theta_{n+1})\wp(\Theta_{n+1},\sigma\bar{w})\right]}{\wp(\bar{w},\Gamma\Theta_{n+1}) + \wp(\Theta_{n+1},\sigma\bar{w})}\\
&\precsim_{i_2} \wp(\bar{w},\Theta_{n+1}) + \wp(\Theta_{n+1},\Theta_{n+2}) + \frac{\beta\left[\Theta\wp(\bar{w},\Theta_{n+2}) + \wp(\Theta_{n+1},\Theta_{n+2})\wp(\Theta_{n+1},\sigma\bar{w})\right]}{\wp(\bar{w},\Theta_{n+2}) + \wp(\Theta_{n+1},\sigma\bar{w})},
\end{aligned}$$

which yields that

$$\begin{aligned}
\|\Theta\| \le\ & \|\wp(\bar{w},\Theta_{n+1})\| + \|\wp(\Theta_{n+1},\Theta_{n+2})\|\\
&+\sqrt{2}\beta\frac{\left[\|\Theta\|\,\|\wp(\bar{w},\Theta_{n+2})\| + \|\wp(\Theta_{n+1},\Theta_{n+2})\|\,\|\wp(\Theta_{n+1},\sigma\bar{w})\|\right]}{\|\wp(\bar{w},\Theta_{n+2}) + \wp(\Theta_{n+1},\sigma\bar{w})\|}.
\end{aligned} \tag{3.12}$$

Taking limit as $n \to \infty$ then $\Theta_n \to \bar{w}$ and from Equation (3.12), we get, $\|\Theta\| \le 0$ which is contradictory as $\|\Theta\| \ge 0$. Therefore, we conclude that $\|\Theta\| = 0 \Rightarrow \|\wp(\bar{w},\sigma\bar{w})\| = 0 \Rightarrow \bar{w} = \sigma\bar{w}$. In an analogous manner, $\bar{w} = \Gamma\bar{w}$ can be demonstrated. Hence σ and Γ have a common fixed point.
We will now demonstrate the common UFP that exists between σ and Γ. Assume, if it is feasible, that $\mu^* \in \bar{P}$ is yet another fixed point that σ and Γ have in common.
Then we have

$$\begin{aligned}
&\wp(\bar{w},\mu^*) = \wp(\sigma\bar{w},\Gamma\mu^*) \precsim_{i_2} \frac{\beta\left[\wp(\bar{w},\sigma\bar{w})\wp(\bar{w},\Gamma\mu^*) + \wp(\mu^*,\Gamma\mu^*)\wp(\mu^*,\sigma\bar{w})\right]}{\wp(\bar{w},\Gamma\mu^*) + \wp(\mu^*,\sigma\bar{w})}\\
&i.e.,\ \|\wp(\bar{w},\mu^*)\| \le \sqrt{2}\beta\frac{\left[\|\wp(\bar{w},\sigma\bar{w})\|\,\|\wp(\bar{w},\Gamma\mu^*)\| + \|\wp(\mu^*,\Gamma\mu^*)\|\,\|\wp(\mu^*,\sigma\bar{w})\|\right]}{\|\wp(\bar{w},\Gamma\mu^*) + \wp(\mu^*,\sigma\bar{w})\|}.\\
&i.e.,\ \|\wp(\bar{w},\mu^*)\| \le 0,
\end{aligned}$$

which is contradictory. Therefore we conclude that

$$\|\wp(\bar{w}, \mu^*)\| = 0 \implies \bar{w} \quad = \mu^*.$$

This concludes the theorem's proof. □

Example 3.7 Let $\dot{P} = \mathbb{R}$ and $\wp(p,q) = |p-q| + |p-q|\, i_2$. Then $(\dot{P}, \wp)$ is a complete BVRMS. Define $\sigma(p) = \frac{p}{2}$ and $\Gamma(p) = \frac{p}{3}$. For all $p, q \in \dot{P}$,

$$\wp(\sigma p, \Gamma q) \preceq \tfrac{1}{2}\, \wp(p,q),$$

so condition (2.10) of Theorem 2.16 holds for $\beta = \frac{1}{2}$. The unique common fixed point of σ and Γ is $u = 0$.

4. Conclusion

This paper presents a novel concept of a complete $BVRMS$, where we have updated the general background of $BVMS$ and demonstrated some renowned fixed point results. The findings of our study demonstrate the singularity of a fixed point under various contraction conditions. We anticipate that these results will make a valuable contribution to future research in this particular field. If we apply the concepts described in this paper to future studies on alternative metric spaces, such as bicomplex valued control metric space and bicomplex valued cone metric space, we may find intriguing results.

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