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Preservation Technology Used Hybrid Price, Stock Dependent Demand Inventory Model with Partial Backlogging and Advance Payment Related Discount Facilities

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ABSTRACT: The objective of the present research is to figure out the best inventory management approach for decaying products with hybrid demand that is depending on credit period and selling price while facing some backlog. Two separate backlog rates are taken in this paper and shortages are partially permitted. The pre-payment policy with discount facilities and set preservation technology are added in this model. This type of model is formed by differential equations, which we may solve to get the model's average profit. Results of the profit functions non-linear the achievement challenge have been calculated with concavity and optimal outcomes need to verify optimum performance. With the help of MATHEMATICA software we determine the best solution of TC (total cost) and also graph of the function. Finally, a sensitivity assessment is performed out to discover how different factor variations affect the best way to take strategy.

Key Words: Hybrid and stock dependent demand, pre-payment, discount facility, partially backlogged shortages.

Contents

1. Introduction

In 1913, Haris built the idea of inventory control, based on three opinions: (1) there is no drop in the product; (2) the need for production is steady; and (3) lower production is not permitted. In actuality, still, everything loses its usefulness and freshness after a specific period of time. The impact continues over time after this. The basis of current corporate policy is competitive marketing tactics. Putting yourselves in a battle for every seller or brand is really difficult. A retailer relies on various kinds of discounts, like price drops, quantity discounts, corporate discounts, seasonal offers, reductions for

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initial deposits, etc., in addition to luring consumers to draw in new purchasers. Among these, volume and corporate discounts are accepted practices in all businesses. These days, the prepayment option has made the discount market an established phenomenon in transportation. Prepayment options are usually given to motivate buyers to buy products for their clients. As a result, the inventory model under prepayment facilities has captured the interest of academics and scholars for the past few decades.

2. Review of Literature

2.1. Hybrid and Stock Dependent Demand

It is found in the retail industry that a variety of factors affect a product's demand. The selling price of the product represents one of these factors. An inventory system that includes demand based on stock levels and selling price has been shown by Hou and Lin [3]. The concept of shortages originated by Yang *et al.* [5], who created an inventory model for price-dependent demand. An inventory model with price-dependent demand during a shortage was solved by Dye and Hsieh [7]. Ramp type demand was taken into consideration in a deteriorating inventory model by Agarwal *et al.* [13]. The trade-related credit inventory approach with price-dependent demand and deterioration was solved by Bhunia and Shaikh [18]. In an inventory system involving the cost choice, Khan *et al.* [27] suggested employing degrading objects with time limits. Price reduction inventory model with stock demand dependence was suggested by Shaikh *et al.* [28]. An inventory model that includes decaying items in a two-warehouse setup under cost and stock demand dependence was examined by Panda *et al.* [29]. A manufacturing inventory system like carbon emissions, price-sensitive demand, and two choices for fixing faults was established by Gautam *et al.* [34]. A model that integrated fluctuating demand, costs related to the environment, and optimum management of inventory was studied by Chaudhary *et al.* [37]. Two production chain models were established by Shekhawat *et al.* [33] for items that are subject to deterioration. These two models are the highest possible level of inventory model (ii) and the most efficient total cost model (i). A system for digital devices with stock-dependent demand and parabolic-time associated holding cost has been examined by Rastogi *et al.* [38] in the context of inflation. Electronic gadget quality reduces with time, therefore the rate of deterioration appears to depend on time, and shortages happening when supply runs out appear to be partially backlogged.

A vital method of reducing its deteriorating impact is preservation technology. In spite of this, it is likely that a lot of organisations and businesses would use preservation innovation in the inventory control systems. The idea of preservation technology was first designed by Hsu *et al.* [6] in their study. They used the storage technique to create an inventory model as there was continually demand. The affect of preservation investing in an industrial inventory model was studied by Hsieh and Dye [10]. The non-instantaneous deteriorating inventory model under preservation procedures was initially proposed by Dye [11]. A model of degrading inventories under stock-dependent demand was successfully solved by Zhang *et al.* [15] using preserved technology. Inventory models related to preservation have been reviewed by Shaikh *et al.* [25] for decaying items. Preservation techniques for non-instantaneously decaying materials was looked at by Li *et al.* [26].

2.2. Partially Backlogged Shortages

Another model with partial backlog and the rate of inflation, where demand varies with product accessibility, has been investigated by Yang *et al.* [9]. A research for non-instantaneous perishable items having a demand that rely on price and time together was presented by Maihami and Kamalabadi [8]. After that, an inventory model containing ramp-type consumer demand was studied by Pal *et al.* [16]. A few years later, Shaikh *et al.* [12] developed a model of inventory based on demand related to price and level of stock under 100% backordered scenarios and inflation. After that, by taking the perishable product rate as a step function along the cycle duration, Mashud *et al.* [22] updated the work of Shaikh *et al.* [20]. The ideal inventory strategy under a linear price decline in demand function and an all-units discount was described by Khan *et al.* [23]. Another model for decaying items with inventory level-related consumer demand with partial backlogging in a cost discount environment has been examined recently by Shaikh *et al.* [25]. Sarkar *et al.* [35] designed an innovative manufacturing process that makes use of pure biofuel to cut the use of energy and carbon emissions in a price-sensitive industry. A production-inventory model was put forward by Moon *et al.* [36] with the intention to lower carbon emissions.

2.3. Pre-payment and Discount Facility

Prepayment options are usually given by sellers to motivate buyers to purchase their goods. Given this, academics and researchers have been especially interested in studying inventory models featuring discounts with pre-payment facilities over the past few decades. Plenty of research on the discount facility may be discovered within the majority of current literature. Here are a number of those that are briefly clarified and discussed. An inventory problem involving two depository multi-item inventories under a discount policy was resolved by Maiti *et al.* [2]. An inventory model that includes a quantity discount policy was examined by Firoozi *et al.* [12]. An inventory issue with two objectives—under-inflation and a discount environment was given by Mousavi *et al.* [19]. Recently, an inventory system that includes a price discount facility has been developed on by Panda *et al.* [29], Khan *et al.* [30], De [31], and others. Zhang [1] offered the notion of advance payment for the initial time in the field of company management. Maiti *et al.* [4] used the generalised reduced gradient (GRG) approach and genetic algorithm (GA) optimizers to analyse the beneficial impact of the prepayment on the inventory system. Under the advance payment technique, Thangam [9] investigated the best lot-sizing policy for the degrading commodities. An EOQ model having a pre-payment and delay payment method was created concurrently by Zhang *et al.* [14]. In the same way, Zia and Taleizadeh [17] developed an additional lot-sizing model with an advanced and delayed payment plan. Pricing and lot-sizing guidelines for degrading commodities under a prepayment plan were discovered by Li *et al.* [21]. The advance payment policy has become the focus of current research by Khan *et al.* [23], Shaikh *et al.* [24], Shi *et al.* [32], and others.

Recent advancements in inventory management have increasingly focused on integrating complex demand patterns, payment schemes, and shortage handling strategies to enhance system realism and efficiency. Rana and Kumar [39] developed a two-warehouse inventory model incorporating hybrid and stock-dependent demand with partially backlogged shortages, highlighting how price and credit-period incentives influence demand. Similarly, Ahmed *et al.* [40] examined a stock-dependent demand system under an advanced payment framework, emphasizing cost variability due to payment timing and partial backlogging. Sharma *et al.* [41] explored a model featuring a nonlinear (quadratic) demand pattern, deterioration, and a mixed cash-advance payment scheme, showing significant cost benefits under hybrid financial terms. Das *et al.* [43] extended this direction by introducing green investment and advance payment policies in a two-warehouse system, noting how sustainability goals and early payment incentives can reshape replenishment strategies. Mondal *et al.* [42] analyzed the impact of trade credit on deteriorating items in a partial backlog environment, demonstrating that credit policies not only affect capital flow but also modify demand responsiveness and backlogging rates. Collectively, these studies underline the importance of combining hybrid or stock-sensitive demand models with pre-payment and discount facilities under partial backlogging, while also revealing a research gap in fully integrated models encompassing all these features.

2.4. Structure of Study

This article presents the notion of a pre-payment strategy that includes set preservation investment and discount options. The objective of this study is to find the best inventory control method for deteriorating products with hybrid demand that is reliant on credit period and selling price while facing some backlog. With the two separate backlog rates, shortages are partially allowed. In the end, a sensitivity analysis is conducted to examine the impact of several variables on the optimal plan of strategy.

Table 1: Related review literature to key words

No.	Authors (Year)	Hybrid	Stock-Dependent Demand	Pre-payment	Discount Facility	Partially Backlogged Shortages
1	Zhang (1996)	✗	✗	✓	✗	✗
2	Maiti *et al.* (2006)	✓	✗	✗	✓	✗
3	Hou and Lin (2006)	✗	✓	✗	✗	✗
4	Maiti *et al.* (2009)	✗	✓	✓	✗	✗
5	Yang *et al.* (2010)	✗	✓	✗	✗	✓
6	Hsu *et al.* (2010)	✗	✗	✗	✗	✗

No.	Authors (Year)	Hybrid	Stock-Dependent Demand	Pre-payment	Discount Facility	Partially Backlogged Shortages
7	Dye and Hsieh (2011)	✗	✓	✗	✗	✗
8	Maihami and Kamalabadi (2012)	✗	✓	✗	✗	✓
9	Thangam (2012)	✗	✗	✓	✓	✗
10	Dye and Hsieh (2012)	✗	✗	✗	✗	✗
11	Dye (2013)	✗	✗	✗	✗	✗
12	Firoozi *et al.* (2013)	✗	✗	✗	✓	✗
13	Agrawal *et al.* (2013)	✗	✓	✗	✗	✓
14	Zhang *et al.* (2014)	✗	✗	✓	✗	✗
15	Zhang *et al.* (2014)	✗	✗	✗	✗	✗
16	Pal *et al.* (2015)	✓	✗	✗	✗	✓
17	Zia and Taleizadeh (2015)	✓	✗	✓	✗	✓
18	Bhunia and Shaikh (2015)	✗	✗	✗	✗	✓
19	Mousavi *et al.* (2016)	✓	✗	✗	✓	✗
20	Shaikh *et al.* (2017)	✗	✓	✗	✗	✓
21	Li *et al.* (2017)	✗	✗	✓	✓	✗
22	Mashud *et al.* (2018)	✗	✓	✗	✗	✓
23	Khan *et al.* (2019)	✗	✗	✓	✗	✓
24	Shaikh *et al.* (2019)	✓	✗	✓	✗	✗
25	Shaikh *et al.* (2019)	✗	✓	✗	✗	✓
26	Li *et al.* (2019)	✗	✗	✗	✗	✗
27	Khan *et al.* (2019)	✗	✗	✗	✗	✗
28	Shaikh *et al.* (2019)	✗	✓	✗	✓	✓
29	Panda *et al.* (2019)	✗	✓	✗	✗	✓
30	Khan *et al.* (2020)	✗	✓	✓	✓	✗
31	De (2020)	✗	✓	✗	✓	✗
32	Shi *et al.* (2020)	✗	✗	✓	✗	✗
33	Rastogi *et al.* (2023)	✗	✗	✗	✗	✓
34	Rana and Kumar(2024)	✓	✓	✗	✗	✓
35	Ahmed (2024)	✗	✓	✓	✗	✓
36	Sharma et al.(2024)	✓	✗	✓	✓	
37	This paper	✓	✓	✓	✓	✓

3. Problem Description, Assumptions, and Notations

3.1. Assumptions

- Demand rate is $D[p,s,n] = \alpha + \beta p + \gamma(a - bs + cn)$ where $\alpha + \beta + \gamma = 1$, and $0 < \alpha, \beta, \gamma \leq 1$. s is selling price and n is credit period, a, b, c are constant parameter.
- Once the lot is received, deterioration begins instantly and continues at a constant pace.
- Shortages are considered partially with two backlogging rate is $B(T-t) = \frac{1}{1+k(T-t)}$ and $B(T-t) = e^{-k(T-t)}$.
- ε-preservation cost.

- There is no limit to the inventory planning horizon.

- To slow down the rate at which the items deteriorate, a deterioration rate with preservation technology is taken as $\theta = \theta_1 e^{-a_1 \varepsilon}$, a_1 is preservation control parameter and θ_1 is constant.

3.2. Notations

- $D[p, s, n]$: the demand function
- $I(t)$: inventory level at time t
- $\theta(t)$: deterioration rate per unit time
- d : discount percentage of advance payment
- p : production price per time unit
- $\boldsymbol{y}$: starting time of deterioration
- $B(T - t)$: backorder rate
- k : backlogging parameter
- Q : initial ordering quantity
- R : maximum shortage level
- T : cycle length
- ε : preservation price per time unit
- O : ordering price per time unit
- C_p : purchase price per time unit
- C_h : holding cost per unit item per unit time
- C_s : shortage price per unit item per unit time
- C_{ls} : lost sale price per unit item per unit time
- t_1 : time of zero ending inventory
- $Z(p, \varepsilon, t_1, T)$: average profit function

4. Mathematical Formulation of Inventory Model

In this model, inventory cycle is divided into three parts, as shown in Figure 1. In first phase $[0, y]$ and for second phase $[y, t_1]$. In third phase $[t_1, T]$ maximum shortage occur and then we take backlogging rate. In 1st phase deterioration rate is not taken into account, after $t = y$ time in 2nd phase deterioration rate taken into action.

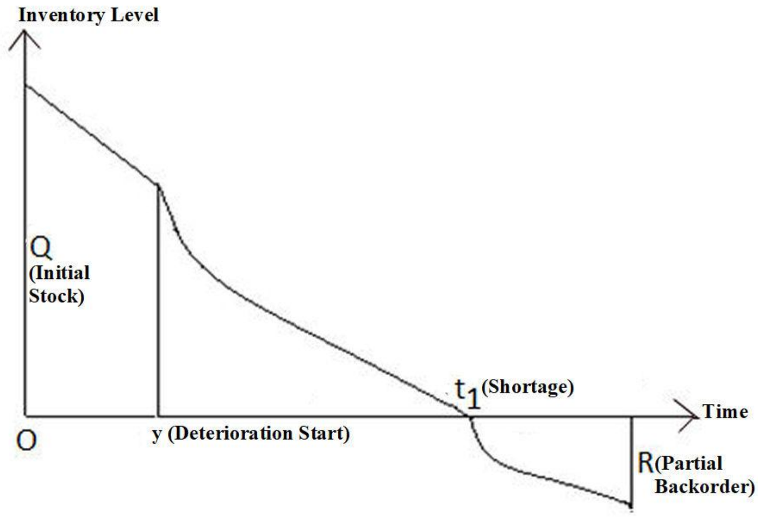

Figure 1: Interpreting inventory stock geometrically across time

$$\frac{dI(t)}{dt} = -D[p,s,n], \qquad 0 < t \le y, \tag{4.1}$$

$$\frac{dI(t)}{dt} = -D[p,s,n] - \theta(t)I(t), \quad y < t \le t_1, \tag{4.2}$$

$$\frac{dI(t)}{dt} = -D[p,s,n]B(T-t), \quad t_1 < t \le T. \tag{4.3}$$

With the boundary conditions $I(0) = Q$, $I(t_1) = 0$ and $I(T) = -R$. Also $I(t)$ is continuous at $t = y$ and $t = t_1$.

The solution of differential Eqs. (4.1)-(4.3) are given by

$$I(t) = -D[p,s,n]t + Q, \qquad 0 < t \le y, \tag{4.4}$$

$$I(t) = \frac{D[p,s,n]}{\theta}[1 + e^{\theta(t-t_1)}], \qquad y < t \le t_1, \tag{4.5}$$

$$I(t) = -D[p,s,n]\int_{t_1}^{t} B(T-t)dt, \quad t_1 < t \le T. \tag{4.6}$$

Now applying the continuity conditions of $I(t)$ at $t = y$ and t_1, we have

$$Q = \frac{D[p,s,n]}{\theta}(\theta + 1 + e^{\theta(y-t_1)}), \tag{4.7}$$

$$R = D[p,s,n]\int_{t_1}^{T} B(T-t)dt, \tag{4.8}$$

Total sales revenue (SR) $= pD[p,s,n]t_1 + pR$,

Total purchase cost (PC) $= (1-d)C_p(Q+R)$,

$$= (1-d)C_p\left(\frac{D[p,s,n]}{\theta}(\theta + 1 + e^{\theta(y-t_1)}) + D[p,s,n]\int_{t_1}^{T} B(T-t)dt\right),$$

The total inventory holding cost (HC)

$$= C_h\left(\int_0^y I(t)dt + \int_y^{t_1} I(t)dt\right)$$

$$= C_h\left[Qy - D[p,s,n]\times\frac{y^2}{2} + \frac{D[p,s,n]}{\theta^2}\{\theta(t_1-y)+1-e^{\theta(y-t_1)}\}\right].$$

The total shortage cost and lost sale cost are $C_sD[p,s,n]\int_{t_1}^{T}\int_{t_1}^{t} B(T-t)dudt$ and $C_{ls}D[p,s,n](y-t_1)-C_{ls}R$, respectively.

Preservation technology cost $=\varepsilon T$.

Therefore, the average cost per cycle is given by

$$\begin{aligned}Z(p,\varepsilon,t_1,T) &= \frac{1}{T}\{\text{sales revenue} - \text{purchase cost} - \text{holding cost} - \text{shortage cost} - \text{lost sale cost}\\ &\quad - \text{ordering cost} - \text{preservation technology cost}\}\\ &= \frac{1}{T}\Bigg\{pD[p,s,n]t_1 + pR - (1-d)C_p(Q+R)C_h\Bigg[Qy - D[p,s,n]\frac{y^2}{2}\\ &\quad + \frac{D[p,s,n]}{\theta^2}\{\theta(t_1-y)+1-e^{\theta(y-t_1)}\}\Bigg]\\ &\quad - C_sD[p,s,n]\int_{t_1}^{T}\int_{t_1}^{t} B(T-t)dudt - C_{ls}D[p,s,n](y-t_1) - C_{ls}R - O - \varepsilon T\Bigg\}\end{aligned}$$

Case 1: When $B(T-t)=\frac{1}{1+k(T-t)}$,

$$\begin{aligned}Z(p,\varepsilon,t_1,T) = \frac{1}{T}\Bigg\{&pD[p,s,n]t_1 + \frac{pD[p,s,n]}{k}\log\{1+k(t_1-T)\}\\ &- (1-d)C_p\left\{\frac{D[p,s,n]}{\theta}(\theta+1+e^{\theta(y-t_1)}) + \frac{D[p,s,n]}{k}\log\{1+k(t_1-T)\}\right\}\\ &- C_h\left[\frac{yD[p,s,n]}{\theta}(\theta+1+e^{\theta(y-t_1)}) - D[p,s,n]\frac{y^2}{2} + \frac{D[p,s,n]}{\theta^2}\{\theta(t_1-y)+1-e^{\theta(y-t_1)}\}\right]\\ &- C_sD[p,s,n]\left[\frac{t_1}{k}\log\{1+k(T-t_1)\} + \frac{T^2}{2} + \frac{T^3}{3} - \frac{t_1^2}{(2+2k(T-t_1))} - \frac{t_1^3}{(3+3k(T-t_1))^2}\right]\\ &- C_{ls}D[p,s,n](y-t_1) - C_{ls}\frac{D[p,s,n]}{k}\log\{1+k(t_1-T)\} - O - \varepsilon T\Bigg\}.\end{aligned}$$

5. Solution Methodology

Optimal solution is obtained by putting, $\frac{\partial Z(p,\varepsilon,t_1,T)}{\partial t_1}=0$ and $\frac{\partial Z(p,\varepsilon,t_1,T)}{\partial T}=0$, neglecting higher power terms

$$\begin{aligned}\frac{\partial Z(p,\varepsilon,t_1,T)}{\partial t_1} = \frac{1}{T}\Bigg[&pD[p,s,n] - \frac{pD[p,s,n]}{\{1+k(t_1-T)\}} - \frac{(1-d)C_pD[p,s,n]}{\{1+k(t_1-T)\}} + (1-d)C_p pe^{\theta(y-t_1)}\\ &- \frac{C_hD[p,s,n]}{\theta} + C_hyD[p,s,n]e^{\theta(y-t_1)} - \frac{C_hD[p,s,n]e^{\theta(y-t_1)}}{\theta}\\ &- \frac{C_sD[p,s,n]\log\{1+k(T-t_1)\}}{k} + \frac{2C_sD[p,s,n]t_1}{\{1+k(T-t_1)\}} + C_{ls}D[p,s,n]\\ &- \frac{C_{ls}D[p,s,n]}{\{1+k(t_1-T)\}}\Bigg] = 0\end{aligned}$$

and

$$\frac{\partial Z(p,\varepsilon,t_1,T)}{\partial T} = \frac{1}{T}\Bigg[-\frac{pD[p,s,n]}{\{1+k(t_1-T)\}} + \frac{(1-d)C_pD[p,s,n]}{\{1+k(t_1-T)\}} - \frac{C_sD[p,s,n]t_1}{\{1+k(t_1-T)\}}$$

$$- C_s D[p,s,n]T - C_s D[p,s,n]T^2 + \frac{C_{ls} D[p,s,n]}{\{1+k(t_1-T)\}} - \varepsilon\Bigg]$$
$$- \frac{1}{T^2} Z(p,\varepsilon,t_1,T) = 0.$$

Total profit of this system will maximize at t_1^* and T^* where $\frac{\partial Z(p,\varepsilon,t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z(p,\varepsilon,t_1,T)}{\partial T} = 0$. Provided that

$$\left(\frac{\partial^2 Z(p,\varepsilon,t_1,T)}{\partial t_1 \partial T}\right) - \frac{\partial^2 Z(p,\varepsilon,t_1,T)}{\partial {t_1}^2} \times \frac{\partial^2 Z(p,\varepsilon,t_1,T)}{\partial T^2} < 0.$$

Case 2: When $B(T-t) = e^{-k(T-t)}$,

$$\begin{aligned} Z(p,\varepsilon,t_1,T) = \frac{1}{T}\Bigg\{ & pD[p,s,n]t_1 + \frac{pD[p,s,n]}{2}(T-t_1)\{2-k(T-t_1)\} \\ & - (1-d)C_p\left(\frac{D[p,s,n]}{\theta}(\theta+1+e^{\theta(y-t_1)}) + \frac{D[p,s,n]}{2}(T-t_1)\{2-k(T-t_1)\}\right) \\ & - C_h\left[\left\{\frac{D[p,s,n]}{\theta}(\theta+1+e^{\theta(y-t_1)})\right\}y - D[p,s,n] \times \frac{y^2}{2} \right. \\ & \left. + \frac{D[p,s,n]}{\theta^2}\{\theta(t_1-y)+1-e^{\theta(y-t_1)}\}\right] \\ & - \frac{C_s D[p,s,n]}{2}\left(T^2 - 2t_1T - t_1^2 - kt_1T^2 + kTt_1^2 - \frac{kT^3}{3} - \frac{kt_1^3}{3}\right) \\ & - C_{ls}D[p,s,n](y-t_1) - C_{ls}\frac{pD[p,s,n]}{2}(T-t_1)\{2-k(T-t_1)\} - O - \varepsilon T\Bigg\}. \end{aligned}$$

Optimal solution of Case 2 can be obtained similar to Case 1. In numerical analysis we calculate optimal value for Case 1 and Case 2 both.

Theorem for Case 1. If $D[p,s,n] < \frac{1}{C_h}\left[C_p p\theta + C_h y D[p,s,n]\theta - \frac{3C_s D[p,s,n]e^{\theta(t_1-y)}}{\{1+k(T-t_1)\}}\right]$, the Hessian matrix for $Z(p,\varepsilon,t_1,T)$ is always negative definite, and $Z(p,\varepsilon,t_1,T)$ attains global maximum at point (t_1^*, T^*) and this point is unique.

Proof. We demonstrate that the second principal minor is positive and the first is negative in order to establish this theorem. To create a Hessian matrix, let's have a look at

$$\begin{aligned} TP(t_1,T) = \frac{1}{T}\Bigg\{ & pD[p,s,n]t_1 + \frac{pD[p,s,n]}{k}\log\{1+k(t_1-T)\} \\ & - (1-d)C_p\left\{\frac{D[p,s,n]}{\theta}(\theta+1+e^{\theta(y-t_1)}) + \frac{D[p,s,n]}{k}\log\{1+k(t_1-T)\}\right\} \\ & - C_h\left[\frac{yD[p,s,n]}{\theta}(\theta+1+e^{\theta(y-t_1)}) - D[p,s,n] \times \frac{y^2}{2} + \frac{D[p,s,n]}{\theta^2}\{\theta(t_1-y)+1-e^{\theta(y-t_1)}\}\right] \\ & - C_s D[p,s,n]\left[\frac{t_1}{k}\log\{1+k(T-t_1)\} + \frac{T^2}{2} + \frac{T^3}{3} - \frac{t_1^2}{(2+2k(T-t_1))} - \frac{t_1^3}{(3+3k(T-t_1))^2}\right] \\ & - C_{ls}D[p,s,n](y-t_1) - C_{ls}\frac{D[p,s,n]}{k}\log\{1+k(t_1-T)\} - O - \varepsilon T\Bigg\} \end{aligned}$$

and $F(t_1, T) = T$.

Differentiating $TP(t_1, T)$ partially with respect to t_1 and T. We get

$$\frac{\partial TP(t_1,T)}{\partial t_1} = \frac{1}{T}\Bigg[pD[p,s,n] - \frac{pD[p,s,n]}{\{1+k(t_1-T)\}} - \frac{(1-d)C_pD[p,s,n]}{\{1+k(t_1-T)\}} + (1-d)C_ppe^{\theta(y-t_1)} - \frac{C_hD[p,s,n]}{\theta} + C_hyD[p,s,n]e^{\theta(y-t_1)} - \frac{C_hD[p,s,n]e^{\theta(y-t_1)}}{\theta} - \frac{C_sD[p,s,n]\log\{1+k(T-t_1)\}}{k} + \frac{2C_sD[p,s,n]t_1}{\{1+k(T-t_1)\}} + C_{ls}D[p,s,n] - \frac{C_{ls}D[p,s,n]}{\{1+k(t_1-T)\}}\Bigg]$$

and

$$\frac{\partial TP(t_1,T)}{\partial T} = \frac{1}{T}\Bigg[-\frac{pD[p,s,n]}{\{1+k(t_1-T)\}} + \frac{(1-d)C_pD[p,s,n]}{\{1+k(t_1-T)\}} - \frac{C_sD[p,s,n]t_1}{\{1+k(t_1-T)\}} - C_sD[p,s,n]T - C_sD[p,s,n]T^2 + \frac{C_{ls}D[p,s,n]}{\{1+k(t_1-T)\}} - \varepsilon\Bigg] - \frac{1}{T^2}TP(t_1,T).$$

Again doing a partial differentiation with regard to both t_1 and T, excluding terms with a greater denominator, we find

$$\frac{\partial^2 TP(t_1,T)}{\partial {t_1}^2} = \frac{1}{T}\Bigg[-(1-d)C_pp\theta e^{\theta(y-t_1)} - C_hyD[p,s,n]\theta e^{\theta(y-t_1)} + C_hD[p,s,n]e^{\theta(y-t_1)} + \frac{3C_sD[p,s,n]}{\{k(T-t_1)+1\}}\Bigg],$$

$$\frac{\partial^2 TP(t_1,T)}{\partial T^2} = \frac{1}{T}[-C_sD[p,s,n] - 2C_sD[p,s,n]T],$$

$$\frac{\partial^2 TP(t_1,T)}{\partial t_1 \partial T} = \frac{1}{T}\Bigg[\frac{C_sD[p,s,n]}{\{k(T-t_1)-1\}}\Bigg],$$

$$\frac{\partial^2 TP(t_1,T)}{\partial T \partial t_1} = \frac{1}{T}\Bigg[-\frac{C_sD[p,s,n]}{\{k(T-t_1)+1\}}\Bigg].$$

Hessian matrix for $TP(t_1, T)$ is

$$\begin{bmatrix} \frac{\partial^2 TP(t_1,T)}{\partial T^2} & \frac{\partial^2 TP(t_1,T)}{\partial T\partial t_1} \\ \frac{\partial^2 TP(t_1,T)}{\partial t_1\partial T} & \frac{\partial^2 TP(t_1,T)}{\partial {t_1}^2} \end{bmatrix}.$$

Here first principal minor is $\frac{\partial^2 TP(t_1,T)}{\partial T^2} = [-C_sD[p,s,n] - 2C_sD[p,s,n]T] < 0$. Moreover, 2nd principal minor is

$$\frac{\partial^2 TP(t_1,T)}{\partial T^2} \times \frac{\partial^2 TP(t_1,T)}{\partial {t_1}^2} - \frac{\partial^2 TP(t_1,T)}{\partial t_1\partial T} \times \frac{\partial^2 TP(t_1,T)}{\partial T\partial t_1}$$

$$= \Bigg[[-C_sD[p,s,n] - 2C_sD[p,s,n]T] \times \Bigg[-C_pp\theta e^{\theta(y-t_1)} - C_hyD[p,s,n]\theta e^{\theta(y-t_1)} + C_hD[p,s,n]e^{\theta(y-t_1)} + \frac{3C_sD[p,s,n]}{\{k(T-t_1)+1\}}\Bigg] - \Bigg[\frac{C_sD[p,s,n]}{\{k(T-t_1)-1\}}\Bigg] \times \Bigg[-\frac{C_sD[p,s,n]}{\{k(T-t_1)+1\}}\Bigg]\Bigg].$$

$C_p p\theta + C_h y D[p,s,n]\theta - C_h D[p,s,n] > \frac{3C_s D[p,s,n] e^{\theta(t_1-y)}}{\{1+k(T-t_1)\}}$, for 2nd principal minor is positive so that the Hessian matrix is negative definite.

From above inequality we find

$$D[p,s,n] < \frac{1}{C_h}\left[C_p p\theta + C_h y D[p,s,n]\theta - \frac{3C_s D[p,s,n] e^{\theta(t_1-y)}}{\{1+k(T-t_1)\}}\right].$$

The function $TP(t_1,T)$ is strictly concave and differentiable. And the function $F(t_1,T) = T$ is strictly positive and affine function. Now in numerical analysis we gave optimal point where $TP(t_1,T)$ attains global maximum value at unique point. Thus the proof is complete.

6. Numerical Analysis

We solve the model using MATHEMATICA software.

Example 1. In this example, the parameter's value $D[p,s,n]$ is supposed as 0.5, and the inventory parameters for case 1 backlogging rate are taken as follows:

$O=200$, $p=30$, $\varepsilon = 5$, $C_p = 15$, $C_h = 3$, $d = 0.3$, $C_s = 14$, $C_{ls} = 16$, $y = 0.21$, $k = 1.48$, $\theta = 0.2$, $D = 40$.

Then $Z(p,\varepsilon,t_1,T) = 1304.41$, $t_1^* = 2.7$, $T^* = 3.1$.

Its graphical representation is shown in Figure 2.

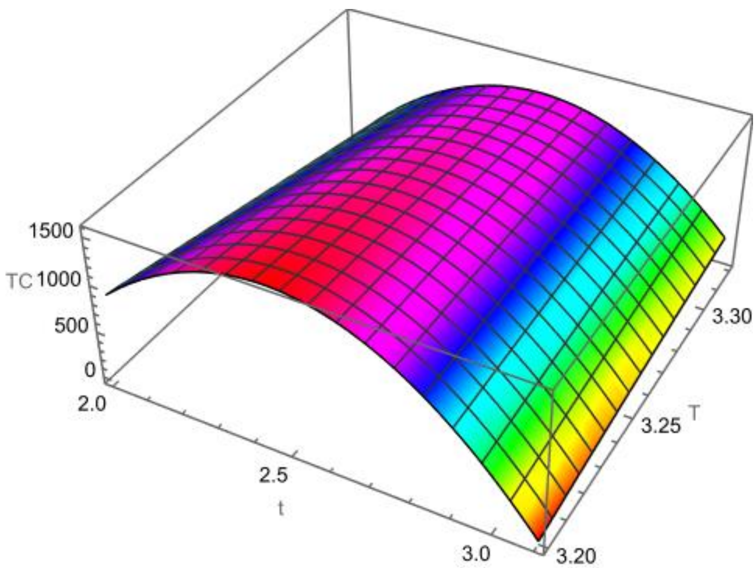

Figure 2: Convexity for TC (total cost function Z) with respect to t_1 and T

We noticed from numerical example is that hybridization of $\alpha + \beta p$ and $(a - bs + cn)$, these are parts of the demand function which shows better results than the single part. The convex combination of these distinct demands is occupied by both functions. Thus, this type of approach may be applied by a decision-maker to create an efficient business with lower risk.

7. Sensitivity Analysis

Here we study the effect of changing the parameters by which the values of t_1, T and profit function $Z(p,\varepsilon,t_1,T)$ also changes. We also look at how Q and R changed as well as what happens when the parameters were changed. We prepared a Table 1 to demonstrate the impact of modifying the parameters both theoretically and geometrically. Also, we studied the changes in preservation cost ε and deterioration cost θ.

Table 2: Look in the impact of changing the known inventory parameters

Parameters	Value of parameters	Value of $Z(p,\varepsilon,t_1,T)$	Value of t_1	Value of T	Value of Q	Value of R
O	210	1110.56	2.8	3.22	363.2	12.79
	200	1304.41	2.7	3.1	365.2	11.26
	190	1518.2	2.6	3	367.2	11.26
	180	1578	2.55	2.9	368.3	10.37

Parameters	Value of parameters	Value of $Z(p,\varepsilon,t_1,T)$	Value of t_1	Value of T	Value of Q	Value of R
p	32	1184.07	2.75	3.1	364.2	10.37
	30	1304.41	2.7	3.1	365.2	11.26
	28	1478.23	2.6	3	367.2	11.26
	26	1543.01	2.55	2.95	368.3	11.26
ε	6	1478	2.8	3	363.2	6.81
	5	1304.41	2.7	3.1	365.2	11.26
	4	1258	2.92	3.15	360.97	7.63
	3	1110	2.93	3.15	360.5	7.7
θ	0.21	1506.05	2.55	3	368.3	12.01
	0.2	1304.41	2.7	3.1	365.2	11.26
	0.19	1258.39	2.8	3.2	363.2	11.26
	0.18	1058.72	2.94	3.24	360.6	9.34
k	1.52	1491.33	2.6	3	367.2	11.26
	1.48	1304.41	2.7	3.1	365.2	11.26
	1.44	1304.26	2.73	3.2	364.6	11.26
	1.40	1259.32	2.75	3.15	364.2	11.26

These conclusions are noted from table:

(i) The total profit is affected by the production cost p, increase in p gives changes in optimal solution. Z, Q decreases and t_1, T and R increase.

(ii) Change in O also affect optimal solution. Increment in O gives changes in optimal solution. Z, Q decreases and t_1, T and R increase.

(iii) The total profit is highly affected by the change in preservation cost, change in value of preservation cost is also calculated in this table.

(iv) The total profit is also changes by deterioration cost and backlogging parameters.

Graph of sensitivity analysis are shown from Figure 3 to Figure 7.

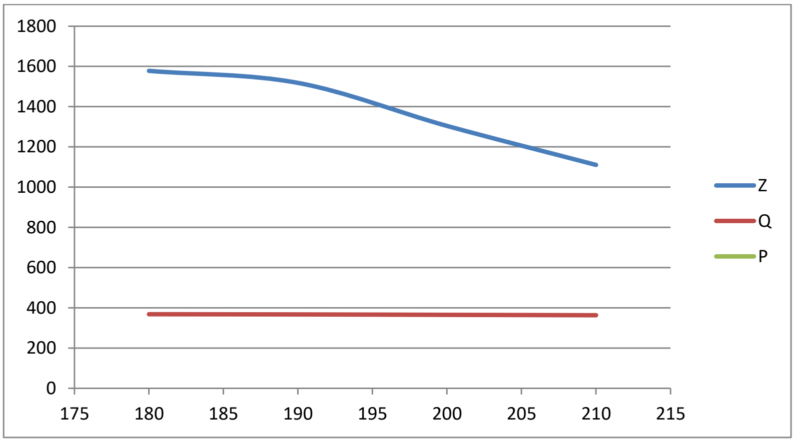

Figure 3: Change with ordering cost O

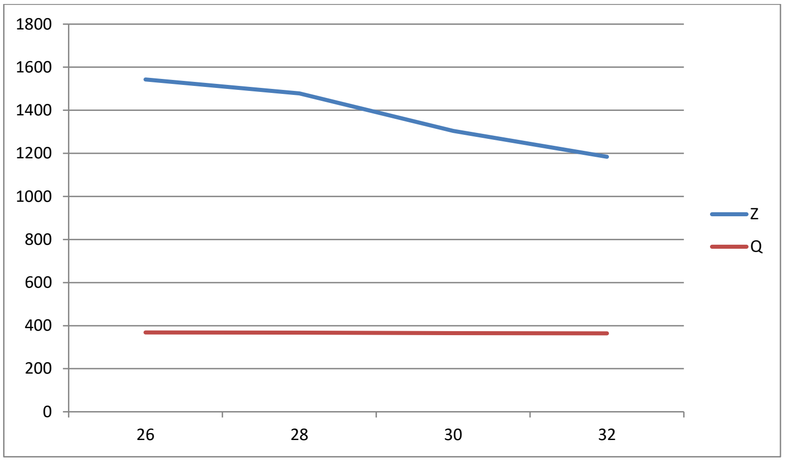

Figure 4: Change with production cost p

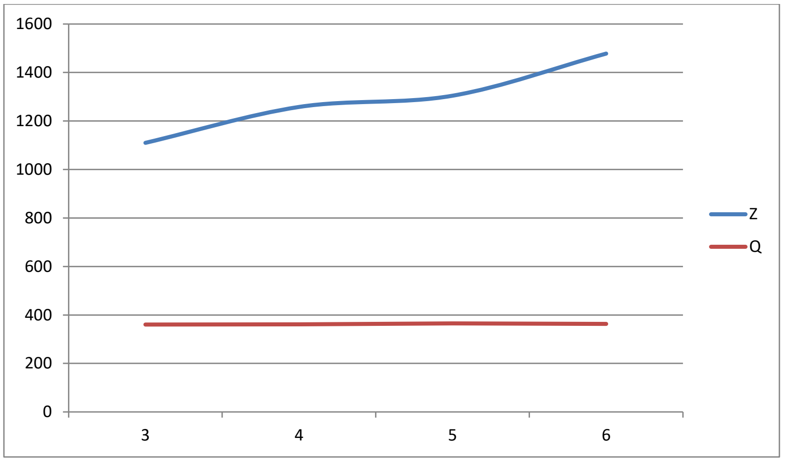

Figure 5: Change with preservation cost ε

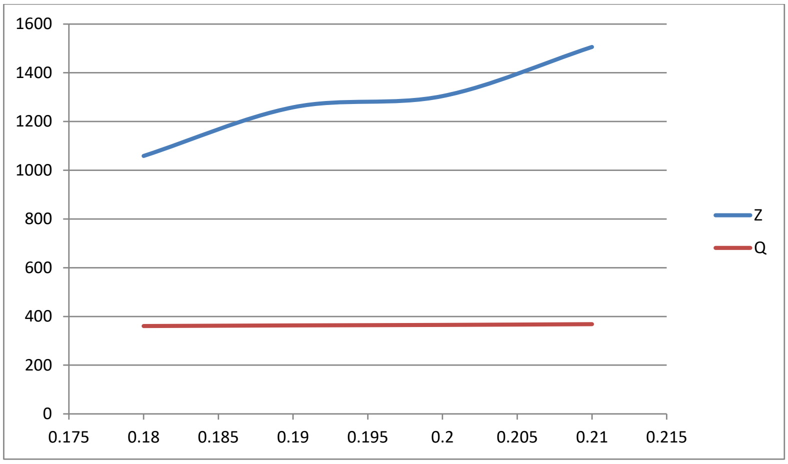

Figure 6: Change with deterioration rate θ

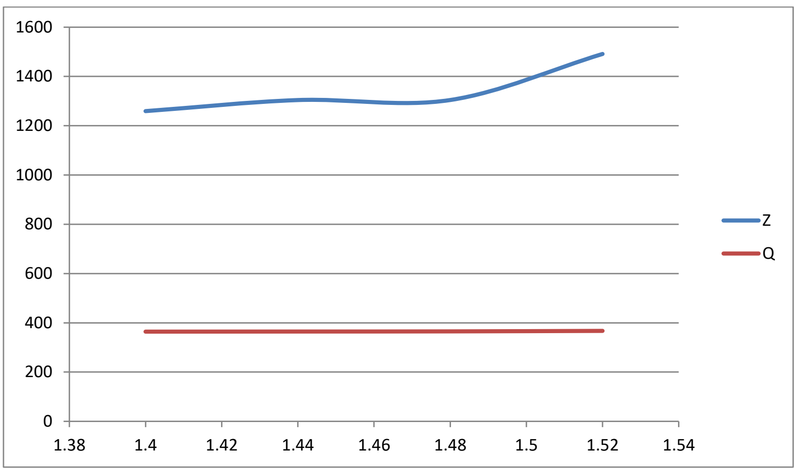

Figure 7: Change with backlogging parameter k

8. Observations

- Figure 3 shows that, in response to a change in ordering cost O, average profit Z is very sensitive, while initial stock Q and maximum shortfall R are only moderately sensitive.

- Figure 4 shows that whereas initial stock Q and maximum shortfall R are only a little flexible to changes in production cost p, average profit Z is quite sensitive.

- Figure 5 shows that, with respect to the change in preservation cost per unit time ε, average profit Z is significantly sensitive, although initial stock Q and maximum shortage R are only moderately sensitive.

- Figure 6 illustrates that whereas initially stock Q and maximum shortfall R are gently sensitive to changes in the decline rate per time unit θ, average profit Z is very sensitive.

- Figure 7 shows that as the backlogging rate parameter (k) is changed, the average profit (Z) is very sensitive, whereas the initial stock (Q) and maximum shortfall (R) are only moderately sensitive.

9. Conclusions and Future Extension

An inventory system for perishable products has been examined during this work. The development of this model depends on the premise of preservation technology, advanced payment services, selling price related hybridized consumer demand, and partially backlogging with a steady rate. The optimality needs of the optimum issue are investigated with the mathematical study of the concave shape of the desired function using the fundamental minor methods of the Hessian's matrix. The optimal standards of the optimum issue are investigated through the mathematical analysis of the concavity of an objective function using the primary minor approaches of the Hessian matrix.

Future research may extend the model by incorporating uncertain demand and deterioration, time-dependent pricing, variable backlogging rates, multi-item and multi-echelon supply chains, sustainability factors, credit risk in advance payments, and numerical or metaheuristic optimization methods, supported by empirical validation using real-world perishable inventory data.

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