Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.81760

On Cyclic Topological Spaces of Graphs

Husniyah Alzubaidi, Diana Dolićanin-Djekić, Ljubiša D. R. Kočinac* and Hakeem A. Othman

ABSTRACT: This paper introduces the innovative concept of cyclic topological spaces for simple graphs, leveraging cyclic subgraphs to define C-vertices. We delve into several properties within these topologies, providing a foundational study. A significant contribution is the establishment of a rigorous framework proving the correspondence between graph isomorphisms and homeomorphisms in cyclic topological spaces. Furthermore, we explore the connectedness of these spaces and its profound implications for graph theory, enhancing our understanding of structural relationships in both fields.

Keywords: Simple graph, topological space, discreteness, density, continuous mapping, connectedness.

Contents

1. Introduction

A significant milestone in the development of graph theory was achieved by Leonhard Euler in 1736 in order to provide solutions to various problems in discrete mathematics, as documented in Bondy and Murty's book [4] from which we use notation and terminology.

A graph $\mathsf{\Gamma}$ is a pair $(\mathsf{V}(\mathsf{\Gamma}), \mathsf{E}(\mathsf{\Gamma}))$, where $\mathsf{V}(\mathsf{\Gamma})$ is called the set of vertices of $\mathsf{\Gamma}$, and $\mathsf{E}(\mathsf{\Gamma})$ is called the set of edges of $\mathsf{\Gamma}$. A graph with finite sets $\mathsf{V}(\mathsf{\Gamma})$ and $\mathsf{E}(\mathsf{\Gamma})$ is called finite. By $\partial(x)$ we mean the degree of $x \in \mathsf{V}(\mathsf{\Gamma})$ which is the number of vertices adjacent with x. A graph $\mathsf{\Gamma}$ is called locally finite if $\partial(x)$ is finite for all $x \in \mathsf{V}(\mathsf{\Gamma})$. If two different vertices x and y in $\mathsf{V}(\mathsf{\Gamma})$ are adjacent by the edge $\xi \in \mathsf{E}(\mathsf{\Gamma})$, then we write $J_\xi = \{x, y\}$. A simple graph is a graph without loops (the edges that start and end at the same vertex) and multiple edges.

A cycle graph C_n, $n > 2$, (a simple graph with n vertices and n edges, where every vertex has degree 2), a complete graph K_n, $n > 0$, (a simple graph with n vertices, where every pair of distinct vertices is connected by a unique edge), a complete bipartite graph $K_{n,m}$,$n, m > 0$, (a simple graph with two disjoint subsets of vertices, V_1 and V_2, containing n and m vertices, respectively, in which no edge has both end points in the same subset, and every vertex in V_1 is connected to every vertex in V_2) are fundamental concepts in graph theory and will be considered in this paper.

A number of researchers have explored how to topologize the vertex or edge set in graphs (see , for example, [8,9]). A number of interesting results on different topologies on graphs has been obtained. For instance, Amiri [1] introduced the notion of graphic topology on the vertex set of a simple graph in 2013. This was accomplished by defining a subbase family $S_{\mathsf{\Gamma}} = \{A_x : x \in \mathsf{V}(\mathsf{\Gamma})\}$, where A_x consists of all vertices adjacent with x. Subsequently, Abdu and Kiliciman [7] structured topologies on the edge set of directed graphs, referred to as incompatible and compatible topologies. In [10], compact compatible topologies for graphs have been considered. The spanning graph topology was studied in [3], and in [18] the total graphic topology was investigated. Closure operators on graphs were considered in [17].

---

* Corresponding author.
2020 *Mathematics Subject Classification:* 05C99, 18F60, 54C05, 54D99.
Submitted February 04, 2026. Published April 09, 2026

In 2019, Nianga and Canoy [11] employed hop neighborhoods of a graph to construct a finite topological space, thereby providing a method for structuring a topology on a graph. Furthermore, they described some topologies induced by the complements of simple undirected graphs. Building upon this work, Gamorez et al. [6] in 2019 and Sari and Kopuzlu [15] in 2020 introduced topologies induced by the corona, edge corona, and tensor product of two graphs, as well as simple undirected graphs on the vertex set, respectively. Most recently, Anabel et al. [2] in 2021 constructed topologies on the vertex set using monophonic eccentric neighborhoods of graphs.

Othman et. al. [13,12,19], used directed graphs $\mathsf{\Gamma}$ to introduce the notion of pathless directed topological space on the set of vertices $\mathsf{V}(\mathsf{\Gamma})$. In [14], Othman, Ayache and Saif introduced the L_2-directed topological spaces on the set of vertices $\mathsf{V}(\mathsf{\Gamma})$ for directed graphs.

Topological structures on graphs have various applications in medical and biomedical sciences. For some applications of topological spaces on graphs in human heart see, for example, [16].

In this paper, Section 2 provides the concept of C-vertices for any simple graph and then we give the notion of cyclic topological space on the set of these C-vertices. Next, we show the openness, minimality and connectedness properties of this topology. We end this section by considering some relationships between claw graphs and their cyclic topological spaces. In Section 3, we introduce and investigate connection between isomorphisms of graphs and homeomorphisms of their corresponding topological spaces.

All graphs throughout this paper ate assumed to be undirected and locally finite.

2. The Cyclic Topological Spaces

In this section we define a novel topology on the set of vertices of a given graph and explore several basic properties of this topology. For topological terminology and notation we follow the book [5].

A vertex x of a graph $\mathsf{\Gamma}$ is said to be a *C-vertex* if for each $y \in A_x$ there is a cycle subgraph C_{n_y} of $\mathsf{\Gamma}$ such that $y \in \mathsf{V}(C_{n_y})$. By $\mathsf{V}_c(\mathsf{\Gamma})$ we mean the set of all C-vertices in $\mathsf{\Gamma}$.

For a vertex $x \in \mathsf{V}(\mathsf{\Gamma})$, we set

$$\mathcal{R}_c(x) = \{y \in \mathsf{V}_c(\mathsf{\Gamma}) : y \text{ is adjacent with } x\}$$
$$\mathcal{R}_c[x] = \mathcal{R}_c(x) \cup \{x\}.$$

$\mathcal{R}_c(x)$ is called the *open C-neighborhood* of x, and $\mathcal{R}_c[x]$ is called the *closed C-neighborhood* of x.

Definition 2.1 Let $\mathsf{\Gamma}$ be a graph. Define the family of subsets of $\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma})$ as follows:

$$\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma}) = \{\mathcal{R}_c[x] : x \in \mathsf{V}(\mathsf{\Gamma})\}.$$

Let $\tau_{\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma})}$ denote a topology on $\mathsf{V}(\mathsf{\Gamma})$ generated by the subbase $\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma})$. The pair $(\mathsf{V}(\mathsf{\Gamma}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma})})$ is called the *cyclic topological space* of the graph $\mathsf{\Gamma}$.

Notice that if x is an isolated vertex in a graph $\mathsf{\Gamma}$, then $\{x\}$ is an open set in the cyclic topological space $(\mathsf{V}(\mathsf{\Gamma}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma})})$. It is also clear that if ξ is an isolated edge, then J_ξ is an open set in $(\mathsf{V}(\mathsf{\Gamma}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma})})$.

Example 2.1 The graph $\mathsf{\Gamma}$ in Figure 1.A is given by

$$\mathsf{V}(\mathsf{\Gamma}) = \{1, 2, \ldots, 9\} \text{ and } \mathsf{E}(\mathsf{\Gamma}) = \{\xi_1, \xi_2, \ldots, \xi_8\}.$$

The subbase $\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma})$ is

$$\mathcal{R}\mathsf{V}_c(\mathsf{\Gamma}) = \{\mathcal{R}_c[i] = \{i\} : i = 1, 2, \ldots, 9\}.$$

The cyclic topological space of the graph $\mathsf{\Gamma}$ is discrete.

In Figure 1.B, the graph G is given by

$$\mathsf{V}(\mathsf{G}) = \{1, 2, 3, 4, 5, 6\} \text{ and } \mathsf{E}(\mathsf{G}) = \{\xi_1, \xi_2, \ldots, \xi_7\}.$$

The subbase $\mathcal{R}\mathsf{V}_c(\mathsf{G})$ is given by

$$\mathcal{R}\mathsf{V}_c(\mathsf{G}) = \{\mathcal{R}_c[i] : i = 1, 2, 3, 4, 5, 6\},$$

where

$$\mathcal{R}_c[1]=\{1,2,3\},\ \ \mathcal{R}_c[2]=\{1,2,3\},\ \ \mathcal{R}_c[3]=\{1,2,3\},\ \ \mathcal{R}_c[4]=\{2,3,4,5,6\},$$
$$\mathcal{R}_c[5]=\{5\},\ \ \mathcal{R}_c[6]=\{6\}.$$

The cyclic topology on the graph G is given by

$$\tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})}=\{\emptyset,\mathsf{V}(\mathsf{G}),\{5\},\{6\},\{5,6\},\{2,3\},\{1,2,3\},\{2,3,5\},$$
$$\{2,3,6\},\{1,2,3,5\},\{1,2,3,6\},\{2,3,5,6\},\{2,3,4,5,6\},\{1,2,3,5,6\}\}.$$

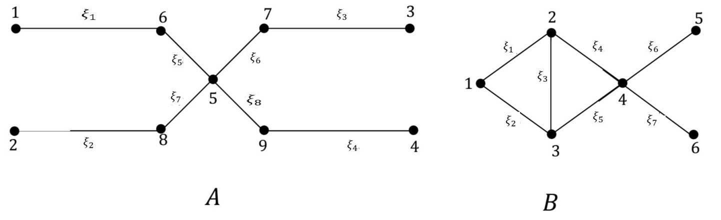

Figure 1: Cyclic topology space of the graphs Γ and G

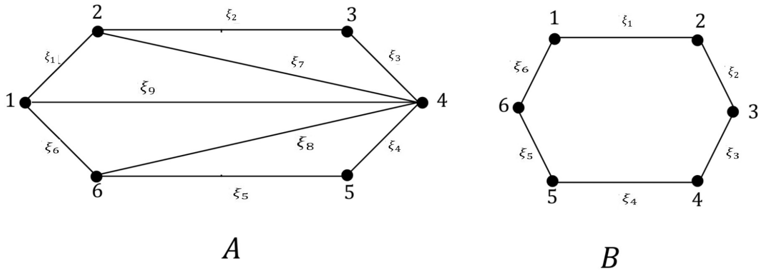

Figure 2: Cyclic topology space of the graphs H and Σ

Example 2.2 The graph H in Figure 2.A is given by

$$\mathsf{V}(\mathsf{H})=\{1,2,3,4,5,6\}\ \text{ and }\ \mathsf{E}(\mathsf{H})=\{\xi_1,\xi_2,\cdots,\xi_9\}.$$

The cyclic topological space of the graph H is induced by the subbase

$$\mathcal{R}\mathsf{V}_c(\mathsf{H})=\{\mathcal{R}_c[i]:i=1,2,3,4,5,6\},$$

where

$$\mathcal{R}_c[1]=\{1,2,4,6\},\ \ \mathcal{R}_c[2]=\{1,2,3,4\},\ \ \mathcal{R}_c[3]=\{2,3,4\},\ \ \mathcal{R}_c[4]=\mathsf{V}(\Gamma),$$
$$\mathcal{R}_c[5]=\{4,5,6\},\ \ \mathcal{R}_c[6]=\{1,4,5,6\}.$$

In Figure 2.B, the graph Σ is given by

$$\mathsf{V}(\Sigma)=\{1,2,3,4,5,6\}\ \text{ and }\ \mathsf{E}(\Sigma)=\{\xi_1,\xi_2,\cdots,\xi_6\}.$$

The subbase $\mathcal{R}\mathsf{V}_c(\Sigma)$ is given by

$$\mathcal{R}\mathsf{V}_c(\Sigma)=\{\mathcal{R}_c[i]:i=1,2,3,4,5,6\},$$

where

$$\mathcal{R}_c[1] = \{1,2,6\},\ \ \mathcal{R}_c[2] = \{1,2,3\},\ \ \mathcal{R}_c[3] = \{2,3,4\},\ \ \mathcal{R}_c[4] = \{3,4,5\},$$
$$\mathcal{R}_c[5] = \{4,5,6\},\ \ \mathcal{R}_c[6] = \{1,5,6\}.$$

The cyclic topology of Σ is discrete.

In Figure 3, the graph Ψ is given by

$$\mathsf{V}(\Psi) = \{1,2,3,4,5,6\} \text{ and } \mathsf{E}(\Psi) = \{\xi_1, \xi_2, \cdots, \xi_{15}\}.$$

The subbase $\mathcal{R}\mathsf{V}_c(\Psi)$ is

$$\mathcal{R}\mathsf{V}_c(\Psi) = \{\mathcal{R}_c[i] : \mathcal{R}_c[i] = \mathsf{V}(\Psi),\ \ (i = 1,2,3,4,5,6,7,8)\},$$

and the cyclic topology of the graph Ψ is indiscrete.

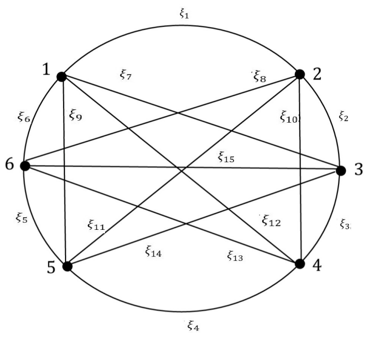

Figure 3: Cyclic topology space of the graph Ψ (Indiscrete)

2.1. Properties of the cyclic topology

This subsection deals with certain properties of cyclic topologies in important classes of graphs.

We begin with complete graphs K_n, $n > 0$. Evidently, if $n = 1$ or $n = 2$, then the cyclic topology of the complete graph K_n is discrete. The situation with K_n, $n \geq 3$, is quite different.

Theorem 2.1 *For each $n \geq 3$ the cyclic topology of the complete graph K_n is indiscrete.*

Proof: It is clear that $\mathsf{V}_c(K_n) = \mathsf{V}(K_n)$ for all $n \geq 3$. Hence, for any $x \in \mathsf{V}(K_n)$, $\mathcal{R}_c[x] = \mathsf{V}(K_n)$. Therefore, the cyclic topological space $(\mathsf{V}(K_n), \tau_{\mathcal{R}\mathsf{V}_c(K_n)})$ is indiscrete. □

Consider now a complete bipartite graph $K_{n,m}$, $m, n > 0$. It is easy to see that for $n = m = 1$ or $n = 2$ and $m = 1$, the cyclic topological space of $K_{n,m}$ is discrete. Also, then cyclic topological space of $K_{2,2}$ is discrete, because $K_{2,2}$ is actually the cycle graph of length 4 (see Theorem 2.3). The following theorem explains other cases.

Theorem 2.2 *The cyclic topology of a complete bipartite graph $K_{n,m}$ for $n, m \geq 2$, $n \neq m$, is*

$$\tau_{\mathcal{R}\mathsf{V}_c(K_{n,m})} = \{\emptyset, \mathsf{V}(K_{n,m}), \mathsf{V}_n, \mathsf{V}_m\},$$

where $\mathsf{V}(K_{n,m}) = \mathsf{V}_n \cup \mathsf{V}_m$.

Proof: Clearly, $\mathsf{V}_c(K_{n,m}) = \mathsf{V}(K_{n,m})$. If x is a vertex in $\mathsf{V}(K_{n,m})$, then either (i) $x \in \mathsf{V}_n$ and $x \notin \mathsf{V}_m$ or (ii) $x \in \mathsf{V}_m$ and $x \notin \mathsf{V}_m$. In the first case, $\mathcal{R}_c(x) = \mathsf{V}_m$, and in the second case, $\mathcal{R}_c(x) = \mathsf{V}_n$. Hence, the cyclic topological space $(\mathsf{V}(K_{n,m}), \tau_{\mathcal{R}\mathsf{V}_c(K_{n,m})})$ is given by

$$\tau_{\mathcal{R}\mathsf{V}_c(K_{n,m})} = \{\emptyset, \mathsf{V}(K_{n,m}), \mathsf{V}_n, \mathsf{V}_m\}.$$

□

Theorem 2.3 *The cyclic topological space of a cycle graph C_n is discrete for all $n \geq 4$, and indiscrete for $n = 3$.*

Proof: The case $n = 3$ is trivial, and thus we prove the case $n \geq 4$. It is clear that $\mathsf{V}_c(C_n) = \mathsf{V}(C_n)$ for all $n \geq 4$. In case $n = 4$, let

$$\mathsf{V}(C_4) = \{x_1, x_2, x_3, x_4\}$$

and

$$x_1 \to x_2 \to x_3 \to x_4 \to x_1.$$

The subbase $\mathcal{R}\mathsf{V}_c(C_4)$ is given by

$$\mathcal{R}\mathsf{V}_c(C_4) = \{\mathcal{R}_c[x_1] = \{x_4, x_1, x_2\}, \quad \mathcal{R}_c[x_2] = \{x_1, x_2, x_3\},$$

$$\mathcal{R}_c[x_3] = \{x_2, x_3, x_4\}, \mathcal{R}_c[x_4] = \{x_3, x_4, x_1\}\}.$$

In case $n > 4$, let

$$\mathsf{V}(C_n) = \{x_1, x_2, \cdots, x_{n-1}, x_n\}$$

and

$$x_1 \to x_2 \to \cdots \to x_{n-1} \to x_n \to x_1.$$

The subbase $\mathcal{R}\mathsf{V}_c(C_n)$ is given by

$$\mathcal{R}\mathsf{V}_c(C_n) = \{\mathcal{R}_c[x_1] = \{x_n, x_1, x_2\}, \quad \dots \mathcal{R}_c[x_k] = \{x_{k-1}, x_k, x_{k+1}\}, \dots$$

$$\mathcal{R}_c[x_n] = \{x_{n-1}, x_n, x_1\}\}$$

for all $k = 2, 3, \dots, n-1$. Note that

$$\{x_1\} = \mathcal{R}_c[x_2] \cap \mathcal{R}_c[x_n] \in \tau_{\mathcal{R}\mathsf{V}_c(C_n)}, \quad \{x_n\} = \mathcal{R}_c[x_{n-1}] \cap \mathcal{R}_c[x_1] \in \tau_{\mathcal{R}\mathsf{V}_c(C_n)},$$

and

$$\{x_k\} = \mathcal{R}_c[x_{k-1}] \cap \mathcal{R}_c[x_{k+1}] \in \tau_{\mathcal{R}\mathsf{V}_c(C_n)}$$

for all $k = 2, 3, \cdots, n-1$. Then the cyclic topology of the graph C_n is discrete. □

2.2. Simple graphs and the cyclic topology

Now we will show that the cyclic topological space of any simple graph is an *Alexandroff space*, i.e., the intersection of an arbitrary collection of open sets in it is open.

Theorem 2.4 *The cyclic topological space $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ of any locally finite simple graph Γ is an Alexandroff space.*

Proof: Let $\{\mathcal{R}_c[x] : x \in A \subseteq \mathsf{V}(\Gamma)\}$ be a collection of elements of $\mathcal{R}\mathsf{V}_c(\Gamma)$. We show that $\bigcap_{x\in A} \mathcal{R}_c[x]$ is an open set. If $y \in \bigcap_{x\in A} \mathcal{R}_c[x]$, then $y \in \mathcal{R}_c[x]$ for each $x \in A$. So, $x \in \mathcal{R}_c[y]$ for all $x \in A$ which means $A \subseteq \mathcal{R}_c[y]$. Since Γ is locally finite, then A is finite. Hence, $\bigcap_{x\in A} \mathcal{R}_c[x]$ is open in $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$. □

The above theorem allows us to consider the smallest open set M_x containing a vertex x in a locally finite simple graph Γ which is the intersection of all open sets in $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ containing x. It is clear that if x is an isolated vertex, and $\xi = yz$ is an isolated edge in Γ, then $M_x = \{x\}$ and $M_y = M_z = J_\xi = \{y, z\}$.

Theorem 2.5 *For any locally finite simple graph Γ and any vertex x in Γ, we have*

$$M_x = \bigcap_{y \in \mathcal{R}_c[x]} \mathcal{R}_c[y].$$

Proof: By definition of $\mathcal{R}\mathsf{V}_c(\Gamma)$, $\mathcal{R}_c[x]$ is an open set in the space $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$. By Theorem 2.4, $\bigcap_{y\in\mathcal{R}_c[x]}\mathcal{R}_c[y]$ is an open set. If $y \in \mathcal{R}_c[x]$, then $x \in \mathcal{R}_c[y]$. Hence, $\bigcap_{y\in\mathcal{R}_c[x]}\mathcal{R}_c[y]$ is an open set containing x. Since M_x is the smallest open set containing x, $M_x \subseteq \bigcap_{y\in\mathcal{R}_c[x]}\mathcal{R}_c[y]$.

On the other side, since M_x is the intersection of all open sets containing x, let $M_x = \bigcap_{y\in A}\mathcal{R}_c[y]$ for some $A \subseteq \mathsf{V}(\Gamma)$. Then $x \in \mathcal{R}_c[y]$ for all $y \in A$ which implies $y \in \mathcal{R}_c[x]$ for each $y \in A$. So, $A \subseteq \mathcal{R}_c[x]$. Hence

$$\bigcap_{y\in\mathcal{R}_c[x]}\mathcal{R}_c[y] \subseteq \bigcap_{y\in A}\mathcal{R}_c[y] = M_x.$$

Therefore,

$$M_x = \bigcap_{y\in\mathcal{R}_c[x]}\mathcal{R}_c[y].$$

□

We end this subsection by some observations on the compactness and connectedness of the cyclic topology.

It is clear that the cyclic topological space $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ of any simple graph Γ is a compact space if $\mathsf{V}(\Gamma)$ is finite, but the converse of this fact is not necessarily true in general. By Theorem 2.3, the cyclic topological space of the complete graph with vertices set $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ is indiscrete, hence compact, but this graph is infinite.

Recall that a graph Γ is called connected if one can move along the edges from any vertex to any other vertex in Γ; otherwise, Γ is called disconnected.

Theorem 2.6 *If Γ is a disconnected simple graph without isolated vertices, then the cyclic topological space $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ is a disconnected space.*

Proof: Let Γ be a disconnected graph. Consider $\mathcal{C} = \{\mathsf{G}_\alpha : \alpha \in I\}$ the family of all components in Γ. Now for all $\alpha \in I$, $\mathsf{V}(\Gamma_\alpha) = \bigcup_{x\in\mathsf{V}(\Gamma_\alpha)}\mathcal{R}_c[x]$. Then $U := \mathsf{V}(\mathsf{G}_\beta)$ is a nonempty proper open subset of $\mathsf{V}(\Gamma)$, where $\beta \in I$. Then

$$H := U^c = [\mathsf{V}(\Gamma_\beta)]^c = \bigcup_{\alpha\in I\setminus\{\beta\}}\mathsf{V}(\Gamma_\alpha)$$

is also a nonempty proper open subset of $\mathsf{V}(\Gamma)$. That is, $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ is a disconnected space. □

The simple graph Γ in Figure 4 is connected, while the cyclic topological space $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ of Γ is a disconnected space because its subbase is

$$\mathcal{R}\mathsf{V}_c(\Gamma) = \{\mathcal{R}_c[i] = \{i\} : \ i = 1, 2, 3, 4, 5, 6, 7\}.$$

2.3. An example: claw graphs and the cyclic topology

Let Γ be a simple graph. A vertex x in Γ adjacent to all the other vertices is called a *claw vertex*. A simple graph Γ is called a *claw graph* if the vertices of $\mathsf{V}_c(\Gamma)$ are not claws and not isolated vertices.

The graph Γ in Figure 5.A is a claw graph, where a subgraph of Γ with the vertices set $\mathsf{V}_c(\Gamma) = \{1, 2, \ldots, 8\}$.

The subbase $\mathcal{R}\mathsf{V}_c(\Gamma)$ is

$$\mathcal{R}\mathsf{V}_c(\Gamma) = \{\mathcal{R}_c[i] : i = 1, 2, \cdots, 12\},$$

where

$$\mathcal{R}_c[1] = \{1, 2, 8\},\ \ \mathcal{R}_c[2] = \{1, 2, 3, 5\},\ \ \mathcal{R}_c[3] = \{2, 3, 4, 6\},\ \ \mathcal{R}_c[4] = \{3, 4, 7\},$$
$$\mathcal{R}_c[5] = \{2, 5, 6, 8\},\ \ \mathcal{R}_c[6] = \{3, 5, 6, 7\},\ \ \mathcal{R}_c[7] = \{4, 6, 7\},\ \ \mathcal{R}_c[8] = \{1, 5, 8\},$$
$$\mathcal{R}_c[9] = \{9\},\ \ \mathcal{R}_c[10] = \{10\},\ \ \mathcal{R}_c[11] = \{11\},\ \ \mathcal{R}_c[12] = \{12\}.$$

The subgraph Δ_c of the graph Δ in Figure 5.B is represented by

$$\mathsf{V}_c(\Delta) = \{1, 2, 3, 4, 5, 6, 7, 8\} \text{ and } \mathsf{E}_c(\Delta) = \{\xi_1, \xi_2, \cdots, \xi_{10}\}.$$

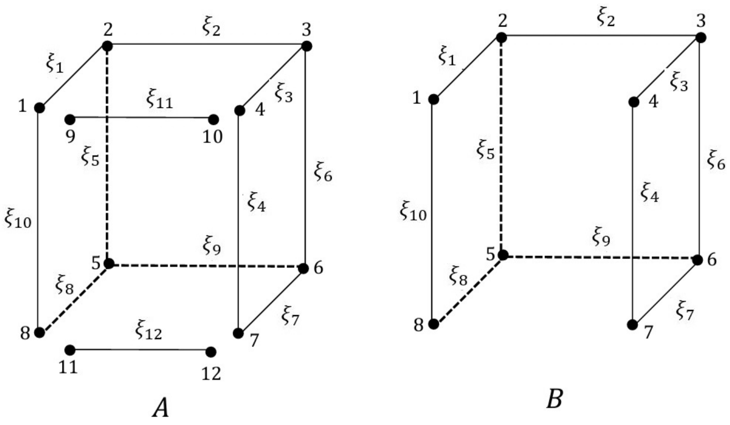

Figure 4: Disconnected space

The subbase $\mathcal{R}\mathsf{V}_c$ of the graphic topological space $(\mathsf{V}(\Delta), \tau_{\mathcal{R}\mathsf{V}_c(\Delta)})$ is given by

$$\mathcal{R}_c[1] = \{1,2,8\},\ \ \mathcal{R}_c[2] = \{1,2,3,5\},\ \ \mathcal{R}_c[3] = \{2,3,4,6\},\ \ \mathcal{R}_c[4] = \{3,4,7\},$$

$$\mathcal{R}_c[5] = \{2,5,6,8\},\ \ \mathcal{R}_c[6] = \{3,5,6,7\},\ \ \mathcal{R}_c[7] = \{4,6,7\},\ \ \mathcal{R}_c[8] = \{1,5,8\}.$$

Note that $\mathcal{R}\mathsf{V}_c = \{\mathcal{R}_c(i) : i \in \mathsf{V}_c(\Delta)\}$.

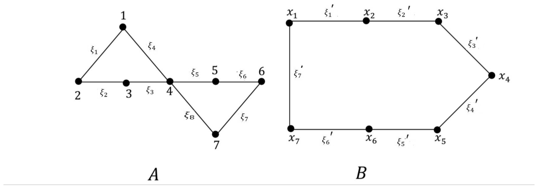

Figure 5: The claw graph and the cyclic topology

Note that the graphic topological space $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ of any claw graph Γ is an Alexandroff space.

Recall that a subset D of a topological space (X, τ) is dense in X if $\overline{D} = X$, or equivalently, if $D \cap O \neq \emptyset$ for all open sets $O \subset X$ [5].

Theorem 2.7 *Let Γ be a claw graph such that $\mathsf{V}_c(\Gamma)$ is a connected graph. Then the set $D = \{x \in \mathsf{V}_c(\Gamma) : \partial(x) > 1\}$ is a dense set in the space $(\mathsf{V}_c(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$.*

Proof: Since M_x is the smallest open set containing x for all $x \in \mathsf{V}_c(\Gamma)$, then to prove that $D \cap O \neq \emptyset$ for all open sets O in $(\mathsf{V}_c(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$, it is enough to prove that $D \cap M_x \neq \emptyset$ for all $x \in \mathsf{V}_c(\Gamma) \setminus D$. Let $x \in \mathsf{V}_c(\Gamma) \setminus D$. Since x is not isolated, there is $y \in \mathsf{V}_c(\Gamma)$ such that $A_x = \{y\}$. Hence, $M_x = A_y$. Since Γ is not a claw, then $\partial(y) > 1$. It follows that there is some $z \in D$ such that $z \in A_y$. Then $z \in D \cap A_y = D \cap M_x$, that is, $D \cap M_x \neq \emptyset$ for all $x \in \mathsf{V}_c(\Gamma) \setminus D$. □

Theorem 2.8 *Let Γ be a claw graph, $\mathcal{F} = \{M_x : x \in \mathsf{V}_c(\Gamma)\}$ and $\mu_{\mathcal{F}}$ be the family of all minimal sets in $\mathcal{F}$. If D is a minimal dense set in the space $(\mathsf{V}_c(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$, then there is an onto mapping $\varphi : \mu_{\mathcal{F}} \to D$ such that $\varphi(M_x) \in M_x$ for all $M_x \in \mu_{\mathcal{F}}$.*

Proof: By the definition of $\mu_{\mathcal{F}}$, $A \cap B = \emptyset$ for any distinct elements A and B of $\mu_{\mathcal{F}}$. As $\overline{D} = \mathsf{V}_c(\Gamma)$, there is $x \in O \cap D$ for any $O \in \mu_{\mathcal{F}}$. This implies $O \subseteq M_x$, which together with $M_x \subseteq O$ gives $M_x = O$.

If $y \in O \cap (D \setminus \{x\})$, then similarly we get $M_x = M_y = O$. Hence, $\overline{\{x\}} = \overline{\{y\}}$. In this way one concludes that $\overline{D \setminus \{y\}} = \mathsf{V}_c(\Gamma)$, which contradicts minimality of D. So, $O \cap D = \{x\}$. Define now a mapping $\varphi : \mu_{\mathcal{F}} \to D$ sending $O \in \mu_{\mathcal{F}}$ to the single element of $O \cap (D \setminus \{x\})$. Let us prove that φ is a surjection. Let $d \in D$. We prove that $M_d \in \mu_{\mathcal{F}}$ and that $\varphi(M_d) = d$. Assume, for a moment, that $M_d \notin \mu_{\mathcal{F}}$. Then there is $x \in \mathsf{V}_c(\Gamma)$ such that M_x is a proper subset of M_d, and $\overline{M_d} = \overline{M_x}$. From here we get $\overline{D \setminus \{d\}} = \mathsf{V}_c(\Gamma)$. This contradicts minimality of D. Hence, $M_d \in \mu_{\mathcal{F}}$ and $\varphi(M_d) = d$. $\square$

Theorem 2.9 *Let Γ be a claw graph, $\mathcal{F} = \{M_x : x \in \mathsf{V}_c(\Gamma)\}$ and $\mu_{\mathcal{F}}$ be the family of minimal sets in $\mathcal{F}$. If $D \subset \mathsf{V}_c(\Gamma)$ and $\Phi : \mu_{\mathcal{F}} \to D$ is a mapping such that $\Phi(M_x) \in M_x$ for each $M_x \in \mu_{\mathcal{F}}$, then $\Phi(\mu_{\mathcal{F}})$ is a minimal dense set in the space $(\mathsf{V}_c(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$.*

Proof: Clearly, for any $x \in \mathsf{V}_c(\Gamma)$ there is $y \in \mathsf{V}_c(\Gamma)$ with $M_y \in \mu_{\mathcal{F}}$ and $M_y \subseteq M_x$. Therefore, $\Phi(M_y) \in M_x \cap \Phi(\mu_{\mathcal{F}})$, that is, $\Phi(\mu_{\mathcal{F}})$ is a dense subset of $(\mathsf{V}_c(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$. It remains to prove that $\Phi(\mu_{\mathcal{F}})$ is a minimal dense set in $(\mathsf{V}_c(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$. Let $\overline{A} = \mathsf{V}_c(\Gamma)$ and $A \subseteq \Phi(\mu_{\mathcal{F}})$. Suppose that $M_x \in \mu_{\mathcal{F}}$ is such that $\Phi(M_x) \notin A$. Then there is $y \in \mathsf{V}_c(\Gamma)$ such that $M_y \in \mu_{\mathcal{F}}$ and $\Phi(M_y) \in M_x \cap A$. Since $\Phi(M_y) \in M_x \cap M_y$ and $\Phi(M_x) \notin A$, that is, $\Phi(M_x) \in M_x \setminus A$, we have then $M_x = M_y$ and so $\Phi(M_x) = \Phi(M_y) \in A$. This is a contradiction. Hence, $A = \Phi(\mu_{\mathcal{F}})$, that is, $\Phi(\mu_{\mathcal{F}})$ is a minimal dense subset of $(\mathsf{V}_c(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$. $\square$

3. Continuity and the Cyclic Topology

In this section we consider conditions under which mappings between cyclic topological spaces are continuous or closed or homeomorphisms. Also, we discuss connectedness of cyclic topological spaces of graphs.

Let Γ and G be simple graphs without isolated vertices. We say that two graphs Γ and G are isomorphic, denoted by $\Gamma \cong \mathsf{G}$, if there is a bijective mapping $f : \mathsf{V}(\Gamma) \to \mathsf{V}(\mathsf{G})$ such that $xy \in \mathsf{E}(\Gamma)$ if and only if $f(x)f(y) \in \mathsf{E}(\mathsf{G})$ for all $x, y \in \mathsf{V}(\Gamma)$.

Theorem 3.1 *Let Γ and G be simple graphs. A mapping $f : (\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)}) \to (\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$ is continuous if and only if for $x, y \in \mathsf{V}(\Gamma)$, $\mathcal{R}_c[x] \subseteq \mathcal{R}_c[y]$ implies $\mathcal{R}_c[f(x)] \subseteq \mathcal{R}_c[f(y)]$.*

Proof: $(\Rightarrow)$ Let $x, y \in \mathsf{V}(\Gamma)$ be such that $\mathcal{R}_c[x] \subseteq \mathcal{R}_c[y]$. Then, $x \in \overline{\{y\}}$. By continuity, of f we have

$$f(x) \in f(\overline{\{y\}}) \subseteq \overline{f(y)\}}.$$

We get $\mathcal{R}_c[f(x)] \subseteq \mathcal{R}_c[f(y)]$.

$(\Leftarrow)$ Suppose that for $x, y \in \mathsf{V}(\Gamma)$, $\mathcal{R}_c[x] \subseteq \mathcal{R}_c[y]$ implies $\mathcal{R}_c[f(x)] \subseteq \mathcal{R}_c[f(y)]$. Let A be a subset of $\mathsf{V}(\Gamma)$ and $x \in \overline{A}$. Then $x \in \overline{\{y\}}$ for some $y \in A$ and thus $\mathcal{R}_c[x] \subseteq \mathcal{R}_c[y]$. By the hypothesis, $\mathcal{R}_c[f(x)] \subseteq \mathcal{R}_c[f(y)]$. This implies $f(x) \in \overline{\{f(y)\}} \subseteq \overline{f(A)}$ which means that f is continuous. $\square$

Theorem 3.2 *Let Γ and G be simple graphs and $f : (\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)}) \to (\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$ an onto mapping. If for all $x, y \in \mathsf{V}(\Gamma)$, $\mathcal{R}_c[f(x)] \subseteq \mathcal{R}_c[f(y)]$ implies $\mathcal{R}_c[x] \subseteq \mathcal{R}_c[y]$, then f is a closed mapping.*

Proof: Let F be a closed set in $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$. As f is a surjection, there exists a mapping $g : \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\Gamma)$ such that $f \circ g = id_{\mathsf{V}(\mathsf{G})}$. We claim that g is continuous. Let $u, v \in \mathsf{V}(\mathsf{G})$ be such that $\mathcal{R}_c[u] \subseteq \mathcal{R}_c[v]$. This implies $\mathcal{R}_c[f(g(u))] \subseteq \mathcal{R}_c[f(g(v))]$. By the assumption, $\mathcal{R}_c[g(u)] \subseteq \mathcal{R}_c[g(v)]$. By Theorem 3.1 one concludes that g is continuous, and so $g^{\leftarrow}(F) = f(F)$ is closed in $(\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$. Hence, f is a closed mapping. $\square$

Theorem 3.3 *Let Γ and G be simple graphs and $f : (\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)}) \to (\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$ be a closed injection. Then for all $x, y \in \mathsf{V}(\Gamma)$, $\mathcal{R}_c[f(x)] \subseteq \mathcal{R}_c[(y)]$ implies $\mathcal{R}_c[x] \subseteq \mathcal{R}_c[y]$.*

Proof: Let $x, y \in \mathsf{V}(\Gamma)$ be such that $\mathcal{R}_c[f(x)] \subseteq \mathcal{R}_c[f(y)]$. Because f is injective, there exists a mapping $g : \mathsf{V}(\mathsf{G} \to \mathsf{V}(\Gamma)$ such that $g \circ f = id_{\mathsf{V}(\Gamma)}$. Using the fact that f is a closed mapping and working as in the proof of the previous theorem, it is easy to prove that g is continuous. Therefore, $\mathcal{R}_c[g(f(x))] \subseteq \mathcal{R}_c[g(f(y))]$, i.e., $\mathcal{R}_c[x] \subseteq \mathcal{R}_c[y]$. □

From Theorems 3.1, 3.2 and 3.3 we obtain the following.

Lemma 3.1 *Let Γ and G be simple graphs and $f : (\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)}) \to (\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$ be a bijective mapping. The following are equivalent:*

(1) *f is a homeomorphism;*

(2) *for all $x, y \in \mathsf{V}(\Gamma)$, $\mathcal{R}_c[x] \subseteq \mathcal{R}_c[y]$ if and only if $\mathcal{R}_c[f(x)] \subseteq \mathcal{R}_c[f(y)]$.*

Theorem 3.4 *Let Γ and G be isomorphic simple graphs without isolated vertices. Then the cyclic topological spaces $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ and $(\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$ are homeomorphic.*

Proof: Let $f : \mathsf{V}(\Gamma) \to \mathsf{V}(\mathsf{G})$ be a bijective mapping witnessing isomorphism between $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ and $(\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$, i.e. satisfying $xy \in \mathsf{E}(\Gamma)$ if and only if $f(x)f(y) \in \mathsf{E}(\mathsf{G})$ for all $x, y \in \mathsf{V}(\Gamma)$. It is clear that if the vertex $x \in \mathsf{V}(\Gamma)$ is a $C-$vertex, then the subgraph of G induced by the open neighborhood $A_{f(x)}$ is complete and $f(z) \in \mathcal{R}_c(A_{f(x)})$, that is, $f(x)$ is a C-vertex in G. By Lemma 3.1, f is a homeomorphism of cyclic topological spaces $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma_1)})$ and $(\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$. □

Note that if there is a homeomorphism between cyclic topological spaces $(\mathsf{V}(\Gamma), \tau_{\mathcal{R}\mathsf{V}_c(\Gamma)})$ and $(\mathsf{V}(\mathsf{G}), \tau_{\mathcal{R}\mathsf{V}_c(\mathsf{G})})$, then $\Gamma \cong \mathsf{G}$ need not be satisfied. For example, in Figure 6, consider the graphs Γ and G given as follows. The graph Γ is given by

$$\mathsf{V}(\Gamma) = \{1, 2, 3, 4, 5, 6, 7\}, \ \ \mathsf{E}(\Gamma) = \{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8\}.$$

The subbase of its topology is

$$\mathcal{R}\mathsf{V}_c(\Gamma) = \{R_c[i] = \{i\} : i = 1, 2, 3, 4, 5, 6, 7\}.$$

The graph G is given by

$$\mathsf{V}(\mathsf{G}) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}, \quad \mathsf{E}(\mathsf{G}) = \{\xi'_1, \xi'_2, \xi'_3, \xi'_4, \xi'_5, \xi'_6, \xi'_7\}.$$

The cyclic topology of G, generated by the subbase

$$\mathcal{R}\mathsf{V}_c(\mathsf{G}) = \{R_c[x_i] = \{x_i\} : i = 1, 2, 3, 4, 5, 6, 7\}.$$

is also discrete.

So, cyclic topologies of Γ and G are both discrete on a set of seven elements, and thus they are homeomorphic.

On the other hand, the graphs Γ and G are not isomorphic since $|\mathsf{E}(\Gamma)|$ and $|\mathsf{E}(\mathsf{G})|$ are finite and $|\mathsf{E}(\Gamma)| \neq |\mathsf{E}(\mathsf{G})|$.

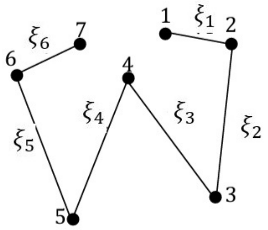

Figure 6: Graphic continuous mappings

Acknowledgment

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 25UQU4350370GSSR05.

References

1. S. M. Amiri, A. Jafarzadeh, H. Khatibzadeh, *An Alexandroff topology on graphs*, Bulletin of Iranian Mathematical Society **39** (2013) 647–662.
2. E. Anabel, R. Sergio, S. Canoy, *On a topological space generated by monophonic eccentric neighborhoods of a graph*, European Journal of Pure and Applied Mathematics **14** (2021) 695–705.
3. A. Aniyan, S. Naduvath, *Spanning graph topology of graphs*, Asian-European Journal of Mathematics **16** (2023) 2350119.
4. J. A. Bondy, U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, Springer, Berlin, 2008.
5. R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, vol. 6, Heldermann, Berlin, 1989.
6. A. Gamorez, C. G. Nianga, S. Canoy, *Topologies induced by neighborhoods of a graph under some binary operations*, European Journal of Pure and Applied Mathematics **12** (2019) 749–755.
7. A. Kiliciman, K. A. Abdu, *Topological spaces associated with simple graphs*, Journal of Mathematics Analysise **9** (2018) 44–52.
8. R. N. Lieberman, *Topologies on directed graphs*, NASA Technical Reports Server TR-214, 1972.
9. S. Nada, A. E. F. El Atik, M. Atef, *New types of topological structures via graphs*, Mathematical Methods in the Applied Sciences **41** (2018) 5801–5810.
10. V. Neumann-Lara, R. G. Wilson, *Compact compatible topologies for graphs with small cycles*, International Journal of Mathematics and Mathematical Sciences **14** (2005) 2195– 2205.
11. C. G. Nianga, S. Canoy, *On a finite topological space induced by Hop neighborhoods of a graph*, Advances and Applications in Discrete Mathematics **21** (2019) 79–89.
12. H. A. Othman, M. Al-Shamiri, A. Saif, S. Acharjee, T. Lamoudan, R. Ismail, *Pathless directed topology in connection to the circulation of blood in the heart of human body*, AIMS Mathematics **7** (2022) 18158–18172.
13. H. A. Othman, A. M. Alzubaidi, *Construction topologies on the edges set of undirected graph*, Palestine Journal of Mathematics **12** (2023) 74–79.
14. H. A. Othman, A. Ayache, A. Saif, *On L_2-directed topological spaces in directed graphs theory*, Filomat **37** (2023) 10005–10013.
15. H. K. Sari, A. Kopuzlu, *On topological spaces generated by simple undirected graphs*, AIMS Mathematics **5** (2020) 5541–5550.
16. M. Shokry, R. E. Aly, *Topological properties on graph vs medical application in human heart*, International Journal of Applied Mathematics **15** (2013) 1103–1109.
17. M. Shokry, Y. Y. Yousif, *Closure operators on graphs*, Australian Journal of Basic and Applied Sciences **5** (2011) 1856–1864.
18. H. O Zomam, *Total graphic topology on the vertex sets of directed graphs*, International Journal of Applied Sciences and Engineering **4** (2024) 18–25.
19. H. O. Zomam, H. A. Othman, M. Dammak, *Alexandroff spaces and graphic topology*, Advances in Mathematics: Scientific Journal **10** (2021): 2653–2662.

Husniyah Alzubaidi,
Department of Mathematics,
AL-Qunfudhah University College,
Umm Al-Qura University,
KSA.
E-mail address: `hazbaidi@uqu.edu.sa`

and

Diana Dolićanin-Djekić,
Faculty of Technical Sciences,
University of Priština in Kosovska Mitrovica,

28000 Kosovska Mitrovica,
Serbia.
E-mail address: `dolicanin_d@yahoo.com`

and

Ljubiša D. R. Kočinac,
Faculty of Sciences and Mathematics,
University of Niš, 18000 Niš,
Serbia.
E-mail address: `lkocinac@gmail.com`

and

Hakeem A. Othman,
Department of Mathematics,
Albaydha University,
Albaydha,
Yemen.
E-mail address: `hakim_albdoie@yahoo.com`