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On the Paranormed Notion of Generalized Δ-Operator Methods

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ABSTRACT: In the present paper, our aim is to study some new spaces of the form $\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$. Also, the determination of complete property will be given. Further, some topological properties will be carried out.

Keywords: Paranormed spaces, modular space, opial property.

Contents

1. Introduction

By Ξ we denote all real valued sequences and any subspace of Ξ is termed as sequence space [1, 2,8,20,33] and references therein. By $\mathbb{R}$ and $\mathbb{C}$, we abbreviate as set of all real and complex numbers, respectively.

A function $F : \daleth \to \mathbb{R}$, where $\daleth$ is a linear space, is called a paranorm, if for all $\varkappa, \varsigma \in \daleth$:
$F(\varkappa) \geq 0$ for all $\varkappa \geq 0$,
$F(\varkappa + \varsigma) \leq F(\varkappa) + F(\varsigma)$,
if $(\varkappa_j)$ is a sequence of vectors with $\lim_{j\to\infty} F(\varkappa_j - \varkappa) = 0$ and (a_j) is a sequence of scalars with $\lim_{j\to\infty} a_j = a$, then $\lim_{j\to\infty} F(a_j\varkappa_j - a\varkappa) = 0$. A paranormed space $(\daleth, F)$ is a linear space equipped with a paranorm F [3,14,17,34,36].

The opial property, Fatou property plays a significant role on complete spaces. In [25], ℓ_p $(1 < p < \infty)$ has shown to have this property but not $L_p[0, 2\pi]$, for $p \neq$. In [29], the brief study of uniform Opial property has been given. Later, it was studied in researches via [5,6,11,23,27] and many others.

For a sequence $\vartheta = (\kappa_r)$ of natural numbers with $\kappa_0 = 0$, $0 < \kappa_r < \kappa_{r+1}$ and $h_r = \kappa_r - \kappa_{r-1} \to \infty$ as $r \to \infty$, we call ϑ as lacunary sequence and denote intervals by $\mathfrak{J}_r = (\kappa_{r-1}, \kappa_r]$ and the quotient $\frac{\kappa_r}{\kappa_{r-1}}$ by q_r and were studied in [9,12] and many others for different domains. In [19,37], the author examined geometric structures connecting lacunary sequences with Cesàro space by equipping Luxemburg norm.

In [21], we see:

$$\mathfrak{S}(\triangle) = \{\chi = (\chi_l) \in \Xi : (\triangle\chi_l) \in \mathfrak{S}\}$$

where $\mathfrak{S} \in \{\ell_\infty, c, c_0\}$ and $\triangle\chi_l = \chi_l - \chi_{l-1}$.

Also for $\mathfrak{w}, r \geq 0$, we see as in [4] that

$$\Delta_r^{\mathfrak{w}}(\mathfrak{S}) = \{\, \chi = (\chi_l) \in \Xi : (\Delta_r^{\mathfrak{w}}\chi_l) \in \mathfrak{S}\} ,$$

where

$$\Delta_r^{\mathfrak{w}}\chi = \left(\Delta_r^{\mathfrak{w}}\chi_l\right) = \left(\Delta_r^{\mathfrak{w}-1}\chi_l - \Delta_r^{\mathfrak{w}-1}\chi_{l+r}\right),\ \triangle_r^0\chi_l = 0 \text{ for each } l \in \mathbb{N},$$

and can be written as

$$\Delta_r^{\mathfrak{w}}\chi_l = \sum_{\mu=0}^{\mathfrak{w}} (-1)^\mu \binom{\mathfrak{w}}{\mu} \chi_{l+r\mu}.$$

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Many interesting structures towards this construction can be found in [10,13,14,16,18], and many others.

As in [31], by ε-separated sequence with $\varepsilon > 0$, one mean a sequence $\{\chi_l\} \subset \Xi$ such that

$$\text{sep}\{\chi_m\} = \inf\{||\chi_l - \chi_m|| : l \neq m\} > \varepsilon.$$

Let Ω^0 be the space of all real sequences and $(\mathfrak{E}, \|\cdot\|) \subset \Omega^0$ and a complete space. For a unit sphere $S(\mathfrak{E})$ and closed unit ball $B(\mathfrak{E})$, one calls a real sequence $(v_n) \subset \mathfrak{E}$ as ε-separated sequence for some $\varepsilon > 0$, if separation of (χ_l) denoted by $\text{sep}(\chi_l) = \inf\{\|\chi_l - \chi_m\| : l \neq m\} > \varepsilon$.

We call a complete space $\mathfrak{E}$ to attain the Opial property provided each weakly null sequence $(\chi_l) \subset \mathfrak{E}$ and each non-zero $\chi \in \mathfrak{E}$, we have [25]

$$\lim \inf_{n\to\infty} \|\chi_l\| < \lim \inf_{l\to\infty} \|\chi_n + \chi\|.$$

As in [7], a Banach sequence lattice $\mathfrak{E}$ attains Fatou property, if for any $\varkappa \in \Omega$ and sequence $(\varkappa_n) \subset \mathfrak{E}_+$ with

$$\mathfrak{E}_+ = \{\varkappa \in \mathfrak{E} : \varkappa \geq 0\}$$

satisfying $0 \leq \varkappa_n(j) \nearrow \varkappa(j)$, that is, $\varkappa_n(j)$ increases to $\varkappa(j)$ as $n \to \infty$ for each $j \in \mathbb{N}$ and $\sup_n \|\varkappa_n\| < \infty$, then, $\varkappa \in \mathfrak{E}$, and $\|\varkappa\|_{\mathfrak{E}} = \lim_{n\to\infty} \|\varkappa_n\|_{\mathfrak{E}}$.

For a real vector space $\mathfrak{E}$, one calles $\daleth : \mathfrak{E} \to [0, \infty]$ a modular if it satisfies:

a) $\daleth(\chi) = 0$ if and only if $\chi = 0$.

b) $\daleth(\kappa\chi) = \daleth(\chi)$ for each $\kappa \in \mathcal{F}$ with $|\kappa| = 1$.

c) $\daleth(\kappa\chi + \lambda\chi) \leq \daleth(\chi) + \daleth(\varkappa)$ for all $\chi, \varkappa \in \mathfrak{E}$ and $\kappa, \lambda \geq 0$ with $\kappa + \lambda = 1$.

Moreover, the modular $\daleth$ is said to be convex if

$$\daleth(\kappa\chi + \lambda\varkappa) \leq \kappa\daleth(\chi) + \lambda\daleth(\varkappa)$$

for all $\chi, \varkappa \in \mathfrak{E}$ and $\kappa, \lambda \geq 0$ with $\kappa + \lambda = 1$.

For any modular $\daleth$ on $\mathfrak{E}$, the space

$$\mathfrak{E}_p = \{\chi \in \mathfrak{E} : \daleth(\kappa\chi) < \infty, \ for \ some \ \kappa > 0\}$$

is called the modular space.

A modular $\daleth$ satisfies δ_2-condition ($\daleth \in \delta_2$) if for any $\varepsilon > 0$, there exists constants $\mathfrak{A} \geq 2$ and $\mathfrak{B} > 0$ such that

$$\daleth(2\chi) \leq \mathfrak{A}\daleth(v) + \varepsilon$$

for all $\chi \in \mathfrak{E}_p$ with $\daleth(v) \leq \mathfrak{B}$.

Also, $\daleth$ meets a strong δ_2-state ($\daleth \in \delta_2^s$) if $\daleth$ satisfies δ_2-condition for all $\mathfrak{B} > 0$ with $\mathfrak{A} \geq 2$ dependent on $\mathfrak{A}$.

For simplicity, we use the following notions:

$$\begin{aligned}
\chi \mid_r &= (\chi(1), \chi(2), \cdots, \chi(r), 0, 0, \cdots) - \ truncation \ of \ \chi \ at \ r, \\
\chi \mid_{\mathbb{N}-r} &= (0, 0, \cdots, 0, \chi(r+1), \chi(r+2), \cdots) - \ truncation \ of \ \chi \ at \ r, \\
\chi \mid_I &= \{\chi = (\chi(r))_{r=1}^{\infty} : \chi(r) \neq 0 \ for \ all \ r \in I \subseteq \mathbb{N} \ and \ \chi(r) = 0 \ for \ all \ r \in \mathbb{N} \backslash I\}, \\
supp\chi &= \{r \in \mathbb{N} : \chi(r) \neq 0\}
\end{aligned}$$

and $cl\mathfrak{E}$ denotes the closure of $\mathfrak{E}$.

2. The New Space $\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$

Here we introduce the space $\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$ and show that it attains uniform Opial property, paranormed structure and some other structures as well.

Following authors as in [5,15,22,24,29,30,35], let $p = (p_i)$ represent sequence of positive real numbers $(p_l \geq 1\ \forall l \in \mathbb{N})$ and $s \geq 0$, we introduce the new space as follows:

$$\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w}) = \left\{\chi \in \Omega^0 : \daleth_{\Delta_r^{\mathfrak{w}}}(K\chi) < \infty \ for\ some\ K > 0\right\},$$

with having Luxemburg norm as

$$\|\chi\| = \inf\left\{\gamma > 0 : \daleth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi}{\gamma}\right) \leq 1\right\},$$

where

$$\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) = \sum_{l=1}^{\mathfrak{w}} |\chi(l)| + \sum_{i=1}^{\infty}\left(\frac{1}{h_i}\sum_{l\in\mathfrak{I}_i} l^{-s}\left|\Delta_r^{\mathfrak{w}}\chi(l)\right|\right)^{p_i},$$

and

$$\Delta_r^{\mathfrak{w}}\chi(l) = \sum_{\mu=0}^{\mathfrak{w}}(-1)^{\mu}\begin{pmatrix}\mathfrak{w}\\ \mu\end{pmatrix}\chi(l + r\mu)\ \forall\ l \in \mathbb{N}.$$

Theorem 2.1 *The functional $\daleth_{\Delta_r^{\mathfrak{w}}}$ on $\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$ is convex modular.*

Proof: We have

$$\begin{aligned}
\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) = 0 &\Leftrightarrow \sum_{l=1}^{\mathfrak{w}} |\chi(i)| + \sum_{l=1}^{\infty}\left(\frac{1}{h_l}\sum_{k\in\mathfrak{I}_l} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_l} = 0\\
&\Leftrightarrow \sum_{l=0}^{\mathfrak{w}} |\chi(l)| = 0 \text{ and } \sum_{l=1}^{\infty}\left(\frac{1}{h_l}\sum_{k\in\mathfrak{I}_l} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_l} = 0\\
&\Leftrightarrow \chi(l) = 0 \text{ for } l = 0, 1, 2, ..., \mathfrak{w}\ \&\ \Delta_r^{\mathfrak{w}}\chi(k) = 0\ \forall k \in \mathfrak{I}_l,\ l \in \mathbb{N}\\
&\Leftrightarrow \chi = 0.
\end{aligned}$$

One can easily see that $\daleth_{\Delta_r^{\mathfrak{w}}}(\lambda\chi) = \daleth_{\Delta_r^{\mathfrak{w}}}(\chi)$ for each scalars λ with $|\lambda| = 1$.

For $\chi, \zeta \in \mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$ with $\lambda \geq 0,\ \mu \geq 0$ and $\lambda + \mu = 1$, we see by linearity of $\Delta_r^{\mathfrak{w}}$ and convexity of map $\daleth \to |\daleth|^{p_i}$ that

$$\begin{aligned}
\daleth_{\Delta_r^{\mathfrak{w}}}(\lambda\chi + \mu\zeta) &= \sum_{l=0}^{\mathfrak{w}} |\lambda\chi(l) + \mu\zeta(l)| + \sum_{l=0}^{\infty}\left(\frac{1}{h_l}\sum_{j\in\mathfrak{I}_l}\left|\lambda\Delta_r^{\mathfrak{w}}\chi(j) + \mu\Delta_r^{\mathfrak{w}}\zeta(j)\right|\right)^{p_l}\\
&\leq \sum_{l=0}^{\mathfrak{w}}(|\lambda\chi(l)| + |\mu\zeta(l)|) + \sum_{l=0}^{\infty}\left(\frac{1}{h_l}\sum_{j\in\mathfrak{I}_l}\left|\lambda\Delta_r^{\mathfrak{w}}\chi(j)\right| + \left|\mu\Delta_r^{\mathfrak{w}}\zeta(j)\right|\right)^{p_l}\\
&\leq \lambda\left[\sum_{l=0}^{\mathfrak{w}} |\chi(l)| + \sum_{l=0}^{\infty}\left(\frac{1}{h_l}\sum_{j\in\mathfrak{I}_l}\left|\Delta_r^{\mathfrak{w}}\chi(j)\right|\right)^{p_l}\right]\\
&\qquad + \mu\left[\sum_{l=0}^{\mathfrak{w}} |\zeta(l)| + \sum_{l=0}^{\infty}\left(\frac{1}{h_l}\sum_{j\in\mathfrak{I}_l}\left|\Delta_r^{\mathfrak{w}}\zeta(j)\right|\right)^{p_l}\right]\\
&= \lambda\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) + \mu\daleth_{\Delta_r^{\mathfrak{w}}}(\zeta).
\end{aligned}$$

This shows that $\daleth_{\Delta_r^{\mathfrak{w}}}$ is a convex modular on $\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$. □

The following results via Theorem 2.2 and Theorem 2.3 are direct consequences of Theorem 2.1:

Theorem 2.2 *For $\chi \in \mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$, we have*

a) if $0 < \kappa < 1$, then $\kappa^{\mathfrak{w}} \daleth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi}{\kappa}\right) \leq \daleth_{\Delta_r^{\mathfrak{w}}}(\chi)$ and $\daleth_{\Delta_r^{\mathfrak{w}}}(\kappa\chi) \leq \kappa \daleth_{\Delta_r^{\mathfrak{w}}}(\chi)$.

b) if $\kappa > 1$, then $\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) \leq \kappa^{\mathfrak{w}} \daleth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi}{\kappa}\right)$.

c) if $\kappa \geq 1$, then $\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) \leq \kappa \daleth_{\Delta_r^{\mathfrak{w}}}(\chi) \leq \daleth_{\Delta_r^{\mathfrak{w}}}(\kappa\chi)$.

d) if $\|\chi\| < 1$, then $\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) \leq \|\chi\|$.

e) if $\|\chi\| > 1$, then $\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) \geq \|\chi\|$.

f) if $\|\chi\| = 1$, then $\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) = 1$.

Theorem 2.3 *If $\chi, \zeta \in \mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$, and $\daleth_{\Delta_r^{\mathfrak{w}}} \in \Delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left|\daleth_{\Delta_r^{\mathfrak{w}}}(\chi + \zeta) - \daleth_{\Delta_r^{\mathfrak{w}}}(\chi)\right| < \varepsilon,$$

with $\daleth_{\Delta_r^{\mathfrak{w}}}(\chi) \leq L$ and $\daleth_{\Delta_r^{\mathfrak{w}}}(\zeta) \leq \delta$.

Theorem 2.4 *a) If $\daleth_{\Delta_r^{\mathfrak{w}}} \in \Delta_2^s$, then for any $\chi \in \mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$, $\|\chi\| = 1$ if and only if $\daleth_{\Delta_r^{\mathfrak{w}}} = 1$.*

b) If $\daleth_{\Delta_r^{\mathfrak{w}}} \in \Delta_2^s$, then for any $(\chi_n) \in \mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$, $\|\chi_n\| \to 0$ if and only if $\daleth_{\Delta_r^{\mathfrak{w}}} \to 0$.

c) If $\daleth_{\Delta_r^{\mathfrak{w}}} \in \Delta_2^s$, then for any $\delta = \delta(\varepsilon) > 0$ such that $\|\chi\| \geq 1 + \delta$ whenever $\daleth_{\Delta_r^{\mathfrak{w}}} \geq 1 + \varepsilon$.

Theorem 2.5 *The space $\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$ is complete paranormed space under*

$$\mathfrak{G}_{\Delta_r^{\mathfrak{w}}}(\xi) = \sum_{j=0}^{\mathfrak{w}} |\xi(j)| + \left(\sum_{i=1}^{\infty}\left(\frac{1}{h_i}\sum_{k \in \mathfrak{I}_i} k^{-s}\left|\daleth_{\Delta_r^{\mathfrak{w}}}\xi(k)\right|\right)^{p_i}\right)^{\frac{1}{\mathcal{M}}},$$

where $\mathcal{M} = max\left(1, \sup_j p_j\right)$.

Proof: The result can be proved by the use of classical techniques. □

Theorem 2.6 *The space $\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$ attains Fatou property.*

Proof: Let $\chi_k \in \mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$ and let $\sup_k \|\chi_k\| < \infty$ with $k \in \mathbb{N}$ and $0 \leq \chi_k(i) \nearrow \chi(i)$ with $k \to \infty$ for each $i \in \mathbb{N}$. Set $\mho = \sup_k \|\chi_k\|$, and since $\|\chi_k\| \leq \mho < \infty$ for $k \in \mathbb{N}$, so that $0 \leq \frac{\chi_k}{\mho} \leq \frac{\chi_k}{\|\chi_k\|}$. Therefore, $\daleth_{\Delta_r^{\mathfrak{w}}} \frac{\chi_k}{\|\chi_k\|} \leq 1$ and since $\daleth_{\Delta_r^{\mathfrak{w}}}$ is monotone, we get

$$\daleth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi_k}{\mho}\right) \leq \daleth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi_k}{\|\chi_k\|}\right) \leq 1.$$

Employing the Beppo Levi theorem (see, [28,31]) and the fact that $\mho^{-1}\chi_k \to \mho^{-1}\chi$ for $k \to \infty$, we thus have

$$\daleth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi}{\mho}\right) = \lim_{k \to \infty} \daleth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi_k}{\mho}\right) = \sup_k \daleth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi_k}{\mho}\right) \leq 1.$$

Consequently, $\|\chi\| \leq \mho$ and $(\|\chi_k\|)$ is non-decreasing, therefore, $\|\chi_k\| \to \mho = \sup_k \|\chi_k\|$ for $k \to \infty$.

Now, by norm definition, we see

$$\begin{aligned}
\|\chi_k\| &= \inf\left\{\lambda>0:\ \beth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi_k}{\lambda}\right)\leq 1\right\}\\
&= \inf\left\{\lambda>0:\ \sum_{j=0}^{\mathfrak{w}}\left|\frac{\chi_k(j)}{\lambda}\right| + \sum_{l=0}^{\infty}\left(\frac{1}{h_l}\sum_{j\in\mathfrak{I}_l} l^{-s}\left|\frac{\Delta_r^{\mathfrak{w}}\chi_k(j)}{\lambda}\right|\right)^{p_l}\leq 1\right\}\\
&\leq \inf\left\{\lambda>0:\ \sum_{j=0}^{\mathfrak{w}}\left|\frac{\chi(j)}{\lambda}\right| + \sum_{l=0}^{\infty}\left(\frac{1}{h_l}\sum_{j\in\mathfrak{I}_l} l^{-s}\left|\frac{\Delta_r^{\mathfrak{w}}\chi(j)}{\lambda}\right|\right)^{p_l}\leq 1\right\}\\
&= \inf\left\{\lambda>0:\ \beth_{\Delta_r^{\mathfrak{w}}}\left(\frac{\chi}{\lambda}\right)\leq 1\right\}\\
&= \|\chi\|.
\end{aligned}$$

Hence, we conclude that $\sup_k \|\chi_k\| \leq \|\chi\|$ and thus $\|\chi\| = \sup_k \|\chi_k\| = \lim_{k\to\infty} \|\chi_k\|$. □

Theorem 2.7 *The space $\mathcal{L}_s^{\vartheta}(p,\Delta,r,\mathfrak{w})$ has uniform Opial property for $\limsup_r p_r < \infty$.*

Proof: Let $\|\chi\| \geq \varepsilon$, where $\varepsilon > 0$ is any number and $\chi \in \mathcal{L}_s^{\vartheta}(p,\Delta,r,\mathfrak{w})$. As, $\limsup_l p_l < \infty$, that is, $\beth_{\Delta_r^{\mathfrak{w}}} \in \delta_2^s$, by Theorem 2.4(ii), for each $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that for each $\chi \in \mathcal{L}_s^{\vartheta}(p,\Delta,r,\mathfrak{w})$ we have $\beth_{\Delta_r^{\mathfrak{w}}}(\chi) \geq \delta$.

Again, since $\beth_{\Delta_r^{\mathfrak{w}}} \in \delta_2^s$, by Theorem 2.3 for any $\varepsilon > 0$, there is a $\delta_1 \in (0,\delta)$ such that

$$\left|\beth_{\Delta_r^{\mathfrak{w}}}(\chi+\zeta) - \beth_{\Delta_r^{\mathfrak{w}}}(\chi)\right| < \frac{\delta}{4}, \tag{2.1}$$

whenever $\beth_{\Delta_r^{\mathfrak{w}}}(\chi) \leq 1$ and $\beth_{\Delta_r^{\mathfrak{w}}}(\zeta) \leq \delta$ and $\chi,\zeta \in \mathcal{L}_s^{\vartheta}(p,\Delta,r,\mathfrak{w})$.

But $\beth_{\Delta_r^{\mathfrak{w}}}(\chi) < \infty$, so there exists a natural number $\mathfrak{I}_0$ such that

$$\sum_{l=0}^{\mathfrak{w}}|\chi(l)| + \sum_{i=\mathfrak{I}_0}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{I}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i} \leq \frac{\delta_1}{4}. \tag{2.2}$$

From (2.2), it follows that

$$\begin{aligned}
\delta &\leq \sum_{l=0}^{\mathfrak{w}}|\chi(l)| + \sum_{i=0}^{\mathfrak{I}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{I}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i} + \sum_{i=\mathfrak{I}_0+1}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{I}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i}\\
&\leq \sum_{i=0}^{\mathfrak{I}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{I}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i} + \frac{\delta_1}{4}
\end{aligned}$$

yielding

$$\begin{aligned}
\sum_{i=0}^{\mathfrak{I}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{I}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i} &\geq \delta - \frac{\delta_1}{4}\\
&> \delta - \frac{\delta}{4} = \frac{3\delta}{4}.
\end{aligned} \tag{2.3}$$

So, by linearity of $\Delta_r^{\mathfrak{w}}$ and weak convergence yields coordinatewise convergence, means, $\chi_n \to 0$ weakly gives $\chi_n(i) \to 0$ for each $i \in \mathbb{N}$, so there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we get

$$\sum_{i=0}^{\mathfrak{I}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{I}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi_n(k) + \Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i} > \frac{3\delta}{4}. \tag{2.4}$$

Again, using the fact that $\chi_n \ ^w \to 0$, we can choose $\mathfrak{J}_0$ such that

$$\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi\left|_{\mathfrak{J}_0}\right.\right) \to 0 \ as \ n \to \infty.$$

So, there exists $n_1 > n_0$ such that

$$\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi\left|_{\mathfrak{J}_0}\right.\right) \le \delta_1 \ for \ all \ n \ge n_1.$$

Since $(\chi_n) \in S\left(\mathcal{L}_s^{\vartheta}(p,\Delta,r,\mathfrak{w})\right)$, that is, $\|\chi\| = 1$, so using Theorem 2.3(i), we see $\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi\left|_{\mathfrak{J}_0}\right.\right) = 1$. So, we can find $\mathfrak{J}_0$ such that

$$\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right) \le 1.$$

Now pick $\mathfrak{v} = \chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right.$ and $\mathfrak{w} = \chi_n\left|_{\mathfrak{J}_0}\right.$. Then, $\beth_{\Delta_r^{\mathfrak{w}}}(\mathfrak{v}) \le 1$ and $\beth_{\Delta_r^{\mathfrak{w}}}(\mathfrak{w}) \le \delta_1$ for $\mathfrak{v}, \mathfrak{w} \in \mathcal{L}_s^{\vartheta}(p,\Delta,r,\mathfrak{w})$. So from equation (2.1), for all $n \ge n_1$, we see

$$\left|\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right. + \chi_n\left|_{\mathfrak{J}_0}\right.\right) - \beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right)\right| < \frac{\delta}{4},$$

yielding

$$\beth_{\Delta_r^{\mathfrak{w}}}(\chi_n) - \frac{\delta}{4} < \beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right) \ for \ all \ n \ge n_1.$$

This shows that

$$\sum_{i=0}^{\mathfrak{w}} |\chi_n(i)| + \sum_{i=\mathfrak{J}_0+1}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi_n(k)\right|\right)^{p_i} > 1 - \frac{\delta}{4} \ for \ all \ n \ge n_1. \tag{2.5}$$

Also, as $\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right) \le 1$ and $\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right) \le \frac{\delta_1}{4} < \delta_1$, so from (2.1), we have

$$\left|\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right. + \chi\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right) - \beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right)\right| < \frac{\delta}{4}$$

this implies that

$$\left|\beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right. + \chi\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right) > \beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right)\right| - \frac{\delta}{4}. \tag{2.6}$$

Now, with the help of (2.4), (2.5), (2.6) and the linearity property of $\Delta_r^{\mathfrak{w}}$, we see

$$\begin{aligned}
\beth_{\Delta_r^{\mathfrak{w}}}(\chi_n + \chi) &= \sum_{j=0}^{\mathfrak{w}} |\chi_n(j) + \chi(j)| + \sum_{i=0}^{\mathfrak{J}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi_n(k) + \Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i} \\
&\quad + \sum_{i=\mathfrak{J}_0+1}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi_n(k) + \Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i} \\
&> \sum_{i=0}^{\mathfrak{J}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i} k^{-s}\left|\Delta_r^{\mathfrak{w}}\chi_n(k) + \Delta_r^{\mathfrak{w}}\chi(k)\right|\right)^{p_i} + \beth_{\Delta_r^{\mathfrak{w}}}\left(\chi_n\left|_{\mathbb{N}-\mathfrak{J}_0}\right.\right) - \frac{\delta}{4} \\
&> \frac{3\delta}{4} + \left(1 - \frac{\delta}{4}\right) - \frac{\delta}{4} = 1 + \frac{\delta}{4}.
\end{aligned}$$

But, $\beth_{\Delta_r^{\mathfrak{w}}} \in \delta_2^s$, so, employing Lemma 2.3(ii), one can find $\lambda > 0$ such that $\|\chi_n - \chi\| \ge 1 + \lambda$, so as $n \to \infty$ we see

$$\liminf \|\chi_n + \chi\| \ge 1 + \lambda.$$

□

Note while picking different values of $\mathfrak{w}, r, s$, we see:

Note 1 *By picking* $\mathfrak{w} = 0$ *and* $r = 1$*, we get the spaces given in [26].*

Note 2 *By picking* $s = 0$*,* $\mathfrak{w} = 0$*,* $r = 1$*,* $p_j = p$ *and* $\vartheta = (2^l)$ *for each* $l \in \mathbb{N}$*, we get the spaces given in [32].*

Note 3 *By picking* $s = 0$*,* $\mathfrak{w} = 0$*,* $r = 1$ *and* $\vartheta = (2^r)$*, we get the spaces given in [35].*

Note 4 *By picking* $s = 0$ *and* $r = 1$*, the space* $\mathcal{L}_s^{\vartheta}(p, \Delta, r, \mathfrak{w})$*, we get the spaces given in [37].*

3. Conclusion

In this study, we have constructed new spaces using modulus function. The paranormed structure has been established. We have established the properties of convex modular, uniform Opial property, and many others. The future work will be to establish some more generalized spaces on the notion of this pattern.

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