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Geometry of Tangent Bundles Equipped with the Deformed Complete Lift Metric

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Abstract: The geometric structure of the tangent bundle (TM, g^{D}) equipped with the deformed complete lift metric g^{D} is investigated. Necessary and sufficient conditions are established for the vertical and horizontal lifts of vector fields to be conformal or Killing vector fields with respect to g^{D}. The conditions under which the tangent bundle, endowed with the horizontal (resp. complete) lift connection, admits a Codazzi or statistical structure are also established. The study also includes the analysis of infinitesimal affine transformations and geodesics associated with g^{D}. Furthermore, explicit examples are given on the Euclidean space to illustrate and validate the obtained characterization of geodesics on the tangent bundle endowed with the deformed complete lift metric.

Keywords: Lie derivative, infinitesimal affine transformation, deformed complete lift metric, Codazzi manifold, statistical manifold, geodesic.

Contents

1. Introduction

The study of geometric structures on the tangent bundle TM of a Riemannian manifold (M, g) was initiated by Sasaki [11, Sasaki 1958] and has since remained an active area of research. The study of the relationship between the geometry of a manifold (M, g) and its tangent bundle TM, equipped with the naturally defined Sasaki metric has revealed certain kinds of rigidity. Over the past few decades, the introduction of various lifted metrics on the tangent bundle TM such as the gradient Sasaki metric, twisted Sasaki metric, Cheeger-Gromoll metric, complete lift metrics and deformed 2nd lift metric (see [2, Belarbi and Elhendi 2023], [12, Sekizawa 1991], [13, Tanno 1974] and [8, Magden *et al.* 2019]) provide an effective framework to extend the geometry of the base manifold (M, g) to its tangent bundle TM, thus enabling the exploration of new geometric, analytic, and physical phenomena on higher-order structures.

Several deformations of the classical lifted metrics have been introduced to enhance their geometric richness and adaptability for applications in different contexts of differential geometry (see [4, Djaa and Zagane 2022], [9, Medjadj *et al.* 2024], [5, Elhendi and Zagane 2022], etc.). Among them, the complete lift metric holds a particular importance, as it naturally arises from the differential structure of the tangent bundle. Among its deformations, the deformed complete lift metric, introduced in [6, Gezer and Özkan 2014], includes a smooth non-vanishing function f on M into the structure. This metric can be viewed as a particular case of the synectic lift metric, such that the lift of the base metric g to TM is given by $\widetilde{G} = g^{\mathrm{C}} + a^{\mathrm{V}}$ with a^{V} denoting the vertical lift of a symmetric $(0,2)$-tensor field a. The deformed complete

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lift metric, defined by $g^{\mathrm{D}} = g^{\mathrm{C}} + (fg)^{\mathrm{V}}$, thus represents a deformation of the complete lift metric through the differentiable function f.

The study of these type of metrics provides additional degree of freedom through the function f. This flexibility allows one to investigate new geometric behaviours that cannot be captured by classical Sasaki or complete lift metrics. Moreover, the metric g^{D} provides a framework for the construction of almost complex and almost product structures on TM, which offers a deeper understanding of curvature properties, geodesic equations, and affine transformations associated with lifted geometries. Hence, it provides a broader framework for analyzing the interplay between the geometry of the base manifold and its tangent bundle. We investigate the geometry of tangent bundles equipped with the deformed complete lift metric g^{D}, and examine the properties of this metric with respect to the horizontal and complete lift connections defined on TM. More specifically, we study the geometric behavior of the pairs $(g^{\mathrm{D}}, \nabla^{\mathrm{D}})$, $(g^{\mathrm{D}}, \nabla^{\mathrm{C}})$ and $(g^{\mathrm{H}}, \nabla^{\mathrm{D}})$.

The paper is organized as follows. In Section 2, we recall some preliminaries on the geometry of tangent bundles and the construction of horizontal and complete lift connections. In Section 3, we study Killing vector fields with respect to the deformed complete lift metric for the vertical and horizontal lift of vector fields and establish the conditions under which the tangent bundle equipped with the deformed complete lift metric and horizontal (resp. complete) lift connection, becomes a Codazzi and statistical manifold. In Section 4, we studied infinitesimal affine transformation for vertical and horizontal lift of vector fields on the tangent bundle $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{D}})$. In Section 5, we study the geodesic structure of the tangent bundle TM equipped with the deformed complete lift metric g^{D}. Furthermore, explicit examples on Euclidean spaces are constructed to verify the corollary concerning the natural lift of geodesics with respect to the deformed complete lift metric.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold and TM denote its tangent bundle. A local chart (U, x^i), $i = \{1, \ldots, n\}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)$, $i = \{1, \ldots, n\}$ on TM. For local coordinates (x^i), $i \in \{1, \ldots, n\}$ on M and the corresponding induced coordinates (x^i, y^i) on the tangent bundle TM, the set $\left\{ \frac{\partial}{\partial x^i}\big|_{(x,y)}, \frac{\partial}{\partial y^i}\Big|_{(x,y)} \right\}$ forms the natural basis of $T_{(x,y)}TM$ at $(x,y) \in TM$. With respect to an affine connection ∇, the tangent space decomposes as $T_{(x,y)}TM = \mathcal{H}_{(x,y)}TM \oplus \mathcal{V}_{(x,y)}TM$, where $\mathcal{H}_{(x,y)}TM$ is spanned by $\left\{ \frac{\delta}{\delta x^i}\big|_{(x,y)} := (\frac{\partial}{\partial x^i})^{\mathrm{H}} = \frac{\partial}{\partial x^i}\big|_{(x,y)} - y^k \Gamma^j_{ki}(x) \frac{\partial}{\partial y^j}\Big|_{(x,y)} \right\}$ and $\mathcal{V}_{(x,y)}TM$ is spanned by $\left\{ \frac{\partial}{\partial y^i}\Big|_{(x,y)} := (\frac{\partial}{\partial x^i})^{\mathrm{V}} \right\}$, with Γ^j_{ki} denoting the connection coefficients of ∇. The projection $\pi : TM \to M$ is given by $\pi(x,y) = x$, and $\mathfrak{F}^1_0(M)$ denotes the set of smooth vector fields on M.

For a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M, the complete, horizontal and vertical lifts of X to the tangent bundle TM are defined by

$$X^{\mathrm{C}} = X^i \frac{\partial}{\partial x^i} + y^a \frac{\partial X^i}{\partial x^a} \frac{\partial}{\partial y^i}, \qquad X^{\mathrm{H}} = X^i \frac{\partial}{\partial x^i} - y^a \Gamma^k_{ai} X^i \frac{\partial}{\partial y^k} \quad \text{and} \quad X^{\mathrm{V}} = X^i \frac{\partial}{\partial y^i},$$

respectively.

The Lie brackets of these horizontal and vertical lifts of vector fields satisfy [14, Yano and Ishihara 1973]

$$[X^{\mathrm{H}}, Y^{\mathrm{H}}] = [X,Y]^{\mathrm{H}} - (R(X,Y)y)^{\mathrm{V}}, \tag{2.1}$$

$$[X^{\mathrm{H}}, Y^{\mathrm{V}}] = (\nabla_X Y)^{\mathrm{V}} - (T(X,Y))^{\mathrm{V}}, \tag{2.2}$$

$$[X^{\mathrm{V}}, Y^{\mathrm{V}}] = 0, \tag{2.3}$$

where T is the torsion tensor field and R is the curvature tensor field of an affine connection ∇.

The horizontal lift connection ∇^{H} and the complete lift connection ∇^{C} associated with an affine connection ∇ are given as follows [14, Yano and Ishihara 1973]

$$\nabla^{\mathrm{H}}_{X^{\mathrm{H}}} Y^{\mathrm{H}} = (\nabla_X Y)^{\mathrm{H}}, \quad \nabla^{\mathrm{H}}_{X^{\mathrm{H}}} Y^{\mathrm{V}} = (\nabla_X Y)^{\mathrm{V}}, \quad \nabla^{\mathrm{H}}_{X^{\mathrm{V}}} Y^{\mathrm{H}} = \nabla^{\mathrm{H}}_{X^{\mathrm{V}}} Y^{\mathrm{V}} = 0,$$

$$\nabla^{C}_{X^{H}}Y^{H} = (\nabla_X Y)^{H} + (R(y,X)Y)^{V}, \quad \nabla^{C}_{X^{V}}Y^{H} = \nabla^{C}_{X^{V}}Y^{V} = 0,$$

$$\nabla^{C}_{X^{H}}Y^{V} = (\nabla_X Y)^{V}, \quad \nabla^{C}_{X^{C}}Y^{C} = (\nabla_X Y)^{C}, \quad \nabla^{C}_{X^{C}}Y^{V} = \nabla^{C}_{X^{V}}Y^{C} = (\nabla_X Y)^{V}.$$

For convenience, throughout the paper we use the notations ∂_i, δ_i and $\partial_{\bar{i}}$ instead of $\frac{\partial}{\partial x^i}$, $\frac{\delta}{\delta x^i}$, and $\frac{\partial}{\partial y^i}$, respectively. It is known that ∇ is flat and torsion free if and only if ∇^H (∇^C) is torsion free [14, Yano and Ishihara 1973].

Definition 2.1 [1, Amari *et al.* 1987] Let (M, g) be a pseudo-Riemannian manifold with pseudo-Riemannian metric g and let ∇ be an affine connection on M. The pair (∇, g) is said to be a *Codazzi couple* on M if the cubic tensor $C := \nabla g$ is totally symmetric, namely, the *Codazzi equations* hold:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) = (\nabla_Z g)(X, Y), \tag{2.4}$$

for all $X, Y, Z \in \mathfrak{F}^1_0(M)$. The triplet (M, g, ∇) is called a *Codazzi manifold* and ∇ is called a *Codazzi connection*. Furthermore, if ∇ is torsion-free, then (M, g, ∇) is a *statistical manifold*, (∇, g) is a *statistical couple* and ∇ is a *statistical connection*.

3. Geometry of Tangent Bundles Equipped with the Deformed Complete Lift Metric

In this section, we investigate the geometric properties of the tangent bundle TM endowed with the deformed complete lift metric g^{D}.

Definition 3.1 [6, Gezer and Özkan 2014] Let (M, g) be a Riemannian manifold and f be a nonzero differentiable function on M. Then, the deformed complete lift metric $g^{D} = g^{C} + (fg)^{V}$ of g on TM is defined by

$$g^{D}(X^{H}, Y^{H}) = fg(X, Y), \tag{3.1}$$

$$g^{D}(X^{H}, Y^{V}) = g^{D}(X^{V}, Y^{H}) = g(X, Y), \tag{3.2}$$

$$g^{D}(X^{V}, Y^{V}) = 0 \tag{3.3}$$

for all vector fields $X, Y \in \mathfrak{F}^1_0(M)$.

Lemma 3.1 *Let (M, g) be a Riemannian manifold and (TM, g^{D}) its tangent bundle equipped with the deformed complete lift metric. Then*

$$X^{V}g^{D}(Y^{V}, Z^{V}) = 0, \tag{3.4}$$

$$X^{V}g^{D}(Y^{H}, Z^{V}) = 0, \tag{3.5}$$

$$X^{V}g^{D}(Y^{H}, Z^{H}) = 0, \tag{3.6}$$

$$X^{H}g^{D}(Y^{V}, Z^{V}) = 0, \tag{3.7}$$

$$X^{H}g^{D}(Y^{H}, Z^{V}) = Xg(Y, Z), \tag{3.8}$$

$$X^{H}g^{D}(Y^{H}, Z^{H}) = X(f)g(Y, Z) + fX(g(Y, Z)), \tag{3.9}$$

where $X, Y, Z \in \mathfrak{F}^1_0(M)$.

3.1. The Levi-Civita connection

In this section, we study the Levi-Civita connection ∇^{D} of TM endowed with the deformed complete lift metric g^{D}. This connection is uniquely characterized by the Koszul formula

$$\begin{aligned} 2g^{D}(\nabla^{D}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) &= \widetilde{X}g^{D}(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y}g^{D}(\widetilde{Z}, \widetilde{X}) - \widetilde{Z}g^{D}(\widetilde{X}, \widetilde{Y}) \\ &\quad + g^{D}(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) + g^{D}(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) - g^{D}(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]) \end{aligned} \tag{3.10}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{F}^1_0(TM)$.

Lemma 3.2 [6, Gezer and Özkan 2014] *Let (M, g) be a Riemannian manifold, ∇^g the Levi-Civita connection of g, and (TM, g^{D}) its tangent bundle equipped with the deformed complete lift metric. Then the Levi-Civita connection ∇^{D} of g^{D} is given by*

$$\nabla^{\mathrm{D}}_{X^{\mathrm{V}}} Y^{\mathrm{V}} = 0, \tag{3.11}$$

$$\nabla^{\mathrm{D}}_{X^{\mathrm{V}}} Y^{\mathrm{H}} = 0, \tag{3.12}$$

$$\nabla^{\mathrm{D}}_{X^{\mathrm{H}}} Y^{\mathrm{V}} = (\nabla^g_X Y)^{\mathrm{V}}, \tag{3.13}$$

$$\nabla^{\mathrm{D}}_{X^{\mathrm{H}}} Y^{\mathrm{H}} = (\nabla^g_X Y)^{\mathrm{H}} + (R(y, X)Y)^{\mathrm{V}} + \frac{1}{2}(X(f)Y + Y(f)X - g(X, Y)\mathrm{grad} f)^{\mathrm{V}} \tag{3.14}$$

for all vector fields $X, Y \in \mathfrak{F}^1_0(M)$.

Definition 3.2 [7, Hsiung 1964], [14, Yano and Ishihara 1973] Let (M, g) be a Riemannian manifold and ∇ an affine connection of M. Then

(1) A vector field X is said to be conformal (resp. Killing) with respect to g, if $\pounds_X g = 2\rho g$ (resp. $\pounds_X g = 0$), where ρ is a function on M and the Lie derivative of g in the direction of X is given by

$$(\pounds_X g)(Y, Z) := Xg(Y, Z) - g(\pounds_X Y, Z) - g(Y, \pounds_X Z). \tag{3.15}$$

(2) A vector field is said to be an infinitesimal affine transformation on M with respect to ∇, if $\pounds_X \nabla = 0$, where the Lie derivative of ∇ in the direction of X is given by

$$(\pounds_X \nabla)(Y, Z) := \pounds_X(\nabla_Y Z) - \nabla_Y(\pounds_X Z) - \nabla_{[X,Y]} Z. \tag{3.16}$$

Definition 3.3 Let (M, g) be a Riemannian manifold with an affine connection ∇, and let $f \in C^\infty(M)$ be a smooth function. The *Hessian* of f is the symmetric $(0, 2)$-tensor field defined by

$$\mathrm{Hess}^f(X, Y) := (\nabla df)(X, Y) = X(Y(f)) - (\nabla_X Y)(f)$$

for all vector fields $X, Y \in \mathfrak{F}^1_0(M)$.

Lemma 3.3 *Let (M, g) be a Riemannian manifold equipped with an affine connection ∇, and (TM, g^{D}) its tangent bundle equipped with the deformed complete lift metric. Then the Lie derivative of the deformed complete lift metric with respect to vertical lift of a vector field X satisfies*

$$(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = 0, \tag{3.17}$$

$$(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = 0, \tag{3.18}$$

$$(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\pounds_X Y, Z) - g(Y, \pounds_X Z) \tag{3.19}$$

for all vector fields $X, Y, Z \in \mathfrak{F}^1_0(M)$.

Proof: From (3.15), we have

$$(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = X^{\mathrm{V}} g^{\mathrm{D}}(Y^{\mathrm{V}}, Z^{\mathrm{V}}) - g^{\mathrm{D}}(\pounds_{X^{\mathrm{V}}} Y^{\mathrm{V}}, Z^{\mathrm{V}}) - g^{\mathrm{D}}(Y^{\mathrm{V}}, \pounds_{X^{\mathrm{V}}} Z^{\mathrm{V}}), \tag{3.20}$$

$$(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = X^{\mathrm{V}} g^{\mathrm{D}}(Y^{\mathrm{H}}, Z^{\mathrm{V}}) - g^{\mathrm{D}}(\pounds_{X^{\mathrm{V}}} Y^{\mathrm{H}}, Z^{\mathrm{V}}) - g^{\mathrm{D}}(Y^{\mathrm{H}}, \pounds_{X^{\mathrm{V}}} Z^{\mathrm{V}}), \tag{3.21}$$

$$(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = X^{\mathrm{V}} g^{\mathrm{D}}(Y^{\mathrm{H}}, Z^{\mathrm{H}}) - g^{\mathrm{D}}(\pounds_{X^{\mathrm{V}}} Y^{\mathrm{H}}, Z^{\mathrm{H}}) - g^{\mathrm{D}}(Y^{\mathrm{H}}, \pounds_{X^{\mathrm{V}}} Z^{\mathrm{H}}). \tag{3.22}$$

In view of (2.3), (3.3) and (3.20), we get (3.17). Next, using (3.2), (3.21) reduces to

$$(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = g^{\mathrm{D}}([Y^{\mathrm{H}}, X^{\mathrm{V}}], Z^{\mathrm{V}}) - g^{\mathrm{D}}(Y^{\mathrm{H}}, [X^{\mathrm{V}}, Z^{\mathrm{V}}]),$$

which in view of (2.2) and (2.3) gives (3.18). Finally, in view of (2.2), (3.2), and (3.22), we get (3.19). □

Lemma 3.4 *Let (M, g) be a Riemannian manifold equipped with an affine connection ∇, and (TM, g^{D}) its tangent bundle equipped with the deformed complete lift metric. Then the Lie derivative of the deformed complete lift metric with respect to horizontal lift of a vector field X satisfies*

$$(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = 0, \tag{3.23}$$

$$(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = (\nabla_X g)(Y, Z) + g(\nabla_Y X + T(X, Y), Z) - g(Y, T(X, Z)), \tag{3.24}$$

$$(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = (\pounds_X fg)(Y, Z) + g(R(X, Y)y, Z) + g(Y, R(X, Z)y) \tag{3.25}$$

for all vector fields $X, Y, Z \in \mathfrak{F}^1_0(M)$.

Proof: From (3.15), we have

$$(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = X^{\mathrm{H}} g^{\mathrm{D}}(Y^{\mathrm{V}}, Z^{\mathrm{V}}) - g^{\mathrm{D}}(\pounds_{X^{\mathrm{H}}} Y^{\mathrm{V}}, Z^{\mathrm{V}}) - g^{\mathrm{D}}(Y^{\mathrm{V}}, \pounds_{X^{\mathrm{H}}} Z^{\mathrm{V}}), \tag{3.26}$$

$$(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = X^{\mathrm{H}} g^{\mathrm{D}}(Y^{\mathrm{H}}, Z^{\mathrm{V}}) - g^{\mathrm{D}}(\pounds_{X^{\mathrm{H}}} Y^{\mathrm{H}}, Z^{\mathrm{V}}) - g^{\mathrm{D}}(Y^{\mathrm{H}}, \pounds_{X^{\mathrm{H}}} Z^{\mathrm{V}}), \tag{3.27}$$

$$(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = X^{\mathrm{H}} g^{\mathrm{D}}(Y^{\mathrm{H}}, Z^{\mathrm{H}}) - g^{\mathrm{D}}(\pounds_{X^{\mathrm{H}}} Y^{\mathrm{H}}, Z^{\mathrm{H}}) - g^{\mathrm{D}}(Y^{\mathrm{H}}, \pounds_{X^{\mathrm{H}}} Z^{\mathrm{H}}). \tag{3.28}$$

In view of (2.2), (3.3) and (3.26), we get (3.23). Next, using (2.1) and (3.2), (3.27) reduces to

$$(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = Xg(Y, Z) - g([X, Y], Z) - g(Y, \nabla_X Z + T(X, Z)),$$

which gives (3.24). Finally, in view of (2.1), (3.1), and (3.28), we get (3.25). □

Corollary 3.1 [3, Cauir and Akpinar 2022] *Let (M, g) be a Riemannian manifold with torsion free affine connection ∇, and (TM, g^{D}) its tangent bundle equipped with the deformed complete lift metric. Then the Lie derivative of the deformed complete lift metric with respect to vertical and horizontal lift of vector fields satisfies*

1. $(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = 0$,
2. $(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = 0$,
3. $(\pounds_{X^{\mathrm{V}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$,
4. $(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = 0$,
5. $(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = (\nabla_X g)(Y, Z) + g(\nabla_Y X, Z)$,
6. $(\pounds_{X^{\mathrm{H}}} g^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = (\pounds_X fg)(Y, Z) + g(R(X, Y)y, Z) + g(Y, R(X, Z)y)$

for all vector fields $X, Y, Z \in \mathfrak{F}^1_0(M)$.

Proof: The proof directly follows from Lemma 3.3 and 3.4. □

Proposition 3.1 *Let (M, g) be a Riemannian manifold, and (TM, g^{D}) its tangent bundle equipped with the deformed complete lift metric. Then the following assertions hold:*

1. *If ∇ is torsion free affine connection on M, then the vertical lift X^{V} of a vector field X is Killing vector field on (TM, g^{D}) if and only if X is Killing vector field on (M, g) (or X is a parallel vector field);*
2. *If ∇ is torsion free affine connection on M, then the horizontal lift X^{H} of a vector field X is Killing vector field on (TM, g^{D}) if and only if X is conformal vector field on M and $(\nabla_X g)(Y, Z) = -g(\nabla_Y X, Z)$, $R(X, Y)Z = 0$ for all $Y, Z \in \mathfrak{F}^1_0(M)$;*

(3) *If ∇ is torsion free affine connection on M and X is parallel, then the horizontal lift X^{H} of a vector field X is Killing vector field on (TM, g^{D}) if and only if X is conformal vector field on (M, g), ∇ be the Levi-Civita connection ∇^g of (M, g) and $R(X, Y)Z = 0$ for all Y, $Z \in \mathfrak{F}^1_0(M)$;*

(4) *If ∇ is torsion free affine connection on M, f is constant and X is parallel, then the horizontal lift X^{H} of a vector field X is Killing vector field on (TM, g^{D}) if and only if X is Killing vector field on (M, g), ∇ be the Levi-Civita connection ∇^g of (M, g) and $R(X, Y)Z = 0$ for all $Y, Z \in \mathfrak{F}^1_0(M)$;*

(5) *If ∇ is the flat Levi-Civita connection on (M, g), f is constant and X is parallel, then the horizontal lift X^{H} of a vector field X is Killing vector field on (TM, g^{D}) if and only if X is Killing vector field on (M, g).*

Proof: The proof directly follows from Corollary 3.1. □

3.2. The components of $\nabla^{\mathbf{H}} g^{\mathbf{D}}$ on the Tangent Bundle $(TM, g^{\mathbf{D}}, \nabla^{\mathbf{H}})$

Let $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{H}})$ be the tangent bundle of an n-dimensional Riemannian manifold (M, g), equipped with the deformed complete lift metric $g^{\mathrm{D}} = g^{\mathrm{C}} + (fg)^{\mathrm{V}}$, and let ∇^{H} denote the horizontal lift of an affine connection ∇ on M. To understand the compatibility of ∇^{H} with the metric g^{D}, we compute the covariant derivative $\nabla^{\mathrm{H}} g^{\mathrm{D}}$ with respect to the adapted frame $\delta_i = (\partial_i)^{\mathrm{H}}$, $\partial_{\bar{i}} = (\partial_i)^{\mathrm{V}}$, $i \in \{1, \dots, n\}$.

A direct computation shows that the component $(\nabla^{\mathrm{H}}_{\delta_i} g^{\mathrm{D}})(\delta_j, \delta_k)$ is given by ([3, Cauir and Akpinar 2022])

$$
\begin{aligned}
\left(\nabla^{\mathrm{H}}_{\delta_i} g^{\mathrm{D}}\right)(\delta_j, \delta_k) &= \delta_i g^{\mathrm{D}}(\delta_j, \delta_k) - g^{\mathrm{D}}\left(\nabla^{\mathrm{H}}_{\delta_i} \delta_j, \delta_k\right) - g^{\mathrm{D}}\left(\delta_j, \nabla^{\mathrm{H}}_{\delta_i} \delta_k\right) \\
&= (\partial_i)^{\mathrm{H}} g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, (\partial_k)^{\mathrm{H}}) - g^{\mathrm{D}}(\nabla^{\mathrm{H}}_{(\partial_i)^{\mathrm{H}}} (\partial_j)^{\mathrm{H}}, (\partial_k)^{\mathrm{H}}) - g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, \nabla^{\mathrm{H}}_{(\partial_i)^{\mathrm{H}}} (\partial_k)^{\mathrm{H}}) \\
&= \partial_i(f g(\partial_j, \partial_k)) - g(\nabla_{\partial_i} \partial_j, \partial_k) - g(\partial_j, \nabla_{\partial_i} \partial_k) \\
&= \partial_i(f) g_{jk} + f(\nabla_{\partial_i} g)(\partial_j, \partial_k).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left(\nabla^{\mathrm{H}}_{\delta_j} g^{\mathrm{D}}\right)(\delta_k, \delta_i) &= \partial_j(f) g_{ki} + f(\nabla_{\partial_j} g)(\partial_k, \partial_i) \\
\left(\nabla^{\mathrm{H}}_{\delta_k} g^{\mathrm{D}}\right)(\delta_i, \delta_k) &= \partial_k(f) g_{ij} + f(\nabla_{\partial_k} g)(\partial_i, \partial_j).
\end{aligned}
$$

In addition, we compute the mixed and vertical components as follows:

$$
\begin{gathered}
\left(\nabla^{\mathrm{H}}_{\partial_{\bar{i}}} g^{\mathrm{D}}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = 0, \quad \left(\nabla^{\mathrm{H}}_{\delta_i} g^{\mathrm{D}}\right)(\delta_j, \partial_{\bar{k}}) = (\nabla_{\partial_i} g)(\partial_j, \partial_k), \\
\left(\nabla^{\mathrm{H}}_{\delta_j} g^{\mathrm{D}}\right)(\partial_{\bar{k}}, \delta_i) = (\nabla_{\partial_j} g)(\partial_k, \partial_i), \quad \left(\nabla^{\mathrm{H}}_{\partial_{\bar{k}}} g^{\mathrm{D}}\right)(\delta_i, \delta_j) = 0, \\
\left(\nabla^{\mathrm{H}}_{\partial_{\bar{i}}} g^{\mathrm{D}}\right)(\partial_{\bar{j}}, \delta_k) = \left(\nabla^{\mathrm{H}}_{\partial_{\bar{j}}} g^{\mathrm{D}}\right)(\delta_k, \partial_{\bar{i}}) = (\nabla^{\mathrm{H}}_{\delta_k} g^{\mathrm{D}})(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0.
\end{gathered}
$$

By the computation of the above components of the connection $\nabla^{\mathrm{H}} g^{\mathrm{D}}$, we have the following theorem.

Theorem 3.1 *Let (M, g) be a Riemannian manifold of dimension n $(n \geq 2)$, ∇ an affine connection on M, and TM the tangent bundle equipped with the deformed complete lift metric g^{D}. Then the following statements hold*:

(1) *The triple $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{H}})$ forms a Codazzi manifold if and only if the function f is constant and the metric g is compatible with the affine connection ∇. Moreover, under these conditions, the horizontal lift connection ∇^{H} is compatible with the metric g^{D}.*

(2) *If $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{H}})$ is a statistical manifold, then the function f is constant, ∇ is flat and coincides with the Levi-Civita connection ∇^g of g. Also, ∇^{H} coincides with the Levi-Civita connection ∇^{D} of the metric g^{D}.*

(3) *If $\nabla = \nabla^g$ the Levi-Civita connection of g and the function f is constant, then ∇^{H} is compatible with the metric g^{D}. In particular, if ∇ is flat, then ∇^{H} coincides with the Levi-Civita connection ∇^{D} of g^{D}.*

3.3. The components of $\nabla^{\mathbf{C}} g^{\mathbf{D}}$ on the Tangent Bundle $(TM, g^{\mathbf{D}}, \nabla^{\mathbf{C}})$

Let $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{C}})$ be the tangent bundle of an n-dimensional Riemannian manifold (M, g) equipped with the deformed complete lift metric $g^{\mathrm{D}} = g^{\mathrm{C}} + (fg)^{\mathrm{V}}$, and let ∇^{C} denote the complete lift of an affine connection ∇ on M. To understand the compatibility of ∇^{C} with the metric g^{D}, we compute the covariant derivative $\nabla^{\mathrm{C}} g^{\mathrm{D}}$ with respect to the adapted frame $\delta_i = (\partial_i)^{\mathrm{H}}$, $\partial_{\bar{i}} = (\partial_i)^{\mathrm{V}}$, $i \in \{1, \ldots, n\}$.

A direct computation shows that the component $(\nabla^{\mathrm{C}}_{\delta_i} g^{\mathrm{D}})(\delta_j, \delta_k)$ is given by

$$\begin{aligned}
(\nabla^{\mathrm{C}}_{\delta_i} g^{\mathrm{D}})(\delta_j, \delta_k) &= \delta_i g^{\mathrm{D}}(\delta_j, \delta_k) - g^{\mathrm{D}}\left(\nabla^{\mathrm{C}}_{\delta_i}\delta_j, \delta_k\right) - g^{\mathrm{D}}\left(\delta_j, \nabla^{\mathrm{C}}_{\delta_i}\delta_k\right) \\
&= (\partial_i)^{\mathrm{H}} g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, (\partial_k)^{\mathrm{H}}) - g^{\mathrm{D}}(\nabla^{\mathrm{C}}_{(\partial_i)^{\mathrm{H}}}(\partial_j)^{\mathrm{H}}, (\partial_k)^{\mathrm{H}}) - g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, \nabla^{\mathrm{C}}_{(\partial_i)^{\mathrm{H}}}(\partial_k)^{\mathrm{H}}) \\
&= (\partial_i)^{\mathrm{H}} g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, (\partial_k)^{\mathrm{H}}) - g^{\mathrm{D}}((\nabla_{\partial_i}\partial_j)^{\mathrm{H}} + (R(y, X)Y)^{\mathrm{V}}, (\partial_k)^{\mathrm{H}}) \\
&\quad - g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, (\nabla_{\partial_i}\partial_k)^{\mathrm{H}} + (R(y, \partial_i)\partial_k)^{\mathrm{V}}) \\
&= (\partial_i)^{\mathrm{H}} g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, (\partial_k)^{\mathrm{H}}) - g^{\mathrm{D}}((\nabla_{\partial_i}\partial_j)^{\mathrm{H}}, (\partial_k)^{\mathrm{H}}) - g^{\mathrm{D}}((R(y, \partial_i)\partial_j)^{\mathrm{V}}, (\partial_k)^{\mathrm{H}}) \\
&\quad - g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, (\nabla_{\partial_i}\partial_k)^{\mathrm{H}}) - g^{\mathrm{D}}((\partial_j)^{\mathrm{H}}, (R(y, \partial_i)\partial_k)^{\mathrm{V}}) \\
&= \partial_i(fg(\partial_j, \partial_k)) - fg(\nabla_{\partial_i}\partial_j, \partial_k) - g(R(y, \partial_i)\partial_j, \partial_k) \\
&\quad - fg(\partial_j, \nabla_{\partial_i}\partial_k) - g(\partial_j, R(y, \partial_i)\partial_k) \\
&= \partial_i(f)g(\partial_j, \partial_k)) + f\partial_i g(\partial_j, \partial_k) - fg(\nabla_{\partial_i}\partial_j, \partial_k) - fg(\partial_j, \nabla_{\partial_i}\partial_k) \\
&\quad - g(R(y, \partial_i)\partial_j, \partial_k) - g(\partial_j, R(y, \partial_i)\partial_k) \\
&= \partial_i(f)g_{jk} + f(\nabla_{\partial_i} g)(\partial_j, \partial_k).
\end{aligned}$$

Similarly,

$$(\nabla^{\mathrm{C}}_{\delta_j} g^{\mathrm{D}})(\delta_k, \delta_i) = \partial_j(f)g_{ki} + f(\nabla_{\partial_j} g)(\partial_k, \partial_i),$$

$$(\nabla^{\mathrm{C}}_{\delta_k} g^{\mathrm{D}})(\delta_i, \delta_k) = \partial_k(f)g_{ij} + f(\nabla_{\partial_k} g)(\partial_i, \partial_j).$$

In addition, we compute the mixed and vertical components as follows:

$$(\nabla^{\mathrm{C}}_{\partial_{\bar{i}}} g^{\mathrm{D}})(\partial_{\bar{j}}, \partial_{\bar{k}}) = 0, \quad (\nabla^{\mathrm{C}}_{\delta_i} g^{\mathrm{D}})(\delta_j, \partial_{\bar{k}}) = (\nabla_{\partial_i} g)(\partial_j, \partial_k),$$

$$(\nabla^{\mathrm{C}}_{\delta_j} g^{\mathrm{D}})(\partial_{\bar{k}}, \delta_i) = (\nabla_{\partial_j} g)(\partial_k, \partial_i), \quad (\nabla^{\mathrm{C}}_{\partial_{\bar{k}}} g^{\mathrm{D}})(\delta_i, \delta_j) = 0,$$

$$(\nabla^{\mathrm{C}}_{\partial_{\bar{i}}} g^{\mathrm{D}})(\partial_{\bar{j}}, \delta_k) = (\nabla^{\mathrm{C}}_{\partial_{\bar{j}}} g^{\mathrm{D}})(\delta_k, \partial_{\bar{i}}) = (\nabla^{\mathrm{C}}_{\delta_k} g^{\mathrm{D}})\left(\partial_{\bar{i}}, \partial_{\bar{j}}\right) = 0.$$

By the computation of the above components of the connection $\nabla^{\mathrm{C}} g^{\mathrm{D}}$, we have the following theorem.

Theorem 3.2 *Let (M, g) be a Riemannian manifold of dimension n $(n \geq 2)$, ∇ an affine connection on M, and TM the tangent bundle equipped with the deformed complete lift metric g^{D}. Then the following statements hold:*

(1) *The triple $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{C}})$ forms a Codazzi manifold if and only if the function f is constant and the metric g is compatible with the affine connection ∇. Moreover, under these conditions, the complete lift connection ∇^{C} is compatible with the metric g^{D}.*

(2) *If $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{C}})$ is a statistical manifold, then the function f is constant, ∇ is flat and coincides with the Levi-Civita connection ∇^g of g. Also, ∇^{C} coincides with the Levi-Civita connection ∇^{D} of the metric g^{D}.*

(3) *If $\nabla = \nabla^g$ the Levi-Civita connection of g and the function f is constant, then ∇^{C} is compatible with the metric g^{D}. In particular, if ∇ is flat, then ∇^{C} coincides with the Levi-Civita connection ∇^{D} of g^{D}.*

4. Infinitesimal Affine Transformation on $(TM, g^{\mathbf{D}})$

In this section, we examine the behaviour of infinitesimal affine transformations under the vertical and horizontal lifts of vector fields to the tangent bundle.

Proposition 4.1 *Let (M, g) be a flat Riemannian manifold and ∇^g the Levi-Civita connection of g. Suppose ∇^{D} denote the Levi-Civita connection of the deformed complete lift metric $g^{\mathrm{D}} = g^{\mathrm{C}} + (fg)^{\mathrm{V}}$ on the tangent bundle TM. Then*

$$(\pounds_{X^{\mathrm{V}}}\nabla^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = 0, \tag{4.1}$$

$$(\pounds_{X^{\mathrm{V}}}\nabla^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = 0, \tag{4.2}$$

$$(\pounds_{X^{\mathrm{V}}}\nabla^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = \{((\nabla^g)^2X)(Y, Z)\}^{\mathrm{V}} \tag{4.3}$$

for all vector fields $X, Y, Z \in \mathfrak{F}^1_0(M)$.

Proof: From (3.16), we have

$$(\pounds_{X^{\mathrm{V}}}\nabla^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = \pounds_{X^{\mathrm{V}}}(\nabla^{\mathrm{D}}_{Y^{\mathrm{V}}}Z^{\mathrm{V}}) - \nabla^{\mathrm{D}}_{Y^{\mathrm{V}}}(\pounds_{X^{\mathrm{V}}}Z^{\mathrm{V}}) - \nabla^{\mathrm{D}}_{[X^{\mathrm{V}}, Y^{\mathrm{V}}]}Z^{\mathrm{V}}, \tag{4.4}$$

$$(\pounds_{X^{\mathrm{V}}}\nabla^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = \pounds_{X^{\mathrm{V}}}(\nabla^{\mathrm{D}}_{Y^{\mathrm{H}}}Z^{\mathrm{V}}) - \nabla^{\mathrm{D}}_{Y^{\mathrm{H}}}(\pounds_{X^{\mathrm{V}}}Z^{\mathrm{V}}) - \nabla^{\mathrm{D}}_{[X^{\mathrm{V}}, Y^{\mathrm{H}}]}Z^{\mathrm{V}}, \tag{4.5}$$

$$(\pounds_{X^{\mathrm{V}}}\nabla^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = \pounds_{X^{\mathrm{V}}}(\nabla^{\mathrm{D}}_{Y^{\mathrm{H}}}Z^{\mathrm{H}}) - \nabla^{\mathrm{D}}_{Y^{\mathrm{H}}}(\pounds_{X^{\mathrm{V}}}Z^{\mathrm{H}}) - \nabla^{\mathrm{D}}_{[X^{\mathrm{V}}, Y^{\mathrm{H}}]}Z^{\mathrm{H}}. \tag{4.6}$$

In view of (2.3), (3.11) and (4.4), we get (4.1). Next, using (2.2), (2.3) and (3.13) in (4.5), we get (4.2). Finally, in view of (2.2), (2.3) and (3.14), (4.6) reduces to

$$(\pounds_{X^{\mathrm{V}}}\nabla^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) = (\nabla^g_Y\nabla^g_Z X - \nabla^g_{\nabla^g_Y Z}X)^{\mathrm{V}},$$

which gives (4.3). □

Theorem 4.1 *Let (M, g) be a flat Riemannian manifold with Levi-Civita connection ∇^g of g. Suppose $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{D}})$ be the tangent bundle equipped with the deformed complete lift metric and the Levi-Civita connection ∇^{D} of g^{D}. Then the vertical lift X^{V} of a vector field X on M is an infinitesimal affine transformation on $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{D}})$ if and only if vector field X is parallel.*

Proposition 4.2 *Let (M, g) be a flat Riemannian manifold and ∇^g is the Levi-Civita connection of g. Suppose ∇^{D} denote the Levi-Civita connection of the deformed complete lift metric $g^{\mathrm{D}} = g^{\mathrm{C}} + (fg)^{\mathrm{V}}$ on the tangent bundle TM. Then*

$$(\pounds_{X^{\mathrm{H}}}\nabla^{\mathrm{D}})(Y^{\mathrm{V}}, Z^{\mathrm{V}}) = 0,$$

$$(\pounds_{X^{\mathrm{H}}}\nabla^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{V}}) = -(\nabla^g_{\nabla^g_Y Z}X + \frac{1}{f^2}g(\mathrm{grad}f, X)\nabla^g_Y Z)^{\mathrm{H}},$$

$$\begin{aligned}(\pounds_{X^{\mathrm{H}}}\nabla^{\mathrm{D}})(Y^{\mathrm{H}}, Z^{\mathrm{H}}) \quad = \quad & \frac{1}{2}(\mathrm{Hess}^f(X, Y)Z + \mathrm{Hess}^f(X, Z)Y - (\pounds_X g)(Y, Z)\mathrm{grad}f \\ & + g(Y, \mathrm{grad}f)\nabla^g_Z X + g(Z, \mathrm{grad}f)\nabla^g_Y X + g(\nabla^g_Z X, \mathrm{grad}f)Y \\ & + g(\nabla^g_Y X, \mathrm{grad}f)Z - g(Y, Z)\nabla^g_X \mathrm{grad}f)^{\mathrm{V}} \\ & + \{((\nabla^g)^2X)(Y, Z)\}^{\mathrm{H}}\end{aligned}$$

for all vector fields $X, Y, Z \in \mathfrak{F}^1_0(M)$.

Proof: The proof is similar to that of Proposition 4.1. □

Theorem 4.2 *Let* (M, g) *be a flat Riemannian manifold with Levi-Civita connection* ∇^g *of* g. *Suppose* (TM, g^{D}) *be the tangent bundle equipped with the deformed complete lift metric and the Levi-Civita connection* ∇^{D} *of* g^{D}. *Then the horizontal lift* X^{H} *of a vector field* X *on* M *is an infinitesimal affine transformation on* $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{D}})$ *if and only if the function* f *is constant, the vector field* X *is parallel.*

Next, we investigate the conditions under which the triple $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{D}})$ defines a statistical manifold. By direct computation, we obtain the following:

$$\begin{aligned}
&(\nabla^{\mathrm{D}}_{\delta_i} g^{\mathrm{D}})(\delta_j, \delta_k) = 0, && (\nabla^{\mathrm{D}}_{\partial_i} g^{\mathrm{D}})(\partial_j, \partial_k) = 0,\\
&(\nabla^{\mathrm{D}}_{\partial_{\bar{i}}} g^{\mathrm{D}})(\partial_{\bar{j}}, \delta_k) = 0, && (\nabla^{\mathrm{D}}_{\partial_{\bar{j}}} g^{\mathrm{D}})(\delta_k, \partial_{\bar{i}}) = 0,\\
&(\nabla^{\mathrm{D}}_{\partial_{\bar{k}}} g^{\mathrm{D}})(\delta_i, \delta_j) = 0, && (\nabla^{\mathrm{D}}_{\delta_k} g^{\mathrm{D}})(\partial_{\bar{i}}, \partial_{\bar{j}}) = -\tfrac{1}{f} g(\mathrm{grad} f, \partial_k).
\end{aligned}$$

By the computation of the above components of the connection $\nabla^{\mathrm{D}} g^{\mathrm{D}}$, we have the following theorem.

Theorem 4.3 *Let* (M, g) *be a flat Riemannian manifold and* ∇^g *be the Levi-Civita connection on* M. *Suppose* (TM, g^{D}) *be the tangent bundle equipped with the deformed complete lift metric and* ∇^{D} *be the Levi-Civita connection of* g^{D}. *Then,* $(TM, g^{\mathrm{D}}, \nabla^{\mathrm{D}})$ *be the statistical manifold if and only the gradient of each component of the metric* g *vanishes, that is,*

$$\mathrm{grad}(g_{ij}) = 0 \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

5. Geodesics of the Deformed Complete Lift Metric

Let (M, g) be a Riemannian manifold and $x : I \to M$ be a curve on M. Define a curve $\gamma : I \to TM$ for all $t \in I$, $\gamma(t) = (x(t), y(t))$, where $y(t) \in T_{x(t)}M$, that is, $y(t)$ is a vector field along $x(t)$.

Definition 5.1 ([10, Salimov and Kazimova 2009], [14, Yano and Ishihara 1973]) Let (M, g) be a Riemannian manifold, if $x(t)$ is a curve on M. The curve $\gamma(t) = (x(t), \dot{x}(t))$ is called the natural lift of the curve $x(t)$.

Definition 5.2 (cf. [14, Yano and Ishihara 1973]) Let (M, g) be a Riemannian manifold and ∇ be an affine connection of M. A curve $\gamma(t) = (x(t), y(t))$ is said to be a horizontal lift of the curve $x(t)$ in M if and only if

$$\nabla_{\dot{x}(t)} y(t) = 0,$$

where $y(t)$ is a vector field along $x(t)$.

Lemma 5.1 [15, Zagane and Djaa 2017] *Let* (M, g) *be a Riemannian manifold and* ∇^g *denote the Levi-Civita connection of* g. *If* $x(t)$ *be a curve on* M *and* $\gamma(t) = (x(t), y(t))$ *be a curve on* TM, *then*

$$\dot{\gamma}(t) = \dot{x}^{\mathrm{H}} + (\nabla_{\dot{x}} y)^{\mathrm{V}}. \tag{5.1}$$

Lemma 5.2 *Let* (M, g) *be a Riemannian manifold and* (TM, g^{D}) *its tangent bundle equipped with the deformed complete lift metric. If* ∇^g *(resp.* ∇^{D}*) denote the Levi-Civita connection of* (M, g) *(resp.* (TM, g^{D})*) and* $\gamma(t) = (x(t), y(t))$ *is the curve on* TM *such* $y(t)$ *is a vector field along* $x(t)$, *then*

$$\nabla^{\mathrm{D}}_{\dot{\gamma}} \dot{\gamma} = (\nabla_{\dot{x}} \dot{x})^{\mathrm{H}} + \frac{1}{2}(2\dot{x}(f)\dot{x} - g(\dot{x}, \dot{x})\mathrm{grad} f + 2\nabla_{\dot{x}} \nabla_{\dot{x}} y + 2R(y, \dot{x})\dot{x})^{\mathrm{V}}. \tag{5.2}$$

Proof: From (5.1), we have

$$\begin{aligned}
\nabla^{\mathrm{D}}_{\dot{\gamma}} \dot{\gamma} &= \nabla^{\mathrm{D}}_{(\dot{x}^{\mathrm{H}} + (\nabla_{\dot{x}} y)^{\mathrm{V}})} (\dot{x}^{\mathrm{H}} + (\nabla_{\dot{x}} y)^{\mathrm{V}})\\
&= \nabla^{\mathrm{D}}_{\dot{x}^{\mathrm{H}}} \dot{x}^{\mathrm{H}} + \nabla^{\mathrm{D}}_{\dot{x}^{\mathrm{H}}} (\nabla_{\dot{x}} y)^{\mathrm{V}} + \nabla^{\mathrm{D}}_{(\nabla_{\dot{x}} y)^{\mathrm{V}}} \dot{x}^{\mathrm{H}} + \widetilde{\nabla}_{(\nabla_{\dot{x}} y)^{\mathrm{V}}} (\nabla_{\dot{x}} y)^{\mathrm{V}}.
\end{aligned} \tag{5.3}$$

In view of Lemma 3.2 and (5.3), we obtain (5.2). □

Theorem 5.1 *Let (M,g) be a Riemannian manifold and $(TM,\widetilde{G})$ its tangent bundle equipped with the deformed complete lift metric. If $\gamma(t)=(x(t),y(t))$ is a curve on TM such $y(t)$ is a vector field along $x(t)$, then $\gamma(t)$ is a geodesic on TM if and only if*

$$\begin{cases} \nabla_{\dot{x}}\dot{x}=0, \\ \dot{x}(f)\dot{x}+\nabla_{\dot{x}}\nabla_{\dot{x}}y+R(y,\dot{x})\dot{x}-\frac{1}{2}g(\dot{x},\dot{x})\mathrm{grad}f=0. \end{cases} \tag{5.4}$$

Proof: This follows directly from Lemma 5.2. □

Corollary 5.1 *Let (M,g) be a Riemannian manifold and $(TM,\widetilde{G})$ its tangent bundle equipped with the deformed complete lift metric. The natural lift $\gamma(t)=(x(t),\dot{x}(t))$ of any unit speed geodesic $x(t)$ is a geodesic on TM if and only if* $\mathrm{grad}f=0$.

Example 5.1 *Consider a Riemannian metric on $\mathbb{R}^2$ given by*

$$g=e^x(dx^2+dy^2).$$

The non-zero Christoffel symbols of the Levi-Civita connection associated with the Riemannian metric g are

$$\Gamma^1_{11}=\frac{1}{2},\quad \Gamma^1_{22}=-\frac{1}{2},\quad \Gamma^2_{12}=\Gamma^1_{21}=\frac{1}{2}.$$

Let $\alpha(t)=(x(t),y(t))$ be the unit speed geodesic with initial conditions

$$x(0)=a,\quad y(0)=b,\quad \dot{x}(0)=u,\quad \dot{y}(0)=v.$$

The equations of geodesic are

$$\begin{cases} \ddot{x}+\frac{1}{2}((\dot{x})^2-(\dot{y})^2)=0, \\ \ddot{y}+\dot{x}\dot{y}=0. \end{cases}$$

Then, the natural lift on $T\mathbb{R}^2$ is

$$\gamma(t)=\left(a+2\ln\left(u+\frac{1}{2}t\right),b+\frac{2v}{u}-\left(\frac{4v}{2u+t}\right),\frac{2}{2u+t},\frac{4v}{(2u+t)^2}\right).$$

In this example, the tangent bundle $T\mathbb{R}^2$ is equipped with the deformed complete lift metric. It follows from Corollary 5.1 that the natural lift $\gamma(t)=(\alpha(t),\dot{\alpha}(t))$ of the unit speed geodesic $\alpha(t)$ is a geodesic on $T\mathbb{R}^2$ if and only if $\mathrm{grad}f=0$, *that is, $\gamma(t)$ is a geodesic on $T\mathbb{R}^2$ only when the deformation function f is constant. For any non-constant function f, the natural lift $\gamma(t)$ fails to satisfy the geodesic equation on $T\mathbb{R}^2$.*

Example 5.2 *Let $M=\mathbb{R}^n$ be a Riemannian manifold equipped with the Euclidean metric*

$$\overline{g}=(dx^1)^2+(dx^2)^2+\cdots+(dx^n)^2,$$

and $TM\cong\mathbb{R}^n\times\mathbb{R}^n$ its tangent bundle endowed with the deformed complete lift metric g^{D}.
Let $f=f(x)$ be a non-zero smooth function on M and let

$$x(t)=(x^1(t),\ldots,x^n(t))$$

be a smooth curve in M. Its natural lift to TM is defined by

$$\gamma(t)=(x(t),\dot{x}(t)).$$

The velocity vector of the curve γ in TM decomposes as (see Lemma 5.1)

$$\dot{\gamma}=\dot{x}^{\mathrm{H}}+\ddot{x}^{\mathrm{V}}.$$

We define the action functional

$$S = \frac{1}{2}\int g^{\mathrm{D}}(\dot{\gamma},\dot{\gamma})\,dt = \frac{1}{2}\int f(x)\|\dot{x}\|^2\,dt + \frac{1}{2}\int \frac{d}{dt}\left(\|\dot{x}\|^2\right)dt. \tag{5.5}$$

Consider the Lagrangian

$$L = \frac{1}{2}g^{\mathrm{D}}(\dot{\gamma},\dot{\gamma}) + \frac{1}{2}\frac{d}{dt}\left(\|\dot{x}\|^2\right).$$

We consider smooth variations of the base curve with fixed endpoints. Hence, any total derivative appearing in the Lagrangian contributes only a boundary term and does not affect the Euler-Lagrange equations. Therefore, the variational problem is equivalent to that determined by the reduced Lagrangian

$$L = \frac{1}{2}f(x)\|\dot{x}\|^2.$$

Next, we have the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) - \frac{\partial L}{\partial x^k} = 0, \quad k \in \{1,\dots,n\},$$

from which we obtain

$$f\ddot{x}^k + (\partial_m f)\dot{x}^m\dot{x}^k - \frac{1}{2}(\partial_k f)\|\dot{x}\|^2 = 0.$$

In vector form this equation becomes

$$f\ddot{x} + \overline{g}(\overline{\nabla}f,\dot{x})\dot{x} - \frac{1}{2}(\overline{\nabla}f)\|\dot{x}\|^2 = 0. \tag{5.6}$$

To test Corollary 5.1, we assume that $x(t)$ is a unit speed geodesic in M. Then, from (5.6), we obtain

$$\overline{g}(\overline{\nabla}f,\dot{x})\dot{x} - \frac{1}{2}(\overline{\nabla}f) = 0.$$

This implies that $\overline{\nabla}f = 0$. From this, in view of (5.5), it follows that $\gamma(t)$ is a geodesic in TM. Conversely, suppose $\overline{\nabla}f = 0$, and if $x(t)$ is a unit speed curve in M, then from (5.6), it follows that $x(t)$ is a geodesic in M.

Thus the natural lift $\gamma(t)$ of a unit speed geodesic $x(t)$ is a geodesic on the tangent bundle (TM, g^{D}) if and only if the deformation function f is constant. This example illustrates the general principle stated in Corollary 5.1.

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