



On the Upper Bounds of Hankel Determinant of q -Starlike Functions Linked to the Bernoulli Lemniscate

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ABSTRACT: Let $\mathcal{SL}_q^*(\varpi, \varphi)$ denote the subclass of analytic functions defined in the open unit disk $\mathbb{U} = \{t \in \mathbb{C} : |t| < 1\}$, normalized by $h(0) = 0$ and $h'(0) = 1$, which satisfy certain subordination conditions

$$\frac{t D_q(\mathcal{R}_q^\varpi(h(t)))}{\mathcal{R}_q^\varpi(h(t))} \prec \left(\frac{2(1+t)}{2+(1-q)t} \right)^{\frac{1}{2}},$$

where \prec is the subordination relation. This class serves as a q -analogue of starlike functions associated with the parameter ϖ . In this paper, we investigate coefficient-related problems for functions in $\mathcal{SL}_q^*(\varpi, \varphi)$. In particular, we derive sharp bounds for the initial Taylor coefficients, the Fekete–Szegő functional, and the second Hankel determinant.

Keywords: Analytic functions, Hankel determinant, subordination, q -difference operator, q -Ruscheweyh derivative operator.

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{t \in \mathbb{C} : |t| < 1\},$$

where \mathbb{C} is the set of complex numbers. Let \mathcal{A} be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form

$$h(t) = t + \sum_{\tau=2}^{\infty} b_{\tau} t^{\tau}, \quad (t \in \mathbb{U}), \tag{1.1}$$

normalized by the conditions $h(0) = 0$ and $h'(0) = 1$. Also, let $\mathcal{S} \subset \mathcal{A}$ denote the class of univalent functions in \mathbb{U} . For analytic functions $h, g \in \mathbb{U}$, we write $h \prec g$ if there exists a Schwarz function w with $w(0) = 0$, $|w(t)| < 1$ ($t \in \mathbb{U}$), such that

$$h(t) = g(w(t)), \quad t \in \mathbb{U}.$$

If g is univalent in \mathbb{U} , then

$$h \prec g \iff h(0) = g(0) \text{ and } h(\mathbb{U}) \subset g(\mathbb{U}).$$

The class of starlike functions in \mathbb{U} is denoted by \mathcal{S}^* , which consists of functions $h \in \mathcal{A}$ given by

$$\mathcal{S}^* = \left\{ h \in \mathcal{A} : \Re \left(\frac{th'(t)}{h(t)} \right) > 0, t \in \mathbb{U} \right\}.$$

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Moreover, let \mathcal{SL}^* denote the class of functions $h \in \mathcal{A}$ such that

$$\mathcal{SL}^* := S^*(\sqrt{1+t}) = \left\{ h \in \mathcal{A} : \left| \left(\frac{th'(t)}{h(t)} \right)^2 - 1 \right| < 1, t \in \mathbb{U} \right\}$$

is the class of *lemniscate starlike functions* studied in [48]. Equivalently, a function $h \in \mathcal{SL}^*$ is characterized by the condition that $\frac{th'(t)}{h(t)}$ takes values in the region bounded by the right half of the lemniscate of Bernoulli as shown in figure 1, given by $|w^2 - 1| < 1$. Rønning [44] introduced *parabolic starlike functions*

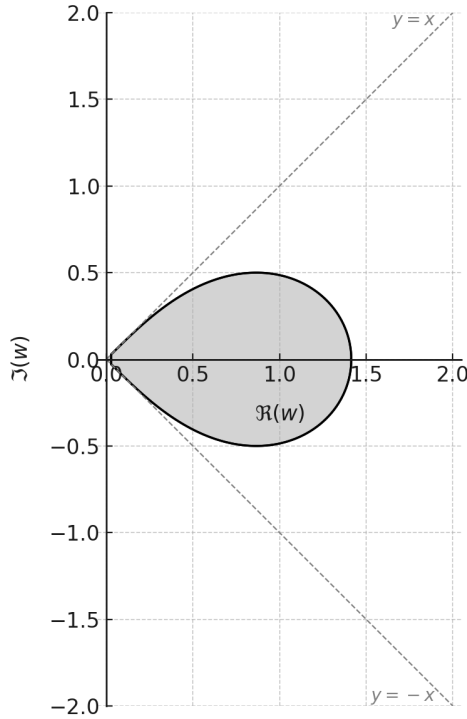


Figure 1: Right-half of the Bernoulli lemniscate region $\{w : |w^2 - 1| < 1, \Re(w) \leq 2\}$ with asymptotes $y = \pm x$.

defined as

$$\mathcal{S}_p^* := S^*(\varphi_{\text{PAR}}) = \left\{ h \in \mathcal{A} : \left| \frac{th'(t)}{h(t)} - 1 \right| < \Re \left(\frac{th'(t)}{h(t)} \right), t \in \mathbb{U} \right\},$$

where

$$\varphi_{\text{PAR}}(h) := 1 + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{h}}{1 - \sqrt{h}} \right)^2.$$

Interpreting this condition geometrically, we see that $\frac{th'(t)}{h(t)}$ lies inside the parabola

$$(\Im mw)^2 < 2\Re ew - 1,$$

which shows that the class of k -starlike functions is closely related to certain conic domains. For more details on k -starlike functions, we refer to [50]. For $0 < \beta \leq 1$, the class

$$S_\beta^* := S^* \left(\left(\frac{1+t}{1-t} \right)^\beta \right) = \left\{ f \in \mathcal{A} : \left| \arg \frac{th'(t)}{h(t)} \right| < \frac{\beta\pi}{2} \right\}$$

is the familiar class of *strongly starlike functions* of order β .

Ma and Minda [31] introduced the function $\frac{1+t}{1-t}$ with a more general analytic function φ which satisfies the conditions $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps \mathbb{U} onto a region that is univalent, starlike with respect to 1, and symmetric with respect to the real axis which is defined as follows

$$\mathcal{S}^*(\varphi) = \left\{ h \in \mathcal{A} : \frac{th'(t)}{h(t)} \prec \varphi(t), t \in \mathbb{U} \right\}.$$

For more information on $\varphi(t)$ and the associated subclasses of $\mathcal{S}^*(\varphi)$, see the following table 1.

Table 1: Various subclasses of $\mathcal{S}^*(\varphi)$

Authors	Class	$\varphi(t)$	Reference
Brannan and Kirwan	\mathcal{S}_β^*	$\left(\frac{1+t}{1-t}\right)^\beta, (0 < \beta \leq 1)$	[8]
Cho et al.	\mathcal{S}_s^*	$1 + \operatorname{sint}$	[10]
Deniz	$\mathcal{S}_T^*(\lambda)$	$e^{t + \frac{\lambda}{2}t^2}$	[11]
Janowski	$\mathcal{S}^*[A, B]$	$\frac{1 + At}{1 + Bt}, (-1 \leq B < A \leq 1)$	[15]
Goel and Kumar	\mathcal{S}_{SG}^*	$\frac{1}{(1 + e^{-t})}$	[13]
Masih et al.	$\mathcal{S}_L^*(\lambda)$	$(1 + t)^\lambda, (0 < \lambda < 1)$	[28]
Mendiretta et al.	\mathcal{S}_e^*	e^t	[29]
Kumar et al.	\mathcal{S}_B^*	$e^{e^{-t-1}}$	[24]
Kumar and Arora	$\mathcal{S}_{sinh^{-1}}^*$	$1 + \sinh^{-1}t$	[23]
Sharma et al.	\mathcal{S}_{car}^*	$1 + \frac{4}{3}t + \frac{2}{3}t^2$	[45]
Sokół	$\mathcal{S}_{q_c}^*$	$\sqrt{ct + 1}, (0 < c \leq 1)$	[49]
Sokół and Stankiewicz	$\mathcal{S}\mathcal{L}^*$	$\sqrt{t + 1},$	[48]
Raina and Sokół	\mathcal{S}_q^*	$\sqrt{1 + t^2} + t,$	[41]
Robertson	\mathcal{S}_α^*	$\frac{1 + (1 - 2\alpha)t}{1 - t}, (\alpha < 1)$	[43]
Paprocki and Sokół	$\mathcal{S}_{q_\alpha}^*$	$\left(\frac{1 + t}{1 + (1 - b)/bt}\right)^{1/\alpha}$	[38]
Wani and Swaminathan	\mathcal{S}_{Ne}^*	$1 + t - \frac{t^3}{3}$	[58]

We recall some basic concepts of q -calculus that will be used in this paper. Throughout, we assume $0 < q < 1$ and adopt the notations

$$\mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \setminus \{0\}, \quad \mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

For $0 < q < 1$, the q -number $[\lambda]_q$ is defined by

$$[\lambda]_q = \begin{cases} \frac{1 - q^\lambda}{1 - q}, & \lambda \in \mathbb{C}, \\ \sum_{\tau=0}^{n-1} q^\tau = 1 + q + q^2 + \dots + q^{n-1}, & \lambda = n \in \mathbb{N}. \end{cases}$$

Let $0 < q < 1$, the q -derivative (q -difference) operator of a complex valued function defined on a subset of \mathbb{C} by Jackson [17,18], is defined as

$$(D_q h)(t) = \begin{cases} \frac{h(t) - h(qt)}{(1 - q)t}, & t \neq 0, \\ h'(0), & t = 0, \end{cases} \quad (1.2)$$

provided that $h'(0)$ exists. It is clear that $D_q h(0) = h'(0)$ and $D_q^2 h(t) = D_q(D_q h(t))$. From (1.2), we obtain

$$D_q h(t) = 1 + \sum_{\tau=2}^{\infty} [\tau]_q b_{\tau} t^{\tau-1}, \quad (1.3)$$

where

$$[\tau]_q = \frac{1 - q^{\tau}}{1 - q}. \quad (1.4)$$

As $q \rightarrow 1^-$, it follows that $[\tau]_q \rightarrow \tau$. For a function $h(t) = t^{\tau}$, we have

$$D_q(h(t)) = D_q(t^{\tau}) = \frac{1 - q^{\tau}}{1 - q} t^{\tau-1} = [\tau]_q t^{\tau-1}.$$

Taking the limit as $q \rightarrow 1^-$, we deduce

$$\lim_{q \rightarrow 1^-} (D_q h(t)) = \lim_{q \rightarrow 1^-} ([\tau]_q t^{\tau-1}) = \tau t^{\tau-1} = h'(t),$$

where h' denotes the usual derivative.

As a right inverse, Jackson [17,18] introduced the q -integral

$$\int_0^z h(t) d_q t = z(1 - q) \sum_{\tau=0}^{\infty} q^{\tau} h(zq^{\tau}),$$

provided the series converges. For $h(t) = t^{\tau}$, we obtain

$$\int_0^z h(t) d_q t = \int_0^z t^{\tau} d_q t = \frac{z^{\tau+1}}{[\tau+1]_q}, \quad (\tau \neq -1).$$

Furthermore,

$$\lim_{q \rightarrow 1^-} \int_0^z h(t) d_q t = \lim_{q \rightarrow 1^-} \frac{z^{\tau+1}}{[\tau+1]_q} = \frac{z^{\tau+1}}{\tau+1} = \int_0^z h(t) dt,$$

where $\int_0^z h(t) dt$ is ordinary integral. It can be easily verified that,

$$D_q \left(\sum_{\tau=1}^{\infty} b_{\tau} t^{\tau} \right) = \sum_{\tau=1}^{\infty} [\tau]_q b_{\tau} t^{\tau-1},$$

where

$$[\tau]_q = \frac{1 - q^{\tau}}{1 - q} = 1 + q + q^2 + \cdots + q^{\tau-1}, \quad [0]_q = 0$$

and

$$(x; q)_{\tau} = \prod_{i=0}^{\tau-1} (1 - xq^i) = \sum_{i=0}^{\tau-1} q^i, \quad (\tau \in \mathbb{N}), \quad x \in \mathbb{C},$$

with

$$(x; q)_0 = 1,$$

The q -calculus provides powerful tools that have been widely applied in the study of various subclasses of \mathcal{A} . Specifically, Srivastava and Bansal [51] investigated the close-to-convexity of q -Mittag-Leffler functions, while Srivastava et al. [52] introduced generalized subclasses of q -starlike functions associated with Janowski functions. Mahmood et al. [32] further studied q -starlike functions in the conic domain. For recent developments on the applications of q -calculus in Geometric Function Theory, see [1,5,33,53,55,57]. Khan et al. [20] recently employed the q -derivative operator to introduce a new subclass of starlike functions associated with the Bernoulli lemniscate, defined by

$$\mathcal{SL}_q^* := \left\{ h \in \mathcal{A} : \frac{t(D_q h)(t)}{h(t)} \prec \sqrt{\frac{2(1+t)}{2+(1-q)t}}, \quad t \in \mathbb{D} \right\}.$$

Equivalently, a function $h \in \mathcal{A}$ belongs to \mathcal{SL}_q^* if and only if it satisfies

$$\left| \left(\frac{t(D_q h)(t)}{h(t)} \right)^2 - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

By setting $\omega = t(D_q h)(t)/h(t)$, the analytic condition of the class \mathcal{SL}_q^* can be rewritten as

$$\left| \omega^2 - \frac{1}{1-q} \right| < \frac{1}{1-q},$$

which describes the interior of the right loop of the Bernoulli lemniscate. A notable feature of this class is that it reduces to the classical class \mathcal{SL}^* as $q \rightarrow 1^-$. Next, we recall the q -analogue of the Ruscheweyh operator introduced by Aldweby and Darus [2], which will be used throughout this paper.

Definition 1.1 Let $h \in \mathcal{A}$. The q -Ruscheweyh operator of order ϖ is defined by

$$\mathcal{R}_q^\varpi h(t) = t + \sum_{\tau=2}^{\infty} \frac{[\tau + \varpi - 1]_q!}{[\varpi]_q! [\tau - 1]_q!} a_\tau t^\tau, \quad (\varpi > -1),$$

where $[\tau]_q!$ is given by

$$[\tau]_q! = \begin{cases} [\tau]_q [\tau - 1]_q \cdots [1]_q, & \tau = 1, 2, \dots, \\ 1, & \tau = 0. \end{cases}$$

From the definition we observe that if $q \rightarrow 1$, we have

$$\lim_{q \rightarrow 1} \mathcal{R}_q^\varpi h(t) = t + \lim_{q \rightarrow 1} \left[\sum_{\tau=2}^{\infty} \frac{[\tau + \varpi - 1]_q!}{[\varpi]_q! [\tau - 1]_q!} b_\tau t^\tau \right] = t + \sum_{\tau=2}^{\infty} \frac{(\tau + \varpi - 1)!}{(\varpi)! (\tau - 1)!} b_\tau t^\tau = \mathcal{R}^\varpi h(t),$$

where \mathcal{R}^ϖ is the Ruscheweyh differential operator defined in [42].

Utilizing the concept of subordination and q -Ruscheweyh differential operator, we introduce a new Ma-Minda type class $\mathcal{SL}_q^*(\varpi, \varphi)$ of starlike functions as follows

Definition 1.2 Let $\varpi > -1$. A function $h \in \mathcal{A}$, is said to be in the class $\mathcal{SL}_q^*(\varpi, \varphi)$, if the following subordination condition holds true:

$$\frac{t D_q (\mathcal{R}_q^\varpi (h(t)))}{\mathcal{R}_q^\varpi (h(t))} \prec \left(\frac{2(1+t)}{2+(1-q)t} \right)^{\frac{1}{2}}, \quad (t \in \mathbb{U}). \quad (1.3)$$

For certain choices of the parameter ϖ and in the limit $q \rightarrow 1^-$, the corresponding well-known classes are given by:

Remark 1.1 For $\varpi = 0$ and $q \rightarrow 1^-$, we conclude that

$$\begin{aligned} \mathcal{S}_e^*(0) &\equiv \mathcal{S}_e^* \text{ [29]}, & \mathcal{SL}^*(0) &\equiv \mathcal{SL}^* \text{ [48]}, & \mathcal{S}_s^*(0) &\equiv \mathcal{S}_s^* \text{ [10]}, \\ \mathcal{S}_{Ne}^*(0) &\equiv \mathcal{S}_{Ne}^* \text{ [58]}, & \mathcal{S}_{SG}^*(0) &\equiv \mathcal{S}_{SG}^* \text{ [13]}. \end{aligned}$$

Figure 2 demonstrates plot of φ for different q -values. For $q \rightarrow 1^-$ we get $\varphi = \sqrt{1+t}$ which is the standard square-root curve.

In 1966, Pommerenke [39] introduced the r^{th} Hankel determinant associated with the Taylor coefficients of a function $h \in \mathcal{A}$ for integers $s, n \in \mathbb{N} = \{1, 2, 3, \dots\}$. It is expressed as

$$\Delta_{s,n}(h) = \begin{vmatrix} \omega_n & \omega_{n+1} & \cdots & \omega_{n+r-1} \\ \omega_{n+1} & \omega_{n+2} & \cdots & \omega_{n+r} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n+r-1} & \omega_{n+r} & \cdots & \omega_{n+2r-2} \end{vmatrix}, \quad (\omega_1 = 1).$$

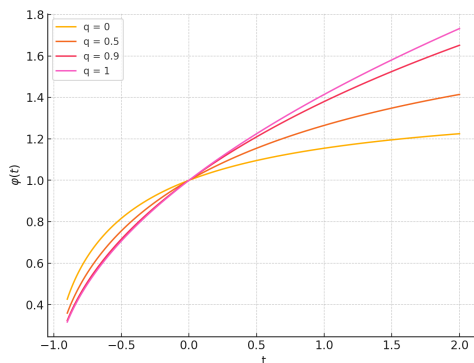


Figure 2: Graph of $\varphi(t)$ for different values of $q = 0, 0.5, 0.9, 1$

The Hankel determinants

$$\Delta_{2,1}(h) = \omega_3 - \omega_2^2, \quad \Delta_{2,2}(h) = \omega_2\omega_4 - \omega_3^2,$$

are widely recognized as the Fekete–Szegő and second Hankel determinant, respectively. For further results concerning this functional, we refer the reader to [3,4]. In 1969, Keogh and Merkes [21] solved the Fekete–Szegő problem for the classes \mathcal{S}^* and \mathcal{K} . Recently, Zaprawa [59,60] investigated the Fekete–Szegő problem for certain subclasses of bi-univalent functions. Furthermore, the generalized Fekete–Szegő functional is defined as $a_3 - \mu a_2^2$, where μ may be real or complex. According to [40], the Hankel determinants of univalent functions satisfy the inequality

$$H_r(\tau) < K\tau^{-\left(\frac{1}{2}+\beta\right)r+\frac{3}{2}}, \quad (\tau = 1, 2, \dots; r = 2, 3, \dots).$$

where $\beta > \frac{1}{4000}$ and K depends only on r . Subsequently, Hayman [14] established that

$$|H_2(\tau)| < A\tau^{1/2}, \quad (\tau = 1, 2, \dots; A \text{ an absolute constant}),$$

for the class of real mean univalent functions. Noor investigated the Hankel determinant for Bazilevic functions in [35], as well as for functions with bounded boundary rotation in [36]. However, for the class of Bazilevic functions, Krishna et al. [22] established the best possible estimate for $H_{2,2}(h)$. Janteng et al. [15,16] determined the absolute sharp bound of the functional $H_{2,2}(h) = |a_2a_4 - a_3^2|$ for each of the subclasses \mathcal{C} , \mathcal{S}^* , and \mathcal{R} . Lee et al. [25] established bounds for the second Hankel determinants of the Ma–Minda starlike and convex subclasses. Bansal [7], examined bounds for a certain class of analytic functions on $\mathcal{H}_2(2)$. Srivastava et al. [54] investigated coefficient estimates for subclasses of bi- λ -convex and bi- μ -starlike functions of the Ma–Minda type. Orhan et al. [37] studied inequalities involving the second Hankel determinant for classes of analytic and bi-univalent functions. On the other hand Çağlar et al. [9] obtained upper bound for bi-univalent functions. Shrigan [46,47] employed the q -differential operator to investigate the second Hankel determinant for the class of bi-univalent functions. Srivastava et al. [55] considered the class of q -starlike functions associated with the conic domain, while in [34], the upper bound of the third Hankel determinant for the class of q -starlike functions was obtained. Furthermore, Ayinla and Opoola [6] introduced the class $\mathcal{C}_\varphi(\eta, \nu)$ and employed the Sălăgean derivative operator to derive inequalities for the Fekete–Szegő functional and the second Hankel determinant. More recently, in 2022, Srivastava et al. [56] derived coefficient bounds and upper estimates for the Fekete–Szegő functional and the Hankel determinant for a class of analytic functions involving the Hohlov operator.

Ma [30] introduced the generalized Zalcman functional, defined by $J_{n,m}(f) = a_n a_m - a_{m+n-1}$, $n, m \in \mathbb{N} \setminus \{1\}$. A particular instance of this is $J_{2,3}(f) = a_2 a_3 - a_4$, which represents a specific case of the generalized Zalcman functional.

In this work, we establish upper bounds for the initial Taylor coefficients $|b_2|$, $|b_3|$, and $|b_4|$, together with the Fekete–Szegő functional $|b_3 - \mu b_2^2|$, for the subclass of analytic functions $\mathcal{S}\mathcal{L}_q^*(\varpi, \varphi)$ defined in the open unit disk. Also, we provide an upper bound for the second Hankel determinant $|\Delta_2(2)|$, particularly when h belongs to the lemniscate of Bernoulli domain.

2. A Set of Lemmas

To establish our main results, we make use of the following auxiliary lemmas.

Lemma 1. [26,27] Let

$$p(h) = 1 + l_1 h + l_2 h^2 + \dots$$

be in the class \mathcal{P} of functions with positive real part in \mathbb{U} . Then

$$2l_2 = l_1^2 + (4 - l_1^2)x, \quad \text{for some } x, |x| \leq 1,$$

and

$$4l_3 = l_1^3 + 2(4 - l_1^2)l_1 x - (4 - l_1^2)l_1 x^2 + 2(4 - l_1^2)(1 - |x|^2)h, \quad \text{for some } h, |h| \leq 1.$$

Lemma 2. [12] Let

$$p(h) = 1 + l_1 h + l_2 h^2 + \dots$$

be in the class \mathcal{P} of functions with positive real part in \mathbb{U} . Then

$$|l_k| \leq 2 \quad (k \in \mathbb{N}),$$

and the inequality is sharp.

Lemma 3. [31] Let

$$p(z) = 1 + l_1 z + l_2 z^2 + \dots$$

be in the class \mathcal{P} of functions with positive real part in \mathbb{U} . Then for any complex number ν ,

$$|l_2 - \nu l_1^2| \leq \begin{cases} -4\nu + 2, & \nu \leq 0, \\ 2, & 0 \leq \nu \leq 1, \\ 4\nu - 2, & \nu \geq 1. \end{cases}$$

3. Main Results

Theorem 3.1 If $h \in \mathcal{SL}_q^*(\varpi, \varphi)$, where h is of the form (1.1), then

$$|b_2| \leq \frac{1+q}{4q[\varpi+1]_q}, \quad (3.1)$$

$$|b_3| \leq \frac{3q^3 - q + 2}{32q^2[\varpi+1]_q[\varpi+2]_q}, \quad (3.2)$$

$$|b_4| \leq \frac{5q^7 - 6q^6 - 3q^5 + 6q^4 - q^3 - 2q^2 - q + 2}{128q^5[\varpi+1]_q[\varpi+2]_q[\varpi+3]_q}. \quad (3.3)$$

Proof: Let $h \in \mathcal{SL}_q^*(\varpi, \varphi)$, then

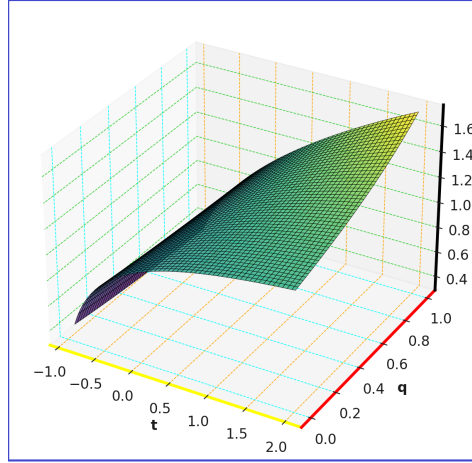
$$\frac{t D_q(\mathcal{R}_q^\varpi(h(t)))}{\mathcal{R}_q^\varpi(h(t))} \prec \varphi(t), \quad (t \in \mathbb{U}). \quad (3.4)$$

where

$$\varphi(t) = \left(\frac{2(1+t)}{2+(1-q)t} \right)^{\frac{1}{2}}.$$

Define a function

$$s(t) = \frac{1+v(t)}{1-v(t)} = 1 + l_1 t + l_2 t^2 + \dots,$$

Figure 3: The 3D plot of the function $\varphi(t)$

where v is a Schwarz function and it is clear that $s \in \mathcal{P}$. This implies that

$$v(t) = \frac{s(t) - 1}{s(t) + 1} = \frac{l_1 t + l_2 t^2 + l_3 t^3 + \dots}{2 + l_1 t + l_2 t^2 + l_3 t^3 + \dots}. \quad (3.5)$$

From (3.4) we have

$$\frac{t D_q(\mathcal{R}_q^\varpi(h(t)))}{\mathcal{R}_q^\varpi(h(t))} \prec \varphi(t), \quad (t \in \mathbb{U}),$$

with

$$\varphi(v(t)) = \left(\frac{4s}{(1+q) + (3-q)s} \right)^{\frac{1}{2}}.$$

It is evident that s is analytic in the unit disk \mathbb{U} , satisfies $s(0) = 1$, and $\Re\{s(t)\} > 0$. By using (3.5), we obtain

$$\begin{aligned} \left(\frac{4s}{(1+q) + (3-q)s} \right)^{\frac{1}{2}} &= 1 + \frac{(1+q)}{8} l_1 t + \frac{(1+q)}{128} [16l_2 + (3q-13)l_1^2] t^2 \\ &+ \frac{(1+q)}{1024} [128l_3 + (48q-208)l_1 l_2 + (5q^2 - 38q + 85)l_1^3] t^3 + \dots \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} \frac{t D_q(\mathcal{R}_q^\varpi h(t))}{\mathcal{R}_q^\varpi h(t)} &= 1 + q[\varpi + 1]_q b_2 t + (q[\varpi + 1]_q [\varpi + 2]_q b_3 - q[\varpi + 1]_q^2 b_2^2) t^2 \\ &+ \left(q[\varpi + 1]_q [\varpi + 2]_q [\varpi + 3]_q b_4 - \left(\frac{q^2 - q + 1}{q} \right) [\varpi + 1]_q^2 [\varpi + 2]_q b_2 b_3 + q[\varpi + 1]_q^3 b_2^3 \right) t^3 + \dots \end{aligned} \quad (3.7)$$

From (3.6), (3.7), and (3.4), it follows that

$$b_2 = \frac{1+q}{8q[\varpi+1]_q} l_1, \quad (3.8)$$

$$b_3 = \frac{1+q}{q[\varpi+1]_q[\varpi+2]_q} \left(\frac{1}{8} l_2 + \frac{3q^2 - 11q + 2}{128q} l_1^2 \right), \quad (3.9)$$

$$b_4 = \frac{1+q}{q[\varpi+1]_q[\varpi+2]_q[\varpi+3]_q} \left(\alpha l_3 + \beta l_1 l_2 + \gamma l_1^3 \right), \quad (3.10)$$

where

$$\alpha = \frac{1}{8}, \quad \beta = \frac{3q^4 - 12q^3 + 1}{64q^3},$$

$$\gamma = \frac{5q^6 - 35q^5 + 72q^4 - 2q^3 + q^2 - 11q + 2}{1024q^4}.$$

By solving the bounds of (3.8), (3.9), and (3.10), and applying Lemma 2, we arrive after straightforward computations at the result stated in Theorem 1. \square

Theorem 3.2 *If $h \in \mathcal{SL}_q^*(\varpi, \varphi)$, where h is of the form (1.1), then*

$$|b_3 - \mu b_2^2| \leq \begin{cases} -\frac{(1+q)^2}{16q^2[\varpi+1]_q^2} \mu + \frac{(1+q)(3q^2 - 3q + 2)}{32q^2[\varpi+1]_q[\varpi+2]_q}, & \mu \leq T_0, \\ \frac{(1+q)^2}{32q^2[\varpi+1]_q^2[\varpi+2]_q^2}, & T_0 \leq \mu \leq T_1, \\ \frac{(1+q)^2}{16q^2[\varpi+1]_q^2} \mu - \frac{(1+q)(3q^2 - 3q + 2)}{32q^2[\varpi+1]_q[\varpi+2]_q}, & \mu \geq T_1. \end{cases}$$

where

$$T_0 = \frac{3q^2 - 11q + 2}{16q[\varpi+2]_q^2},$$

and

$$T_1 = \frac{3q^2 + 5q + 2}{16q[\varpi+2]_q^2}.$$

Proof: Using (3.8) and (3.9), we find that

$$\begin{aligned} b_3 - \mu b_2^2 &= \frac{1+q}{8q[\varpi+1]_q[\varpi+2]_q} \left[l_2 + \frac{3q^2 - 11q + 2}{16q} l_1^2 - \frac{\mu(1+q)[\varpi+2]_q}{8q[\varpi+1]_q} l_1^2 \right] \\ &= \frac{1+q}{8q[\varpi+1]_q[\varpi+2]_q} (l_2 - \kappa l_1^2), \end{aligned} \quad (3.11)$$

where

$$\kappa = \frac{\mu(1+q)[\varpi+2]_q}{8q[\varpi+1]_q} - \frac{3q^2 - 11q + 2}{16q}.$$

from (3.11) and Lemma 2 we get required result. \square

Theorem 3.3 *If $h \in \mathcal{SL}_q^*(\varpi, \varphi)$, where h is of the form (1.1), then*

$$|b_2 b_4 - b_3^2| \leq \frac{(1+q)^2}{q^2[\varpi+1]_q^2} \left(\frac{1}{32[\varpi+2]_q[\varpi+3]_q} + \frac{1}{16[\varpi+2]_q^2} \right). \quad (3.12)$$

Proof: Using (3.8), (3.9) and (3.10), we have

$$b_2 b_4 - b_3^2 = \frac{(1+q)^2}{q^2 [\varpi + 1]_q^2} \left[\frac{l_1 l_3}{64 [\varpi + 2]_q [\varpi + 3]_q} + \left(\frac{3q^4 - 12q^3 + 1}{512q^3 [\varpi + 2]_q [\varpi + 3]_q} - \frac{3q^2 - 11q + 2}{512q [\varpi + 2]_q^2} \right) l_1^2 l_2 \right. \\ \left. + \left(\frac{5q^6 - 35q^5 + 72q^4 - 2q^3 + q^2 - 11q + 2}{8192q^4 [\varpi + 2]_q [\varpi + 3]_q} - \frac{(3q^2 - 11q + 2)^2}{16384q^2 [\varpi + 2]_q^2} \right) l_1^4 - \frac{l_2^2}{64 [\varpi + 2]_q^2} \right].$$

By substituting the values of l_2 and l_3 from Lemma 2, and then applying the triangular inequality, we replace $|x| < 1$ by ρ and l_1 by l . Consequently, we obtain

$$|b_2 b_4 - b_3^2| \leq K \left[\Lambda l^4 + \Pi (4 - l^2) + \Omega \rho l^2 (4 - l^2) + \Xi (4 - l^2)^2 \rho^2 \right] = \psi(l, \rho), \quad (3.13)$$

where

$$K = \frac{(1+q)^2}{q^2 [\varpi + 1]_q^2},$$

and

$$\begin{aligned} \Pi &= \frac{1}{128 [\varpi + 2]_q [\varpi + 3]_q}, \\ \Omega &= \frac{1}{2} \left[\frac{1}{64 [\varpi + 2]_q [\varpi + 3]_q} + \frac{3q^4 - 12q^3 + 1}{512q^3 [\varpi + 2]_q [\varpi + 3]_q} - \frac{3q^2 - 11q + 2}{512q [\varpi + 2]_q^2} + \frac{1}{64 [\varpi + 2]_q^2} \right], \\ \Xi &= \frac{1}{256 [\varpi + 2]_q^2}, \\ \Lambda &= \frac{5q^6 - 35q^5 + 72q^4 - 2q^3 + q^2 - 11q + 2}{8192q^4 [\varpi + 2]_q [\varpi + 3]_q} - \frac{(3q^2 - 11q + 2)^2}{16384q^2 [\varpi + 2]_q^2} \\ &\quad + \frac{1}{256 [\varpi + 2]_q [\varpi + 3]_q} + \frac{3q^4 - 12q^3 + 1}{1024q^3 [\varpi + 2]_q [\varpi + 3]_q} - \frac{3q^2 - 11q + 2}{1024q [\varpi + 2]_q^2} + \frac{1}{256 [\varpi + 2]_q^2}. \end{aligned}$$

Differentiating (3.13) with respect to ρ , we have

$$\frac{\partial \psi}{\partial \rho} \leq 2 \left[\Omega c^2 (4 - c^2) + \Xi (4 - c^2)^2 \rho \right] = \psi(l, \rho). \quad (3.14)$$

It follows from the fact that

$$\frac{\partial \psi(l, \rho)}{\partial \rho} \geq 0,$$

which shows that $\psi(l, \rho)$ is an increasing function on the closed interval $[0, 1]$. Hence, the maximum value of $\psi(l, \rho)$ occurs at $\rho = 1$. Therefore, for the maximum case we set

$$\psi(l, 1) = \varsigma(l).$$

Now

$$\varsigma(l) = K \left[(\Lambda - \Omega + \Xi) l^4 + (-\Pi + 4\Omega - 8\Xi) l^2 + (4\Pi + 16\Xi) \right]. \quad (3.15)$$

Differentiating (3.15) with respect to l we have

$$\varsigma'(l) = 2Kl \left[2(\Lambda - \Omega + \Xi) l^2 + (-\Pi + 4\Omega - 8\Xi) \right].$$

Differentiating again with respect to l we have

$$\varsigma''(l) = 12K(\Lambda - \Omega + \Xi) l^2 + 2K(-\Pi + 4\Omega - 8\Xi).$$

For $l = 0$, it follows that the maximum of $\zeta(l)$ occurs at $l = 0$. Hence, we obtain

$$|b_2b_4 - b_3^2| \leq \frac{(1+q)^2}{q^2[\varpi+1]_q^2} \left(\frac{1}{32[\varpi+2]_q[\varpi+3]_q} + \frac{1}{16[\varpi+2]_q^2} \right).$$

□

Theorem 3.4 *If $h \in \mathcal{SL}_q^*(\varpi, \varphi)$, where h is of the form (1.1), then*

$$|b_2b_3 - b_4| \leq \frac{1536q^3(1+q)^2}{[\varpi+1]_q[\varpi+2]_q}. \quad (3.16)$$

Proof: Using (3.8), (3.9) and (3.10), we have

$$\begin{aligned} b_2b_3 - b_4 &= \frac{1+q}{q[\varpi+1]_q[\varpi+2]_q} \left[-\frac{\alpha}{[\varpi+3]_q} l_3 + \left(\frac{1+q}{64q[\varpi+1]_q} - \frac{\beta}{[\varpi+3]_q} \right) l_1l_2 \right. \\ &\quad \left. + \left(\frac{(1+q)(3q^2 - 11q + 2)}{1024q^2[\varpi+1]_q} - \frac{\gamma}{[\varpi+3]_q} \right) l_1^3 \right], \end{aligned}$$

where

$$\alpha = \frac{1}{8}, \quad \beta = \frac{3q^4 - 12q^3 + 1}{64q^3}, \quad \gamma = \frac{5q^6 - 35q^5 + 72q^4 - 2q^3 + q^2 - 11q + 2}{1024q^4}.$$

Using Lemma 2 and, since $l_1 \leq 2$ by Lemma 3, we set $l_1 = c$ and assume without loss of generality that $l \in [0, 2]$. Taking absolute values and applying the triangle inequality with $\rho = |x|$, we obtain

$$|b_2b_3 - b_4| \leq \frac{1+q}{\zeta(q)} (A(q)l_3 + B(q)(4-l^2) + C(q)l(4-l^2)\rho + D(q)(l-2)(4-l^2)\rho^2) = \Psi(\rho)$$

where

$$\zeta(q) = 1024q^5[\varpi+1]_q[\varpi+2]_q[\varpi+3]_q$$

and

$$\begin{aligned} A(q) &= |(1+q)(3q^2 - 11q + 2)q^2[\varpi+3]_q - (5q^6 - 35q^5 + 72q^4 - 2q^3 + q^2 - 11q + 2)[\varpi+1]_q|, \\ B(q) &= 128q^4(1+q), \\ C(q) &= |(1+q)^2q^2[\varpi+3]_q - (3q^4 - 12q^3 + 1 + q)[\varpi+1]_q|, \\ D(q) &= 256q^4(1+q). \end{aligned}$$

Differentiating $\Psi(\rho)$ with respect to ρ we have

$$\Psi'(\rho) = \frac{1+q}{\zeta(q)} (C(q)l(4-l^2) + 2D(q)(l-2)(4-l^2)\rho) > 0.$$

This observation shows that $F(\rho)$ is an increasing function of ρ on the closed interval $[0,1]$. Consequently, we have

$$F(\rho) \leq F(0), \quad \forall \rho \in [0, 1].$$

Thus we have

$$\Psi(\rho) \leq \frac{1+q}{\zeta(q)} (A(q)l^3 + B(q)(4-l^2)) = E(l).$$

Differentiating $E(l)$ with respect to l we have

$$E'(l) = \frac{1+q}{\zeta(q)} (3A(q)l^2 - 2B(q)l).$$

Again differentiating the above equation with respect to l we have

$$E''(l) = \frac{1+q}{\zeta(q)} (6A(q)l - 2B(q)).$$

Since $l \in [0, 2]$, it follows from the assumption that $E(l)$ attains its maximum at $l = 0$, which corresponds to $\rho = 0$. Hence, this yields the desired upper bound. \square

4. Conclusion

In this work, we derived sharp upper bounds for the initial Taylor coefficients $|b_2|$, $|b_3|$, and $|b_4|$, together with the Fekete–Szegő functional $|b_3 - \mu b_2^2|$, for functions belonging to the subclass $\mathcal{SL}_q^*(\varpi, \varphi)$ defined in the open unit disk. Furthermore, an upper bound for the second Hankel determinant was obtained, with particular emphasis on the case when the function h lies in the lemniscate of Bernoulli domain. These results contribute to the growing body of work on coefficient estimates in q -starlike and related analytic function classes, and may stimulate further investigations into Hankel determinants and Fekete–Szegő type problems for other subclasses of analytic and univalent functions.

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