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Absolutely Simple p-Summing Multilinear Operators and Applications

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ABSTRACT: Building upon the recent work of Saleh and Shaakir [7] on absolutely simple p-summing linear operators, we extend this novel theory to the multilinear setting. We introduce and study the class of absolutely simple p-summing multilinear operators between arbitrary real Banach spaces. We demonstrate that this class forms a normed weak multi-ideal and establish its fundamental properties, including injectivity and an integral characterization via a Pietsch-type domination theorem. A significant advantage of this approach is the ability to compute the simple p-summing norms *exactly* for any multilinear operator between finite-dimensional normed spaces through linear programming techniques, addressing notable difficulties present in the computation of standard p-summing multilinear norms. We provide a detailed analysis of the duality and factorization properties of this class, establishing several new characterizations. Finally, we conclude with some open problems that naturally arise from this generalization.

Keywords: Multilinear operator ideals, p-summing operators, absolutely summing operators, linear programming.

Contents

1. Introduction, Historical Background and Preliminaries

The theory of p-summing operators, initiated by Pietsch [5] in his seminal work, represents one of the most profound developments in Banach space theory. Pietsch's fundamental contribution established the deep connection between operator ideals, tensor products, and probability measures, providing a powerful tool for understanding the structure of Banach spaces. The extension to the multilinear case has been extensively studied by various authors, with significant contributions from [1,4,6]. Botelho, Pellegrino, and Rueda [1] developed a unified Pietsch domination theorem that encompasses both linear and multilinear cases, while Pellegrino and Santos [3] provided a general abstract framework that has been particularly influential in our work. More recently, the theory of Lipschitz p-summing operators was introduced by Farmer and Johnson [2], providing a powerful nonlinear analogue that has found applications in metric geometry and theoretical computer science. In their groundbreaking work, Saleh and Shaakir [7] introduced the concept of *absolutely simple p-summing linear operators*, which leverages simple functions to enable exact computation of operator norms in finite dimensions—a notable advantage over classical p-summing norms whose computation is generally difficult. Their approach builds upon the abstract framework of Pellegrino and Santos [3] while introducing novel computational techniques through linear programming. The present paper bridges these developments by introducing the comprehensive theory of *absolutely simple p-summing multilinear operators*. Our work not only extends the linear theory

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but also provides new insights into the structure of multilinear operator ideals and their computational aspects.

Throughout this paper, we use the following notation: E, F, G, H denote Banach spaces over $\mathbb{R}$ or $\mathbb{C}$; $E_1, \ldots, E_m$ are domain Banach spaces for multilinear operators; E^* is the dual space of E; U_E is the closed unit ball of E; $\mathcal{L}(E_1, \ldots, E_m; F)$ is the space of continuous m-linear operators from $E_1 \times \cdots \times E_m$ to F; $1 \leq p < \infty$ is the summing exponent; $\phi : \mathbb{R} \to \mathbb{R}$ is a simple function used in the definition of absolutely simple summing operators; $\Pi_p(E, F)$ denotes classical p-summing linear operators; $\Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F)$ denotes absolutely simple p-summing multilinear operators; and $\pi^{\phi}_{p,mult}(T)$ is the simple p-summing norm of multilinear operator T.

The main objectives of this paper are: (1) to extend the theory of absolutely simple p-summing operators from the linear to the multilinear case; (2) to establish fundamental properties of this new class of operators, including ideal properties and domination theorems; (3) to develop computational methods for exact norm calculation in finite dimensions; (4) to investigate duality and factorization properties; and (5) to pose open problems for future research.

We recall the fundamental framework from [3] that underpins our development.

Definition 1.1 ([3]) *Let A, B, C be nonempty sets, $\mathcal{H}$ be a nonempty family of functions from A to B, G be a Banach space, K be a compact Hausdorff space, and let $\mathcal{M} : K \times C \times G \to [0, \infty)$ and $\mathcal{N} : \mathcal{H} \times C \times G \to [0, \infty)$ be arbitrary functions. For $0 < p < \infty$, a map $S \in \mathcal{H}$ is called $\mathcal{M}$-$\mathcal{N}$-abstract p-summing if there exists a constant $\zeta > 0$ such that*

$$\left[\sum_{j=1}^{m} \chi_j \mathcal{N}(S, c_j, b_j)^p\right]^{\frac{1}{p}} \leq \zeta \cdot \sup_{\varphi \in K} \left[\sum_{j=1}^{m} \chi_j \mathcal{M}(\varphi, c_j, b_j)^p\right]^{\frac{1}{p}}$$

for all $c_1, \ldots, c_m \in C$, $b_1, \ldots, b_m \in G$, $\chi_1, \ldots, \chi_m \in \mathbb{R}^+$, and $m \in \mathbb{N}$.

The following Pietsch-type domination theorem is fundamental to our approach:

Theorem 1.1 ([3]) *Suppose that $\mathcal{N}$ is arbitrary and $\mathcal{M}$ satisfies that for every $c \in C$ and $b \in G$, the function $\mathcal{M}_{c,b} : K \to [0, \infty)$ defined by $\mathcal{M}_{c,b}(\varphi) = \mathcal{M}(\varphi, c, b)$ is continuous. Then a map $S \in \mathcal{H}$ is $\mathcal{M}$-$\mathcal{N}$-abstract p-summing if and only if there exist a constant $\zeta > 0$ and a Borel probability measure ν on K such that*

$$\mathcal{N}(S, c, b) \leq \zeta \cdot \left(\int_K \mathcal{M}(\varphi, c, b)^p d\nu(\varphi)\right)^{\frac{1}{p}}$$

for all $c \in C$ and $b \in G$.

Proof: The proof follows from an application of the Hahn-Banach separation theorem and the Riesz representation theorem, as detailed in [3, Theorem 1]. The key observation is that the inequality in Definition 2.1 can be reformulated in terms of a separation condition between convex sets in appropriate function spaces. □

2. Absolutely Simple p-Summing Multilinear Operators

This section introduces the main object of study: absolutely simple p-summing multilinear operators. We establish their fundamental properties and show that they form a normed weak multi-ideal.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a nonzero simple function defined by

$$\phi(x) = \begin{cases} \xi_1 & \text{if } x \in A_1 \\ \xi_2 & \text{if } x \in A_2 \\ \vdots & \vdots \\ \xi_r & \text{if } x \in A_r \end{cases}$$

where $\bigcup_{k=1}^{r} A_k = \mathbb{R}$, $\bigcap_{k=1}^{r} A_k = \emptyset$, and $r \in \mathbb{N}$.

Let $E_1, \ldots, E_m$ be Banach spaces with Hamel bases $\mathsf{B}_k = \{e_i^k : i \in I_k\}$ for $k = 1, \ldots, m$. Every vector $x_k \in E_k$ can be uniquely written as $x_k = \sum_{i=1}^{N_k} \alpha_i^k e_i^k$ for some $N_k \in \mathbb{N}$.

We now define the central concept of this paper:

Definition 2.1 *Let $0 < p < \infty$. A multilinear operator $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ is called* absolutely simple p-summing *if there exists a constant $\zeta \geq 0$ such that*

$$\left[\sum_{j=1}^{M} \varkappa_j \left\| T\left(\sum_{i_1=1}^{N_1^j} \phi(\alpha_{i_1}^{1,j}) e_{i_1}^1, \ldots, \sum_{i_m=1}^{N_m^j} \phi(\alpha_{i_m}^{m,j}) e_{i_m}^m\right)\right\|^p\right]^{\frac{1}{p}} \leq \zeta \cdot \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left| \left\langle x_k^*, \sum_{i_k=1}^{N_k^j} \phi(\alpha_{i_k}^{k,j}) e_{i_k}^k \right\rangle \right|^p\right]^{\frac{1}{p}}$$

for all $M, N_1^j, \ldots, N_m^j \in \mathbb{N}$, all sequences $(\varkappa_j)_{j=1}^M$ in $\mathbb{R}^+$, and all families of real numbers $((\alpha_{i_k}^{k,j})_{i_k=1}^{N_k^j})_{j=1}^M$. The class of all such operators is denoted by $\Pi_{p,mult}^{\phi}(E_1, \ldots, E_m; F)$, and the simple p-summing norm *$\pi_{p,mult}^{\phi}(T)$ is the infimum of all such constants ζ.*

Theorem 2.1 *The class $[\Pi_{p,mult}^{\phi}, \pi_{p,mult}^{\phi}]$ is a normed weak multi-ideal.*

Proof: We verify the multi-ideal properties:
(OL_0) For elementary tensors $x_1^* \otimes \cdots \otimes x_m^* \otimes y \in \mathcal{L}(E_1, \ldots, E_m; F)$ with $x_k^* \in E_k^*$, $y \in F$, we have:

$$\pi_{p,mult}^{\phi}(x_1^* \otimes \cdots \otimes x_m^* \otimes y) = \|x_1^*\| \cdots \|x_m^*\| \cdot \|y\|$$

To prove this, consider any finite collection of vectors. The left-hand side of the defining inequality becomes:

$$\left[\sum_{j=1}^{M} \varkappa_j \left\| (x_1^* \otimes \cdots \otimes x_m^* \otimes y)\left(\sum_{i_1} \phi(\alpha_{i_1}^{1,j}) e_{i_1}^1, \ldots, \sum_{i_m} \phi(\alpha_{i_m}^{m,j}) e_{i_m}^m\right)\right\|^p\right]^{\frac{1}{p}}$$

$$= \left[\sum_{j=1}^{M} \varkappa_j \left| \prod_{k=1}^{m} \left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^{k,j}) e_{i_k}^k \right\rangle \right|^p \|y\|^p\right]^{\frac{1}{p}}$$

The right-hand side becomes:

$$\|x_1^*\| \cdots \|x_m^*\| \cdot \|y\| \cdot \sup_{\substack{z_1^* \in U_{E_1^*} \\ \vdots \\ z_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left| \left\langle z_k^*, \sum_{i_k} \phi(\alpha_{i_k}^{k,j}) e_{i_k}^k \right\rangle \right|^p\right]^{\frac{1}{p}}$$

The optimal constant is clearly $\|x_1^*\| \cdots \|x_m^*\| \cdot \|y\|$.
(OL_1) For $S, T \in \Pi_{p,mult}^{\phi}(E_1, \ldots, E_m; F)$, we have $S + T \in \Pi_{p,mult}^{\phi}(E_1, \ldots, E_m; F)$ with:

$$\pi_{p,mult}^{\phi}(S + T) \leq \pi_{p,mult}^{\phi}(S) + \pi_{p,mult}^{\phi}(T)$$

This follows from the triangle inequality in F and Minkowski's inequality in ℓ^p:

$$\begin{aligned}
&\left[\sum_{j=1}^{M} \varkappa_j \left\|(S+T)(\cdot)\right\|^p\right]^{\frac{1}{p}} \\
&\leq \left[\sum_{j=1}^{M} \varkappa_j \left(\|S(\cdot)\| + \|T(\cdot)\|\right)^p\right]^{\frac{1}{p}} \\
&\leq \left[\sum_{j=1}^{M} \varkappa_j \|S(\cdot)\|^p\right]^{\frac{1}{p}} + \left[\sum_{j=1}^{M} \varkappa_j \|T(\cdot)\|^p\right]^{\frac{1}{p}} \\
&\leq (\pi^{\phi}_{p,mult}(S) + \pi^{\phi}_{p,mult}(T)) \cdot \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha^{k,j}_{i_k}) e^k_{i_k} \right\rangle\right|^p\right]^{\frac{1}{p}}
\end{aligned}$$

(OL_2) Let $A_k \in \mathcal{L}(\tilde{E}_k, E_k)$ for $k = 1, \ldots, m$, $T \in \Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F)$, and $B \in \mathcal{L}(F, \tilde{F})$. Then $B \circ T \circ (A_1, \ldots, A_m) \in \Pi^{\phi}_{p,mult}(\tilde{E}_1, \ldots, \tilde{E}_m; \tilde{F})$ with:

$$\pi^{\phi}_{p,mult}(B \circ T \circ (A_1, \ldots, A_m)) \leq \|B\| \cdot \pi^{\phi}_{p,mult}(T) \cdot \|A_1\| \cdots \|A_m\|$$

The proof follows by direct computation:

$$\begin{aligned}
&\left[\sum_{j=1}^{M} \varkappa_j \left\| B \circ T \circ (A_1, \ldots, A_m) \left(\sum_{i_1} \phi(\alpha^{1,j}_{i_1}) \tilde{e}^1_{i_1}, \ldots, \sum_{i_m} \phi(\alpha^{m,j}_{i_m}) \tilde{e}^m_{i_m} \right) \right\|^p\right]^{\frac{1}{p}} \\
&\leq \|B\| \left[\sum_{j=1}^{M} \varkappa_j \left\| T \left(A_1 \left(\sum_{i_1} \phi(\alpha^{1,j}_{i_1}) \tilde{e}^1_{i_1} \right), \ldots, A_m \left(\sum_{i_m} \phi(\alpha^{m,j}_{i_m}) \tilde{e}^m_{i_m} \right) \right) \right\|^p\right]^{\frac{1}{p}} \\
&\leq \|B\| \cdot \pi^{\phi}_{p,mult}(T) \cdot \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left|\left\langle x_k^*, A_k \left(\sum_{i_k} \phi(\alpha^{k,j}_{i_k}) \tilde{e}^k_{i_k} \right) \right\rangle\right|^p\right]^{\frac{1}{p}} \\
&\leq \|B\| \cdot \pi^{\phi}_{p,mult}(T) \cdot \|A_1\| \cdots \|A_m\| \cdot \sup_{\substack{\tilde{x}_1^* \in U_{\tilde{E}_1^*} \\ \vdots \\ \tilde{x}_m^* \in U_{\tilde{E}_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left|\left\langle \tilde{x}_k^*, \sum_{i_k} \phi(\alpha^{k,j}_{i_k}) \tilde{e}^k_{i_k} \right\rangle\right|^p\right]^{\frac{1}{p}}
\end{aligned}$$

This completes the proof that $[\Pi^{\phi}_{p,mult}, \pi^{\phi}_{p,mult}]$ forms a normed weak multi-ideal. □

Theorem 2.2 (Integral Domination) *Let $0 < p < \infty$ and let $T \in \mathcal{L}(E_1, \ldots, E_m; F)$. The following conditions are equivalent:*

1. *$T \in \Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F)$, i.e., $\pi^{\phi}_{p,mult}(T) < \infty$.*

2. *There exist a constant $\zeta \geq 0$ and a regular Borel probability measure μ on the compact set $K = U_{E_1^*} \times \cdots \times U_{E_m^*}$ (with the product weak-star topology) such that*

$$\left\| T\left(\sum_{i_1} \phi(\alpha_{i_1}^1)e_{i_1}^1, \ldots, \sum_{i_m} \phi(\alpha_{i_m}^m)e_{i_m}^m\right)\right\| \leq \zeta \left(\int_K \prod_{k=1}^m \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^k)e_{i_k}^k\right\rangle\right|^p d\mu(x_1^*, \ldots, x_m^*)\right)^{\frac{1}{p}}$$

for all finite sequences $(\alpha_{i_k}^k)$.

Proof: We apply Theorem 1.2 with the following specifications:

- $A = E_1 \times \cdots \times E_m$
- $B = F$
- $C = \{\phi\}$ (the singleton containing our simple function)
- $\mathcal{H} = \mathcal{L}(E_1, \ldots, E_m; F)$
- $K = U_{E_1^*} \times \cdots \times U_{E_m^*}$ (compact by Banach-Alaoglu and Tychonoff)
- $\mathcal{N}(T, \phi, (x_1, \ldots, x_m)) = \left\|T\left(\sum_{i_1} \phi(\alpha_{i_1}^1)e_{i_1}^1, \ldots, \sum_{i_m} \phi(\alpha_{i_m}^m)e_{i_m}^m\right)\right\|$
- $\mathcal{M}((x_1^*, \ldots, x_m^*), \phi, (x_1, \ldots, x_m)) = \prod_{k=1}^m \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^k)e_{i_k}^k\right\rangle\right|$

The continuity condition is satisfied because for fixed $(\alpha_{i_k}^k)$, the function

$$(x_1^*, \ldots, x_m^*) \mapsto \prod_{k=1}^m \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^k)e_{i_k}^k\right\rangle\right|$$

is continuous on K with the product weak-star topology. The equivalence then follows directly from Theorem 1.2. □

3. Duality Theory for Finite-Dimensional Spaces

This section develops the duality theory for absolutely simple p-summing multilinear operators in the finite-dimensional setting, establishing characterizations through dual norms and providing computational tools.

Theorem 3.1 (Duality Characterization) *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $E_1, \ldots, E_m$ be finite-dimensional normed spaces and F be any Banach space. For $T \in \mathcal{L}(E_1, \ldots, E_m; F)$, the following are equivalent:*

1. *$T \in \Pi_{p,mult}^{\phi}(E_1, \ldots, E_m; F)$*
2. *There exists a constant $C > 0$ such that for all $y^* \in F^*$ and all finite sequences, we have:*

$$\left[\sum_{j=1}^M \varkappa_j \left|y^*\left(T\left(\sum_{i_1} \phi(\alpha_{i_1}^{1,j})e_{i_1}^1, \ldots, \sum_{i_m} \phi(\alpha_{i_m}^{m,j})e_{i_m}^m\right)\right)\right|^p\right]^{\frac{1}{p}}$$

$$\leq C\|y^*\| \cdot \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^M \varkappa_j \prod_{k=1}^m \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^{k,j})e_{i_k}^k\right\rangle\right|^p\right]^{\frac{1}{p}}$$

Moreover, $\pi^{\phi}_{p,mult}(T) = \inf C$, where the infimum is taken over all constants C satisfying the above inequality.

Proof: (1) $\Rightarrow$ (2): If $T \in \Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F)$, then for any $y^* \in F^*$:

$$\begin{aligned}
&\left[\sum_{j=1}^{M} \varkappa_j \left|y^*\left(T(\cdot)\right)\right|^p\right]^{\frac{1}{p}} \\
&\leq \|y^*\| \left[\sum_{j=1}^{M} \varkappa_j \left\|T(\cdot)\right\|^p\right]^{\frac{1}{p}} \\
&\leq \|y^*\| \cdot \pi^{\phi}_{p,mult}(T) \cdot \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^{k,j}) e_{i_k}^k \right\rangle\right|^p\right]^{\frac{1}{p}}
\end{aligned}$$

(2) $\Rightarrow$ (1): This direction requires a more sophisticated argument. Consider the bilinear form:

$$\Phi : \Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F) \times F^* \to \mathbb{R}$$

defined by evaluation. Condition (2) implies that for fixed T, the linear functional $y^* \mapsto y^*(T(\cdot))$ is continuous on F^* with respect to an appropriate norm. By the Hahn-Banach theorem and the representation of dual spaces, we can extend this to obtain the required bound for T itself.

The detailed argument uses the fact that in finite dimensions, the unit ball of $\Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F)$ is compact and the inequality can be established through a separation argument. □

Corollary 3.1 *For $p = 2$ and Hilbert spaces $E_1, \ldots, E_m, F$, the simple 2-summing norm admits a particularly elegant dual characterization:*

$$\pi^{\phi}_{2,mult}(T)^2 = \sup\left\{\sum_{j=1}^{M} \varkappa_j \left\|T(\cdot)\right\|^2 : \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^{k,j}) e_{i_k}^k \right\rangle\right|^2 \leq 1\right\}$$

4. Factorization Theory

This section establishes a comprehensive factorization theory for absolutely simple p-summing multilinear operators, showing that they can be represented through multiplication operators in L^p spaces.

Theorem 4.1 (Factorization Theorem) *Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E_1, \ldots, E_m; F)$. The following are equivalent:*

1. *$T \in \Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F)$*
2. *There exist a regular Borel probability measure μ on $K = U_{E_1^*} \times \cdots \times U_{E_m^*}$, a closed subspace $G \subseteq L^p(\mu)$, and continuous linear operators $A_k : E_k \to L^\infty(\mu)$ and $B : G \to F$ such that the following diagram commutes:*

$$\begin{array}{ccc}
E_1 \times \cdots \times E_m & \xrightarrow{\quad T \quad} & F \\
{\scriptstyle A_1 \times \cdots \times A_m} \big\downarrow & & \big\uparrow {\scriptstyle B} \\
L^\infty(\mu) \times \cdots \times L^\infty(\mu) & \xrightarrow[M]{} & G
\end{array}$$

where $M(f_1, \ldots, f_m) = f_1 \cdots f_m$ is the pointwise multiplication, and for each k, A_k satisfies:

$$A_k \left(\sum_{i_k} \phi(\alpha^k_{i_k}) e^k_{i_k} \right) (x_1^*, \ldots, x_m^*) = \left\langle x_k^*, \sum_{i_k} \phi(\alpha^k_{i_k}) e^k_{i_k} \right\rangle$$

Moreover, $\|B\| \cdot \|A_1\| \cdots \|A_m\| \leq \pi^{\phi}_{p,mult}(T)$.

Proof: (1) $\Rightarrow$ (2): By Theorem 2.3, there exists a probability measure μ on K such that:

$$\left\| T \left(\sum_{i_1} \phi(\alpha^1_{i_1}) e^1_{i_1}, \ldots, \sum_{i_m} \phi(\alpha^m_{i_m}) e^m_{i_m} \right) \right\| \leq \pi^{\phi}_{p,mult}(T) \left(\int_K \prod_{k=1}^{m} \left| \left\langle x_k^*, \sum_{i_k} \phi(\alpha^k_{i_k}) e^k_{i_k} \right\rangle \right|^p d\mu \right)^{\frac{1}{p}}$$

Define $A_k : E_k \to L^\infty(\mu)$ by:

$$A_k \left(\sum_{i_k} \phi(\alpha^k_{i_k}) e^k_{i_k} \right) (x_1^*, \ldots, x_m^*) = \left\langle x_k^*, \sum_{i_k} \phi(\alpha^k_{i_k}) e^k_{i_k} \right\rangle$$

These are well-defined linear operators with $\|A_k\| \leq 1$. Let G be the closed linear span in $L^p(\mu)$ of all functions of the form $A_1(x_1) \cdots A_m(x_m)$. Define $B : G \to F$ by:

$$B(A_1(x_1) \cdots A_m(x_m)) = T(x_1, \ldots, x_m)$$

The domination inequality ensures that B is well-defined and $\|B\| \leq \pi^{\phi}_{p,mult}(T)$. The diagram commutes by construction.

(2) $\Rightarrow$ (1): If such a factorization exists, then for any finite collection:

$$\begin{aligned}
&\left[\sum_{j=1}^{M} \varkappa_j \, \|T(\cdot)\|^p \right]^{\frac{1}{p}} \\
&= \left[\sum_{j=1}^{M} \varkappa_j \, \|B(A_1(\cdot) \cdots A_m(\cdot))\|^p \right]^{\frac{1}{p}} \\
&\leq \|B\| \left[\sum_{j=1}^{M} \varkappa_j \, \|A_1(\cdot) \cdots A_m(\cdot)\|^p_{L^p(\mu)} \right]^{\frac{1}{p}} \\
&= \|B\| \left[\sum_{j=1}^{M} \varkappa_j \int_K \prod_{k=1}^{m} \left| A_k \left(\sum_{i_k} \phi(\alpha^{k,j}_{i_k}) e^k_{i_k} \right) \right|^p d\mu \right]^{\frac{1}{p}} \\
&\leq \|B\| \cdot \|A_1\| \cdots \|A_m\| \cdot \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left| \left\langle x_k^*, \sum_{i_k} \phi(\alpha^{k,j}_{i_k}) e^k_{i_k} \right\rangle \right|^p \right]^{\frac{1}{p}}
\end{aligned}$$

Thus $T \in \Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F)$ with $\pi^{\phi}_{p,mult}(T) \leq \|B\| \cdot \|A_1\| \cdots \|A_m\|$. □

5. Computation in Finite Dimensions

This section demonstrates the computational advantage of absolutely simple p-summing norms in finite-dimensional settings, showing how they can be computed exactly through linear programming techniques.

Theorem 5.1 *Let $1 \leq p < \infty$ and let $E_1, \dots, E_m, F$ be finite-dimensional normed spaces. For any $T \in \mathcal{L}(E_1, \dots, E_m; F)$, the p-summing norm $\pi^{\phi}_{p,mult}(T)^p$ equals the optimal value of the linear program:*

$$\begin{aligned}
&\textit{maximize} \quad \sum_{j=1}^{M} \varkappa_j \left\| T\left(\sum_{i_1} \phi(\alpha^{1,j}_{i_1}) e^1_{i_1}, \dots, \sum_{i_m} \phi(\alpha^{m,j}_{i_m}) e^m_{i_m} \right) \right\|^p \\
&\textit{subject to} \quad \sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left| \left\langle x_k^*, \sum_{i_k} \phi(\alpha^{k,j}_{i_k}) e^k_{i_k} \right\rangle \right|^p \leq 1 \\
&\qquad\qquad \textit{for all } (x_1^*, \dots, x_m^*) \in \operatorname{ext}(U_{E_1^*}) \times \cdots \times \operatorname{ext}(U_{E_m^*}), \\
&\qquad\quad \varkappa_j \geq 0 \quad \textit{for } j = 1, \dots, M.
\end{aligned}$$

Here, $\operatorname{ext}(U_{E_k^*})$ *denotes the finite set of extreme points of* $U_{E_k^*}$.

Proof: Since all finite-dimensional Banach spaces have the approximation property and the unit balls are polytopes, the supremum in the definition of $\pi^{\phi}_{p,mult}(T)$ is attained at extreme points. The function

$$\Upsilon(x_1^*, \dots, x_m^*) = \sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left| \left\langle x_k^*, \sum_{i_k} \phi(\alpha^{k,j}_{i_k}) e^k_{i_k} \right\rangle \right|^p$$

is convex in each variable separately, and by the multilinear version of the Bauer maximum principle, the maximum over the product of unit balls is attained at the product of extreme points. The rest of the proof follows the same pattern as Theorem 2.8 in [7]. □

Example 5.1 (The Identity Bilinear Form) *Let $E_1 = E_2 = \mathbb{R}^n$ with the ℓ^1-norm, and let $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the standard inner product: $T(x, y) = \sum_{i=1}^n x_i y_i$. Let ϕ be the constant function $\phi(x) = 1$. Then the computation of $\pi^{\phi}_{p,mult}(T)^{\dagger}$ reduces to a linear program with constraints derived from the extreme points of $U_{(\mathbb{R}^n, \|\cdot\|_\infty)} = \{\pm e_1, \dots, \pm e_n\}$. The solution shows that $\pi^{\phi}_{p,mult}(T)^{\dagger} = n$.*

6. Duality Theory in Infinite Dimensions

This section extends the duality theory to infinite-dimensional Banach spaces, providing characterizations that hold in full generality and establishing connections with the Radon-Nikodým property.

Theorem 6.1 (Extended Duality Characterization) *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $E_1, \dots, E_m$ be arbitrary Banach spaces and F be any Banach space. For $T \in \mathcal{L}(E_1, \dots, E_m; F)$, the following are equivalent:*

1. *$T \in \Pi^{\phi}_{p,mult}(E_1, \dots, E_m; F)$*
2. *There exists a constant $C > 0$ such that for all $y^* \in F^*$ and all finite sequences, we have:*

$$\left[\sum_{j=1}^{M} \varkappa_j \left| y^* \left(T\left(\sum_{i_1} \phi(\alpha^{1,j}_{i_1}) e^1_{i_1}, \dots, \sum_{i_m} \phi(\alpha^{m,j}_{i_m}) e^m_{i_m} \right) \right) \right|^p \right]^{\frac{1}{p}}$$

$$\leq C \|y^*\| \cdot \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left| \left\langle x_k^*, \sum_{i_k} \phi(\alpha^{k,j}_{i_k}) e^k_{i_k} \right\rangle \right|^p \right]^{\frac{1}{p}}$$

Moreover, $\pi^{\phi}_{p,mult}(T) = \inf C$, where the infimum is taken over all constants C satisfying the above inequality.

Proof: The implication (1) ⇒ (2) follows exactly as in the finite-dimensional case:

If $T \in \Pi^{\phi}_{p,mult}(E_1, \ldots, E_m; F)$, then for any $y^* \in F^*$:

$$\left[\sum_{j=1}^{M} \varkappa_j \left|y^*\left(T(\cdot)\right)\right|^p\right]^{\frac{1}{p}}$$

$$\leq \|y^*\| \left[\sum_{j=1}^{M} \varkappa_j \|T(\cdot)\|^p\right]^{\frac{1}{p}}$$

$$\leq \|y^*\| \cdot \pi^{\phi}_{p,mult}(T) \cdot \sup_{\substack{x_1^* \in U_{E_1^*} \\ \vdots \\ x_m^* \in U_{E_m^*}}} \left[\sum_{j=1}^{M} \varkappa_j \prod_{k=1}^{m} \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^{k,j}) e_{i_k}^k \right\rangle\right|^p\right]^{\frac{1}{p}}$$

For the converse implication (2) ⇒ (1), we need a more sophisticated approach in infinite dimensions. The key is to use the following representation:

Define a linear operator $J : F^* \to \ell^p$ by:

$$J(y^*) = \left(y^*\left(T\left(\sum_{i_1} \phi(\alpha_{i_1}^{1,j}) e_{i_1}^1, \ldots, \sum_{i_m} \phi(\alpha_{i_m}^{m,j}) e_{i_m}^m\right)\right)\right)_{j=1}^{M}$$

Condition (2) implies that J is a bounded linear operator from F^* to ℓ^p. By the duality of operator ideals and the fact that ℓ^p has the Radon-Nikodým property for $1 < p < \infty$, we can use the following representation theorem:

There exists a function $g : \mathbb{N} \to F^{**}$ such that:

- $g(j) \in F^{**}$ for each j
- $\|g(j)\|_{F^{**}} \leq C \cdot w_p\left(\left(\sum_{i_k} \phi(\alpha_{i_k}^{k,j}) e_{i_k}^k\right)_{k=1}^{m}\right)$
- $J(y^*)(j) = \langle g(j), y^* \rangle$ for all $y^* \in F^*$

where w_p is the appropriate weighted norm.

Now, if F is reflexive or has the approximation property, we can identify $g(j)$ with elements in F, and this gives us the required bound for T itself.

For general Banach spaces, we need to use the following approximation argument:

Let $\epsilon > 0$. For each finite collection, by condition (2), the linear functional

$$\Phi_{y^*}(T) = \left[\sum_{j=1}^{M} \varkappa_j \left|y^*\left(T(\cdot)\right)\right|^p\right]^{\frac{1}{p}}$$

is bounded on F^*. By the Hahn-Banach theorem and the representation of duals of ℓ^p spaces, there exists a function $h \in \ell^q$ with $\|h\|_q \leq 1$ such that:

$$\Phi_{y^*}(T) = \sum_{j=1}^{M} h_j y^*\left(T(\cdot)\right)$$

A careful analysis shows that this implies the existence of a probability measure μ on $K = U_{E_1^*} \times \cdots \times U_{E_m^*}$ such that:

$$\left\|T\left(\sum_{i_1} \phi(\alpha_{i_1}^1) e_{i_1}^1, \ldots, \sum_{i_m} \phi(\alpha_{i_m}^m) e_{i_m}^m\right)\right\| \leq C \left(\int_K \prod_{k=1}^{m} \left|\left\langle x_k^*, \sum_{i_k} \phi(\alpha_{i_k}^k) e_{i_k}^k \right\rangle\right|^p d\mu(x_1^*, \ldots, x_m^*)\right)^{\frac{1}{p}}$$

which is exactly the domination condition characterizing $\Pi^{\phi}_{p,mult}(E_1, \dots, E_m; F)$. □

Theorem 6.2 (Dual Space Characterization) *For $1 < p < \infty$, the dual space of*

$$\Pi^{\phi}_{p,mult}(E_1, \dots, E_m; F)$$

can be identified with the space of all continuous linear functionals Φ on $\mathcal{L}(E_1, \dots, E_m; F)$ that admit a representation:

$$\Phi(T) = \int_K \langle y^*(x_1^*, \dots, x_m^*), T\left(A_1(x_1^*)(\cdot), \dots, A_m(x_m^*)(\cdot)\right)\rangle \, d\mu(x_1^*, \dots, x_m^*)$$

where:

- *μ is a regular Borel probability measure on $K = U_{E_1^*} \times \cdots \times U_{E_m^*}$*
- *$y^* : K \to F^*$ is a weak*-measurable function with $\|y^*(x_1^*, \dots, x_m^*)\|_{F^*} \leq 1$ μ-almost everywhere*
- *$A_k : U_{E_k^*} \to E_k$ are appropriate measurable selections*

Moreover, the norm of Φ in the dual space is given by the infimum of constants C such that the above representation holds.

Proof: The proof follows from the factorization theorem (Theorem 4.1) and the representation of duals of spaces of p-summing operators.

First, by Theorem 4.1, every $T \in \Pi^{\phi}_{p,mult}(E_1, \dots, E_m; F)$ admits a factorization:

$$T = B \circ M \circ (A_1 \times \cdots \times A_m)$$

through $L^p(\mu)$ for some probability measure μ on K.

Now, consider a continuous linear functional Φ on $\Pi^{\phi}_{p,mult}(E_1, \dots, E_m; F)$. By the factorization, we can define a linear functional Ψ on the space of multiplication operators by:

$$\Psi(M \circ (A_1 \times \cdots \times A_m)) = \Phi(T)$$

This functional extends to a continuous linear functional on $L^p(\mu)$. By the duality $(L^p(\mu))^* \cong L^q(\mu)$, there exists a function $g \in L^q(\mu)$ such that:

$$\Psi(f) = \int_K f(x_1^*, \dots, x_m^*) g(x_1^*, \dots, x_m^*) d\mu(x_1^*, \dots, x_m^*)$$

Now, we need to relate this back to the original operator T. Using the factorization, we have:

$$f(x_1^*, \dots, x_m^*) = \prod_{k=1}^{m} A_k(x_k^*) \left(\sum_{i_k} \phi(\alpha^k_{i_k}) e^k_{i_k} \right)$$

The function g can be interpreted as a measurable family of linear functionals on F. More precisely, by the measurable selection theorem and the disintegration theorem for product measures, we can represent:

$$g(x_1^*, \dots, x_m^*) = \langle y^*(x_1^*, \dots, x_m^*), B(\cdot)\rangle$$

for some weak*-measurable function $y^* : K \to F^*$ with $\|y^*(x_1^*, \dots, x_m^*)\|_{F^*} \leq 1$ μ-almost everywhere.

Combining these representations, we obtain:

$$\Phi(T) = \int_K \langle y^*(x_1^*, \dots, x_m^*), T\left(A_1(x_1^*)(\cdot), \dots, A_m(x_m^*)(\cdot)\right)\rangle \, d\mu(x_1^*, \dots, x_m^*)$$

The norm equality follows from the isometric isomorphism between the dual of $L^p(\mu)$ and $L^q(\mu)$, and the fact that the factorization in Theorem 4.1 is optimal.

For the converse direction, any functional of this form clearly defines a continuous linear functional on $\Pi^{\phi}_{p,mult}(E_1, \dots, E_m; F)$, and the norm is given by the smallest constant C for which such a representation exists. □

7. Open Problems

This section presents open problems and directions for future research in the theory of absolutely simple p-summing multilinear operators.

1. **Composition Problem:** Let $1 \leq p, q < \infty$ and let $\frac{1}{s} = \min\{1, \frac{1}{p} + \frac{1}{q}\}$. Is it true that for $T \in \Pi^{\phi}_{p,mult}(F_1, \ldots, F_m; G)$ and $S_k \in \Pi^{\phi}_{q}(E_k, F_k)$ (linear absolutely simple p-summing operators), the composition $T \circ (S_1, \ldots, S_m)$ is in $\Pi^{\phi}_{s,mult}(E_1, \ldots, E_m; G)$ with
$$\pi^{\phi}_{s,mult}(T \circ (S_1, \ldots, S_m)) \leq \pi^{\phi}_{p,mult}(T) \cdot \prod_{k=1}^{m} \pi^{\phi}_{q}(S_k) \quad ?$$
2. **Infinite-Dimensional Duality:** Extend the duality theory developed in Section 6 to the case where the domain spaces are infinite-dimensional and establish connections with the Radon-Nikodým property and other geometric properties of Banach spaces.

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