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On Periodic Solutions for $p(x)$-Laplacian Problems with Leray-Schauder Degree Theory

Hamza Jourhmane, Ali El Mfadel and Abderrazak Kassidi

ABSTRACT: In this paper, we delve into the existence of nonnegative periodic solutions for a distinct class of $p(x)$-Laplacian problems with nonlocal terms. Employing the parabolic regularization method coupled with Leray-Schauder degree theory, we establish significant results that advance the understanding of these complex mathematical structures. Our results not only demonstrate the robustness of these methods but also open new pathways for further research in the field of nonlinear analysis and differential equations.

Keywords: $p(x)$-Laplacian operator, periodic solutions, Leray-Schauder degree theory.

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1. Introduction

Recently, significant attention has been directed towards nonlocal operators, both in theoretical mathematical research and practical applications, such as optimization, finance, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces, and recreation theory. These operators commonly arise from the stochastic stabilization of Lévy processes; see [2,3,11,15] and the references therein. The representation of physical processes with pointwise distinct properties, initially emerging from nonlinear elasticity theory, leads to problems with variable exponent growth conditions; see [5,10,12]. Additionally, various authors have examined issues with nonlocal terms, see [14,13]. Inspired by the work of Allegretto and Nistri on the equation

$$\frac{\partial u}{\partial t} - \triangle u = f(x,t,\Psi[u],u,m)u,$$

which illustrates rapid diffusion speeds under typical conditions where $f(x,t,\Psi[u],u,m)$ is $m-\Psi[u]$, our study considers a different type of equation: the $p(x)$-Laplacian equation. This equation indicates a slower diffusion speed, making it more applicable to scenarios involving gas or fluid flow media.

In this paper, we examine the periodic problem for a periodic $p(x)$-Laplacian with nonlocal terms as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = (g-\Psi[u])u, & (x,t)\in Q_T, \\ u(x,t)=0, \quad (t,x)\in\partial\Omega\times(0,T), \\ u(x,0)=u(x,T), \quad x\in\Omega, \end{cases} \tag{1.1}$$

where $p(x)>2$, Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary, and $Q_T=\Omega\times(0,T)$. This problem is motivated by models proposed for mathematical biology and fisheries management issues, where $g=g(x,t)$ represents the maximal rate of natural increase at location x and time t, $u=u(x,t)$ represents the species density at position x and time t, the nonlocal term $\Psi[u]:L^2\to\mathbb{R}$ is a bounded continuous functional, and $g-\Psi[u]$ denotes the actual growth rate with self-limitation. This suggests that the growth rate is influenced not by the local density of the species but by the total population.

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Our paper is organized as follows. Section 2 introduces the functional framework for seeking periodic solutions and defines the weak periodic solution. Section 3 demonstrates the existence of a nontrivial nonnegative time-periodic solution for the problem (1.1) using an auxiliary abstract problem and Leray-Schauder degree theory.

2. Abstract Framework

In this section, we will introduce an adequate functional space where problems of type (1.1) can be studied. Such a space will be called Sobolev spaces with variable exponent $W^{1,p(x)}(\Omega)$.

Let Ω be a bounded open subset of $\mathbb{R}^N(N \geq 2)$, we say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|\log|x-y||} \quad \forall x, y \in \overline{\Omega} \quad \text{such that} \quad |x-y| < \frac{1}{2},$$

with C a constant. We denote

$$\mathcal{C}_+(\overline{\Omega}) = \{p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R}\text{log-Hölder continuous function, such that } 1 < p^- \leq p^+ < N\},$$

where

$$p^- = \min_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^+ = \max_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto \mathbb{R}$ for which the convex modular:

$$\varrho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx,$$

is finite, then

$$\|u\|_{p(x)} = \inf\{ \lambda > 0 : \varrho_{p(x)}(\frac{u}{\lambda}) \leq 1 \},$$

defines a norm in $L^{p(\cdot)}(\Omega)$ called the Luxemburg norm.
The space $\left(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)}\right)$ is a separable and reflexive Banach space, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.
The generalized Hölder type inequality:

$$\Big|\int_\Omega uv \quad dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{p(x)}|v|_{p'(x)},$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

Notice that if (u_n) and $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold true (see [6])

1. $\|u\|_{p(x)} < 1\ (=1; > 1) \quad \Leftrightarrow \varrho_{p(x)}(u) < 1\ (=1; > 1)$,
2. $\|u\|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \leq \varrho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}$,
3. $\|u\|_{p(x)} < 1 \Rightarrow \|u\|^{p^+}_{p(x)} \leq \varrho_{p(x)}(u) \leq \|u\|^{p^-}_{p(x)}$,
4. $\lim\limits_{n\to\infty} \|u_n - u\|_{p(x)} = 0 \Leftrightarrow \lim\limits_{n\to\infty} \varrho_{p(x)}(u_n - u) = 0$.

From (2) and (3), we can deduce the inequalities

$$\|u\|_{p(x)} \leq \varrho_{p(x)}(u) + 1,$$
$$\varrho_{p(x)}(u) \leq \|u\|^{p^-}_{p(x)} + \|u\|^{p^+}_{p(x)}.$$

If $p, q \in \mathcal{C}_+(\bar{\Omega})$ and $p(x) \le q(x)$ for any $x \in \overline{\Omega}$, then we have the continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$. The Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$ is defined by:

$$W^{1,p(\cdot)}(\Omega) = \big\{u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\big\},$$

which is a Banach space when equipped with the following norm:

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $\big(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(x)}\big)$ is a separable and reflexive Banach space. We define $W_0^{1,p(x)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [5].

Remark 2.1 Recall that the definition of these spaces requires only the measurability of $p(\cdot)$, while the Poincaré and the Sobolev-Poincaré inequality are proved for $p(\cdot)$ log-Hölder continuous, (see [5,10]).

3. Main Results

First, we present the assumptions and the definitions of solutions.

(A_1) Let $\Psi : L^2(\Omega)^+ \to \mathbb{R}^+$ be a bounded continuous functional, $\Psi[0] = 0$, and

$$C_1\|\xi\|^2_{L^2(\Omega)} \le \Psi[\xi] \le C_2\|\xi\|^2_{L^2(\Omega)},$$

where $0 < C_1 \le C_2$ are constants independent of ξ, $\mathbb{R}^+ = [0, +\infty)$ and

$$L^2(\Omega)^+ = \{u \in L^2(\Omega) \mid u \ge 0, \text{ a. e. } \Omega\}.$$

(A_2) $g \in \mathcal{C}_T(\overline{Q}_T)$ may change sign, but $\Big\{x \in \Omega : \dfrac{1}{T}\displaystyle\int_0^T g(x,t)dt > 0\Big\} \ne \emptyset$, where $\mathcal{C}_T(\overline{Q}_T)$ is a class of functions that are continuous in $\overline{\Omega} \times \mathbb{R}$ and T-periodic with respect to t.

By (A_2) and the continuity of the function $g(x,t)$, there exist $x_0 \in \Omega, r_0 > 0$ and constant $g_0 > 0$ such that $\dfrac{1}{T}\displaystyle\int_0^T g(x,t)dt \ge g_0$ for all $x \in B(x_0, r_0) \subset \Omega$.

Let μ_1 be the first eigenvalue of the following problem

$$-\triangle v = \mu v, \text{ in } B(x_0, \frac{1}{2}r_0),$$

$$v = 0, \text{ on } \partial B(x_0, \frac{1}{2}r_0).$$

Our main efforts center around the discussion of generalized solutions, since the regularity follows from a quite standard approach. Hence we give the following definition of the generalized solutions of the problem (1.1).

Definition 3.1 *A function u is called a generalized solution of the problem (1.1), if $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap \mathcal{C}_T(\overline{Q}_T)$, and u satisfies*

$$\iint_{Q_T} \Big(-u\frac{\partial \varphi}{\partial t} + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \big(|u|^{p-2} - g + \Psi[u]\big)u\varphi\Big)dxdt = 0, \tag{3.1}$$

for any $\varphi \in C^1(\overline{Q}_T)$ with $\varphi(x,0) = \varphi(x,T)$ on $\partial\Omega \times (0,T)$.

Due to the degeneracy of the equation, we will consider the following regularized problem

$$\begin{cases} \dfrac{\partial u_h}{\partial t} - div\Big(\big(|\nabla u_h|^2 + h\big)^{\frac{p(x)-2}{2}} \nabla u_h\Big) = \big(g - |u_h|^{p(x)-2} - \Psi[u_h]\big)u_h^+, \quad (x,t) \in Q_T, \\ u_h(x,t) = 0, \quad (t,x) \in \partial\Omega \times (0,T), \\ u_h(x,0) = u_h(x,T), \quad x \in \Omega, \end{cases} \tag{3.2}$$

where $p(x) > 2, h$ is a positive constant. The desired solution of the problem (1.1) will be obtained as a limit point of the nonnegative solutions u_h of the problem (3.2).

For fixed $h > 0$, the first equation in (3.2) is uniformly parabolic. However, though u_h is smooth enough, we cannot ensure $\Psi[u_h]$ is smoother than C^0. So we would not expect the right side of the first equation in (3.2) to have C^α smoothness. Furthermore, we do not expect the problem (3.2) to have a classical solution. Now we should define the strong generalized solution of the problem (3.2).

Definition 3.2 *A function u_h is called a strong generalized solution of problem (3.2), if $u_h \in W_q^{2,1}(Q_T) \cap \mathcal{C}_T(\overline{Q}_T)$ and u satisfies the first equation in (3.2) almost everywhere.*

Now we introduce a map and deduce some useful lemmas. Consider the problem

$$\begin{cases} \dfrac{\partial u_h}{\partial t} - div\Big(\big(|\nabla u_h|^2 + h\big)^{\frac{p(x)-2}{2}} \nabla u_h\Big) = f, \quad (x,t) \in Q_T, \\ u_h(x,t) = 0, \quad (t,x) \in \partial\Omega \times (0,T), \\ u_h(x,0) = u_h(x,T), \quad x \in \Omega, \end{cases}$$

Utilizing the results in [14], we know that for all given functions $f \in \mathcal{C}_T(\overline{Q}_T)$, there exists a unique generalized solution $u_h \in \mathcal{C}_T(\overline{Q}_T) \cap C^1(\overline{Q}_T)$, which satisfies $\dfrac{\partial u_h}{\partial t} \in L^2(Q_T)$. We define a map $u_h = Gf$ with $G : \mathcal{C}_T(\overline{Q}_T) \to \mathcal{C}_T(\overline{Q}_T)$. Using methods similar to those in [14], we can get that the map is completely continuous. Let $f(v) = \big(g - |v_h|^{p(x)-2} - \Psi[v_h]\big)v_h^+$, where $v^+ = \max\{v, 0\}$, by the condition (A_1), we can see when v is continuous with respect to $t, \Psi[v]$ is continuous with respect to t. So we can study the existence of the fixed point of the complete continuous map $u_h = G\big(g - |v_h|^{p(x)-2} - \Psi[v_h]\big)v_h^+$ instead of obtaining the existence of the solution of (3.2). First, we prove the nonnegativity of the solutions of the regularized problem.

Lemma 3.1 *If $u_h \in \mathcal{C}_T(\overline{Q}_T)$ is a non-trivial periodic solution of problem (3.2), then*

$$u_h(x,t) > 0, \qquad x \in \Omega, \ t \geq 0.$$

Proof: We first prove $u_h \geq 0$. Multiplying u_h^- by the first equation in (3.2), and integrating over Q_T, we can see

$$\int_{Q_T} \frac{\partial u_h}{\partial t} u_h^- dtdx + \int_{Q_T} \big(|\nabla u_h|^2 + h\big)^{\frac{p(x)-2}{2}} \nabla u_h \nabla u_h^- dtdx$$
$$= \int_{Q_T} (g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+ u_h^- dtdx,$$

where $u_h^- = \min\big\{u_h, 0\big\}$. Considering the above terms respectively,

$$\begin{aligned} &\int_{Q_T} \frac{\partial u_h}{\partial t} u_h^- dtdx = -\frac{1}{2}\int_\Omega (u_h^-)^2\Big|_0^T dx = 0, \\ &\int_{Q_T} (|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} \nabla u_h \nabla u_h^- dtdx = \int_{Q_T} (|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} |\nabla u_h^-|^2 dtdx, \\ &\int_{Q_T} \big(g - |u_h|^{p(x)-2} - \Psi[u_h]\big)u_h^+ u_h^- dtdx = 0. \end{aligned}$$

Then

$$\int_{Q_T} (|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} |\nabla u_h^-|^2 dtdx = 0.$$

Notice that $(|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} > 0$, leaving us with

$$\int_\Omega \int_0^T |\nabla u_h^-|^2 dtdx = 0. \tag{3.3}$$

By Poincarè's inequality and (3.3), we get

$$\int_\Omega |u_h^-|^2 dx \le C \int_\Omega |\nabla u_h^-|^2 dx.$$

This combined with (3.3) implies $u_h \ge 0$.

Next we prove $u_h > 0$. By the fact that $u_h \in \mathcal{C}_T(\overline{Q}_T)$ is non-trivial, there exist $\xi \in (0, T]$ and $x \in \Omega$, such that $u_h(x, \xi) \ne 0$. Let $0 \le \psi(x) \in \mathcal{C}_0^\infty(\Omega)$ be non-trivial with $\psi(x) < u_h(x, \xi)$. For a constant $\sigma > 0$, let v solve the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - div\Big((|\nabla v|^2 + h)^{\frac{p(x)-2}{2}} \nabla v \Big) + \sigma v = 0, \quad x \in \Omega,\ t > \xi, \\ v(x,t) = 0, \quad (t,x) \in \partial\Omega \times [\xi, T], \\ v(x,0) = \psi(x), \quad x \in \Omega, \end{cases}$$

Noticing that $u_h \in \mathcal{C}_T(\overline{Q}_T)$, and by the condition (A_1), we can obtain $\Psi[u_h] \in \mathcal{C}_T(\overline{Q}_T)$. Combining this with $g \in \mathcal{C}_T(\overline{Q}_T)$, we know $g - |u|^{p(x)-2} - \Psi[u_h] \in \mathcal{C}_T(\overline{Q}_T)$. By the comparison theorem, we can see that when D is large enough, we have $u_h(x,t) \ge v(x,t)$. And by using the maximum principle, we have for all $x \in \Omega$ and $t > \xi$, $v(x,t) > 0$. Finally, utilizing the periodicity of u_h, we can find that $u_h(x,t) \ge v(x,t) > 0$, that is for all $x \in \Omega$ and $t > 0$, $u_h(x,t) > 0$. The proof is completed. □

Lemma 3.2 *There exists a constant $r > 0$, such that no solutions u of the problem (3.2) satisy*

$$0 < \|u_h\|_{L^\infty(Q_T)} \le r.$$

Proof: Let u_h be the solution of the problem (3.2), and $0 < \|u_h\|_{L^\infty(Q_T)} \le r$. By Lemma 3.1, we know that for all $(x,t) \in Q_T$, $u_h(x,t) > 0$. Now for all $\varphi(x) \in \mathcal{C}_0^\infty(\Omega)$, we choose $\dfrac{\varphi^2}{u_h}$ as the test function. Then, multiplying $\dfrac{\varphi^2}{u_h}$ by the first equation in (3.2), before integrating over Q_T, we have

$$\int_{Q_T} \frac{\partial u_h}{\partial t} \Big(\frac{\varphi^2}{u_h} \Big) dtdx + \int_{Q_T} (|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} \nabla u_h \nabla \Big(\frac{\varphi^2}{u_h} \Big) dtdx$$
$$= \int_{Q_T} \varphi^2 \big(g - |u_h|^{p(x)-2} - \Psi[u_h] \big) dtdx.$$

By the periodicity of u_h, the first term of the equation can be written as

$$\int_{Q_T} \frac{\partial u_h}{\partial t} \Big(\frac{\varphi^2}{u_h} \Big) dtdx = \int_\Omega \varphi^2 \int_0^T \frac{\partial (\ln u_h)}{\partial t} dtdx = 0,$$

and the second term can be considered as

$$\begin{aligned}
&\int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} \nabla u_h \nabla\Big(\frac{\varphi^2}{u_h}\Big) dtdx = \int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} \nabla u_h \nabla\Big(\varphi \frac{\varphi}{u_h}\Big) dtdx \\
&= \int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} \frac{\nabla\varphi}{u_h} \varphi \nabla u_h dtdx + \int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} \varphi \nabla u_h \nabla\Big(\frac{\varphi}{u_h}\Big) dtdx \\
&= \int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} \frac{\nabla\varphi}{u_h} \Big(u_h \nabla\varphi - u_h^2 \nabla\Big(\frac{\varphi}{u_h}\Big)\Big) dtdx \\
&+ \int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} \varphi \nabla u_h \nabla\Big(\frac{\varphi}{u_h}\Big) dtdx \\
&= \int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} |\nabla\varphi|^2 dtdx - \int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} u_h^2 \Big|\nabla\Big(\frac{\varphi}{u_h}\Big)\Big|^2 dtdx.
\end{aligned}$$

So the inequality

$$\begin{aligned}
\int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} |\nabla\varphi|^2 dtdx - \int_{Q_T} \varphi^2 (g - |u_h|^{p(x)-2} - \Psi[u_h]) dtdx& \\
= \int_{Q_T} (|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} u_h^2 \Big|\nabla\Big(\frac{\varphi}{u_h}\Big)\Big|^2 dtdx \geq 0&,
\end{aligned}$$

follows.

By [4, Theorem 5.1] and some remarks in (page 238, page 243), it follows that there exists a constant $\gamma / = \gamma/(N,\ p)$ such that

$$\sup_{[(x_0,t_0)+Q(\tau\theta,\tau\rho)]} |\nabla u| \leq \frac{\gamma\sqrt{\theta/\rho^2}}{(1-\tau)^{(N+2)/2}} \Big(\int_{[(x_0,t_0)+Q(\theta,\rho)]} |\nabla u|^{p(x)} dxdt \Big)^{\frac{1}{2}} \wedge \Big(\frac{\rho^2}{\theta}\Big)^{\frac{1}{p^- -2}}, \tag{3.4}$$

for any $(x_0,t_0) \in Q_{(T,3T)} = \Omega \times (T,3T),\ [(x_0,t_0)+Q(\theta,\rho)] \subset Q_{(T,3T)}$, and any $\tau \in (0,1)$.
Substituting $\theta = r_0,\ \rho = \min\Big\{T, \frac{\sqrt{g_0 r_0}}{2^{\frac{p^-+6}{2}}}\Big\},\ \tau = \frac{1}{2}$ in (3.4) gives

$$\begin{aligned}
\sup_{[(x_0,t_0)+Q(\frac{1}{2}r_0,\frac{1}{2}\rho)]} |\nabla u| \leq C(N,p,r_0,g_0,\mu_1) \Big(\int_{[(x_0,t_0)+Q(r_0,\rho)]} |\nabla u|^{p(x)} dxdt \Big)^{\frac{1}{2}}& \\
\wedge \frac{1}{2}\Big(\frac{g_0}{4\mu_1}\Big)^{\frac{1}{p^- -2}}&.
\end{aligned}$$

On the other hand, by (3.2), we have

$$\int_{Q_T} |\nabla u_h|^{p(x)} dtdx \leq \max|g(x,t)| Q_T \int_{Q_T} |u_h|^2 dtdx.$$

So

$$\sup_{[(x_0,t_0)+Q(\frac{1}{2}r_0,\frac{1}{2}\rho)]} |\nabla u_h| \leq C(N,p,r_0,g_0,\mu_1) \Big(\int_{Q_T} |u_h|^2 dtdx \Big) \wedge \frac{1}{2}\Big(\frac{g_0}{4\mu_1}\Big)^{\frac{1}{p^- -2}},$$

which implies

$$\|\nabla u_h\|_{L^\infty(B(x_0,\frac{1}{2}r_0)\times(0,T))} \leq C \|u_h\|_{L^\infty(Q_T)} \wedge \frac{1}{2}\Big(\frac{m0}{4\mu_1}\Big)^{\frac{1}{p^- -2}},$$

where C is a constant independent of h.
Since

$$(|\nabla u_h|^2+h)^{\frac{p(x)-2}{2}} \leq 2^{\frac{p(x)-2}{2}} \Big(|\nabla u_h|^{p(x)-2} + h^{\frac{p(x)-2}{2}} \Big),$$

we have

$$\int_{Q_T} 2^{\frac{p(x)-2}{2}} \Big(|\nabla u_h|^{p(x)-2} + h^{\frac{p(x)-2}{2}}\Big)|\nabla\varphi|^2 dtdx - \int_{Q_T} \varphi^2(g - |u_h|^{p(x)-2} - \Psi[u_h])dtdx$$
$$\geq \int\int_{Q_T} (|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} |\nabla\varphi|^2 dtdx - \int_{Q_T} \varphi^2(g - |u_h|^{p(x)-2} - \Psi[u_h])dtdx \geq 0.$$

By the approximating process, taking $\varphi = \varphi_1$, where φ_1 is an eigenfunction of the first eigenvalue μ_1, we get

$$\begin{aligned}\int_{B(x_0,\frac{1}{2}r_0)\times(0,T)} \varphi_1^2(g - |u_h|^{p(x)-2} - \Psi[u_h])dtdx &\leq \int_{B(x_0,\frac{1}{2}r_0)\times(0,T)} 2^{\frac{p(x)-2}{2}}(|\nabla u_h|^{p(x)-2} \\ &\quad + h^{\frac{p(x)-2}{2}})|\nabla\varphi_1|^2 dtdx \\ &\leq \int\int_{B(x_0,\frac{1}{2}r_0)\times(0,T)} (Cr^{p(x)-2} \wedge \frac{g_0}{4\mu_1} + 2^{\frac{p(x)-2}{2}} h^{\frac{p(x)-2}{2}})|\nabla\varphi_1|^2 dtdx \\ &= T\Big(C\mu_1 r^{p(x)-2} \wedge \frac{g_0}{4} + 2^{\frac{p(x)-2}{2}}\mu_1 h^{\frac{p(x)-2}{2}}\Big)\int_{B(x_0,\frac{1}{2}r_0)} \varphi_1^2 dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{B(x_0,\frac{1}{2}r_0)\times(0,T)} \varphi_1^2\big(g - |u_h|^{p(x)-2} - \Psi[u_h]\big)dtdx \\ &\qquad\geq \int_{B(x_0,\frac{1}{2}r_0)} \varphi_1^2(x)\Big(\int_0^T g(x,t)dt - Tr^{p(x)-2} - TC_2r^2\Big)dx \\ &\qquad\qquad\geq \big(Tg_0 - Tr^{p^- -2} - TC_2r^2\big)\int_{B(x_0,\frac{1}{2}r_0)} \varphi_1^2(x)dx.\end{aligned}$$

Therefore we obtain

$$g_0 \leq C_2r^2 + r^{p^- -2} + C\mu_1 r^{p^- -2} \wedge \frac{g_0}{4} + 2^{\frac{p^- -2}{2}}\mu_1 h^{\frac{p^- -2}{2}}.$$

Obviously, this inequality does not hold if

$$h \leq \frac{1}{2}\Big(\frac{g_0}{4\mu_1}\Big)^{\frac{2}{p^- -2}}, \qquad r \leq \min\Big\{\Big(\frac{g_0}{4C_2}\Big)^{\frac{1}{2}}, \Big(\frac{g_0}{4}\Big)^{\frac{1}{p^- -2}}, \Big(\frac{g_0}{4C\mu_1}\Big)^{\frac{1}{p^- -2}}\Big\}.$$

Therefore there exists one positive constant r, such that no solution u_h of the problem (3.2) satisfy

$$0 < \|u_h\|_{L^\infty(Q_T)} \leq r.$$

The proof is hence completed. □

In this paper, we should use the theory of topological degree to find the solution of the problem (3.2). We will consider the problem in the set

$$\Lambda = \big\{u_h \in \mathcal{C}_T(\overline{Q}_T)|u_h \in B_R,\ u_h \notin B_r\big\},$$

where B_R is a ball centered at the origin with radius R in $\mathcal{C}_T(\overline{Q}_T)$, $0 < r < R$, where r, R are all constants independent of h.

Now we will discuss the topological degree in B_R and B_r separately. In order to apply the homotopy invariant of Leray-Schauder degree, let $\lambda \in [0,1]$, and consider the map $u_h = F\big(\lambda(g - |v|^{p(x)-2} - \Psi[v])v^+\big)$. First we verify that $u_h = F\big(\lambda(g - |v|^{p(x)-2} - \Psi[v])v^+\big)$ is an homotopic map of $u_h = F\big(g - |v|^{p(x)-2} - \Psi[v])v^+\big)$, meaning that the value of the topological degree deg $\big(u_h - F\big(\lambda(g - |v|^{p(x)-2} - \Psi[v])u_h^+\big), B_R, 0\big)$ is identical whatever might be the parameter $\lambda \in [0,1]$.

Lemma 3.3 *Let $\lambda \in [0,1]$. Then there exists a positive constant R independent of h, such that no solutions $u_h \in \mathcal{C}_T(\overline{Q}_T)$ satisy both the following problem*

$$\begin{cases} \dfrac{\partial u_h}{\partial t} - div\big((|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} \nabla u_h\big) + \lambda |u_h|^{p(x)-2} u_h = \lambda(g - \Psi[u_h]) u_h, \ (x,t) \in Q_T, \\ u_h(x,t) = 0, \quad (t,x) \in \partial\Omega \times (0,T), \\ u_h(x,0) = u_h(x,T), \quad x \in \Omega, \end{cases} \tag{3.5}$$

and the condition $\|u_h\|_{L^\infty(Q_T)} = R$.

Proof: Like for the problem (3.2), we can also define the strong generalized solution of the problem (3.5). Let u_h be the strong generalized solution of the problem (3.5). Multiplying u_h by the first equation in (3.5), and then integrating over Q_T, we have

$$\int_{Q_T} u_h \frac{\partial u_h}{\partial t} dtdx + \int_{Q_T} (|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} |\nabla u_h|^2 dtdx$$
$$= \int_{Q_T} \lambda(g - |u_h|^{p(x)-2} - \Psi[u_h]) u_h^2 dtdx.$$

By the periodicity of u_h, we obtain

$$\int\int_{Q_T} u_h \frac{\partial u_h}{\partial t} dtdx = 0.$$

Since

$$\int\int_{Q_T} (|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} |\nabla u_h|^2 dtdx \geq 0,$$

then

$$\int\int_{Q_T} \lambda(g - |u_h|^{p(x)-2} - \Psi[u_h]) u_h^2 dtdx \geq 0.$$

Setting $M = \sup_{x\in\Omega, t\in[0,T]} g(x,t)$, and combining with the assumption (A_1) and $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega)$, we can see that

$$\begin{aligned} 0 &\leq \int_{Q_T} \lambda(g - |u_h|^{p(x)-2} - \Psi[u_h]) u_h^2 dtdx \\ &\leq \int_{Q_T} \big(g - |u_h|^{p(x)-2} - \Psi[u_h]\big) u_h^2 dtdx \\ &\leq \int_{Q_T} M u_h^2 dtdx - \int_{Q_T} |u_h|^{p(x)} dtdx - \int_{Q_T} \big(u_h^2 \Psi[u_h]\big) dxdt \\ &\leq M \int_{Q_T} u_h^2 dtdx - C' \int_{Q_T} |u_h|^2 dtdx - C \int_0^T \Big(\int_\Omega u_h^2 dx\Big)^2 dt, \\ &\leq C'' \int_{Q_T} u_h^2 dtdx - C \int_0^T \Big(\int_\Omega u_h^2 dx\Big)^2 dt, \end{aligned}$$

that is

$$\int_0^T \Big(\int_\Omega u_h^2 dx\Big)^2 dt \leq C \int_0^T \Big(\int_\Omega u_h^2 dx\Big) dt, \tag{3.6}$$

where C is a constant independent of λ and h. Furthermore, utilizing Cauchy's inequality, we have

$$\int_\Omega u_h^2 dx \leq \frac{1}{4\varepsilon^2} + \varepsilon^2 \Big(\int_\Omega u_h^2 dx\Big)^2,$$

that is

$$\int_0^T \int_\Omega u_h^2 dxdt \leq \frac{1}{4\varepsilon^2} + \varepsilon^2 \int_0^T \Big(\int_\Omega u_h^2 dx\Big)^2 dt. \tag{3.7}$$

Combining (3.6) with (3.7), we have

$$\|u_h\|_{L^2(Q_T)} \le C,$$

where C is a constant independent of λ and h.
By (3.2) and $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega)$, we have

$$\int_{Q_T} |u_h|^{p(x)} dtdx \le \int\int_{Q_T} |\nabla u_h|^{p(x)} dtdx \le C \max_{Q_T} |g(x,t)| \int_{Q_T} |u_h|^2 dtdx.$$

The assumption (A_1) implies that there exists a constant $K > 0$, such that u_h satisfies

$$\frac{\partial u_h}{\partial t} - div\big((|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} \nabla u_h\big) \le K u_h.$$

Therefore by the Young inequality and [4, Theorem 3.2] with $\delta = p(x)$, $q = \dfrac{p(x)(N+2)}{N}$ and $\kappa = \dfrac{p(x)}{N}$, we have the following estimate

$$u_h(x,t) \le C\Big(\int_{\frac{T}{2}}^{\frac{3T}{2}} \int_\Omega u_h^2 dxdt\Big),$$

for $t \in \left[\frac{T}{2}, \frac{3T}{2}\right]$. Then by the periodicity of u_h, we have

$$\|u_h\|_{L^\infty(Q_T)} \le C,$$

where C is independent of λ and h. So the proof is completed if $R > C$. □

Utilizing the above result, and the result in [15] with the case $\lambda = 0$, we can apply the homotopy invariant of Leray-Schauder degree and discuss the topological degree of the problem (3.2) in B_R.

Lemma 3.4 *There exists a constant R such that*

$$\deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big), B_R, 0\big) = 1.$$

Proof: Using Lemma 3.3, and the existence and uniqueness of the solution of problem (3.2) when $\lambda = 0$, we apply the homotopy invariance of the Leray-Schauder degree and obtain

$$1 = \deg\big(u_h, B_R, 0\big) = \deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big), B_R, 0\big).$$

□

We now explore the topological degree of problem (3.2) in B_r. Due to the fact that problem (3.2) has the solution $u_h = 0$, we cannot handle the problem directly. Therefore, for any given constant $\gamma \ge 0$ and a smooth function $\theta = \theta(x) > 0$ in Ω, we introduce a map $u_h = F\big((g - \Psi[v])v^+\big) + \gamma F(\theta)$. By the Leray-Schauder degree theory, it follows that for $\gamma \ge 0$ and $\theta > 0$, the map $u_h = F\big((g - \Psi[v])v^+\big) + \gamma G(\theta)$ is homotopic to the map $u_h = F\big((g - \Psi[v])v^+\big)$, meaning that for all $\theta > 0$, the topological degree $\deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big), B_r, 0\big)$ is the same for all $\gamma \ge 0$.

Lemma 3.5 *For any given smooth function $\theta = \theta(x) > 0$ in Ω and a constant $\gamma > 0$, all solutions $u_h \in \mathcal{C}_T(\overline{Q}_T)$ of the following problem*

$$\begin{cases} \dfrac{\partial u_h}{\partial t} - \operatorname{div}\big((|\nabla u_h|^2 + h)^{\frac{p(x)-2}{2}} \nabla u_h\big) + |u_h|^{p(x)-2} u_h = (g - \Psi[u_h])u_h + \gamma\theta, & (x,t) \in Q_T \\ u_h(x,t) = 0, & (t,x) \in \partial\Omega \times (0,T), \\ u_h(x,0) = u_h(x,T), & x \in \Omega, \end{cases}$$

satisfy

$$\|u_h\|_{L^\infty(Q_T)} > r > 0,$$

where r is a constant independent of h, γ, and θ.

Proof: Similarly to Lemma 3.2, we can find a constant r independent of h, γ, and θ, such that when $0 \leq \|u_h\|_{L^\infty(Q_T)} \leq r$, it contradicts condition (A_2). □

Noting that after introducing the map $u_h = F\big((g - |v|^{p(x)-2} - \Psi[v])v^+\big) + \gamma G(\theta)$, $u_h = 0$ is not the solution of the problem, we obtain the following lemma.

Lemma 3.6 *There exists a constant r such that*

$$\deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big), B_r, 0\big) = 0.$$

Proof: Using Lemma 3.5 and the homotopy invariance of the Leray-Schauder degree, we have

$$\begin{aligned} 0 &= \deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big) - \gamma F(\theta), B_r, 0\big) \\ &= \deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big), B_r, 0\big). \end{aligned}$$

The proof is completed. □

Summing up the results above, we can see that when $\Psi[u_h]$ and m satisfy conditions (A_1) and (A_2), there exist constants $r, R > 0$ such that

$$\begin{aligned} \deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big), B_R, 0\big) &= 1, \\ \deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big), B_r, 0\big) &= 0, \end{aligned}$$

that is

$$\deg\big(u_h - F\big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h^+\big), \Lambda, 0\big) = 1.$$

By the theory of Leray-Schauder degree, we can conclude that problem (3.2) has a non-trivial periodic solution in u_h. Then by Lemma 3.1, we see that problem (3.2) has a nonnegative non-trivial periodic solution in Q_T.

Finally, we can state our main result:

Theorem 3.1 *Problem (1.1) has a nonnegative non-trivial periodic solution u.*

Proof: Using Lemma 3.2 and the proof of Lemma 3.3, we know that all solutions $u_h \in \Lambda$ of problem (3.2) satisfy

$$r < \|u_h\|_{L^\infty(Q_T)} < R,$$

where r and R are constants independent of h. In the proof of Lemma 3.2, we have

$$\|\nabla u_h\|_{L^\infty(Q_T)} \leq C,$$

where C is independent of h. Combining this with conditions (3.3) and (3.4), it follows that

$$u_h \in L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)) \cap \mathcal{C}_T(\overline{Q}_T).$$

To obtain the uniform estimate of $\dfrac{\partial u_h}{\partial t}$, we multiply the first equation in (1.1) by $(u_h)_t$ and integrate over Q_T. By the periodicity of u_h, condition (A_1), and Cauchy's inequality,

$$\int_{Q_T} (u_h)_t^2 \, dx\, dt = \int_{Q_T} (g - |u_h|^{p(x)-2} - \Psi[u_h])u_h(u_h)_t \, dx\, dt - \int_{Q_T} A(\nabla u_h)\nabla u_h \nabla (u_h)_t \, dx\, dt,$$

holds. Noting the periodicity of u_h, we have

$$\int_{Q_T} A(\nabla u_h)\nabla u_h \nabla (u_h)_t \, dx\, dt = \int_0^T \frac{\partial}{\partial t}\left(\int_\Omega B(\nabla u)\, dx\right) dt = 0,$$

where $B(s) = \int_0^s A(\xi)\xi\, d\xi$. Thus,

$$\begin{aligned}\int_{Q_T} (u_h)_t^2\, dx\, dt &= \int_{Q_T} (g - |u_h|^{p(x)-2} - \Psi[u_h])u_h(u_h)_t\, dx\, dt \\ &= \int_{Q_T} (g - |u_h|^{p(x)-2} - \Psi[u_h])u_h(u_h)_t\, dx\, dt \\ &\le \frac{1}{2}\int_{Q_T} \big((g - |u_h|^{p(x)-2} - \Psi[u_h])u_h\big)^2\, dx\, dt + \frac{1}{2}\int_{Q_T} (u_h)_t^2\, dx\, dt.\end{aligned}$$

By the boundedness of $\Psi[\omega]$, $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega)$, and u_h, we have

$$\int_{Q_T} (u_h)_t^2\, dx\, dt \le C,$$

with the constant C independent of h. Thus, we obtain the uniform estimate of $\dfrac{\partial u_h}{\partial t}$,

$$\left\| \frac{\partial u_h}{\partial t} \right\|_{L^2(Q_T)} \le C.$$

From the above estimates of u_h, and noting that the constant C is independent of h, we conclude that there exists a subsequence $\{u_{h_i}\}$ in Λ and $u \in \Lambda$ such that

$$\begin{aligned}&|\nabla u_{h_i}|^{p(x)-2}\nabla u_{h_i} \to |\nabla u|^{p(x)-2}\nabla u, && \text{weakly in } L^{p(x)}(Q_T), \\ &\frac{\partial u_{h_i}}{\partial t} \to \frac{\partial u}{\partial t}, && \text{weakly in } L^2(Q_T), \\ &u_{h_i} \to u, && \text{in } C(Q_T).\end{aligned}$$

It is straightforward to see that u is the nonnegative non-trivial periodic solution of problem (1.1). The proof is completed. □

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Ethical Approval

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Availability of data and materials

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References

1. Allegretto, W.; Nistri, P.: Existence and optimal control for periodic parabolic equations with nonlocal terms. IMA Journal of Mathematical Control and Information, **16**(1),43-58 (1999)
2. Calsina, A.; Perello, C.: Equations for biological evolution. Proceedings of the Royal Society of Edinburgh Section A: Mathematics. **125**(5), 939-958 (1995)
3. Calsina, Á.; Perelló, C.; Saldaña, J.: Non-local reaction-diffusion equations modelling predator-prey coevolution. Publicacions matematiques. 315-325 (1994)

4. Dibenedetto, E.: Degenerate Parabolic Equations, Springer-Verlag, New York, (1993)

5. Diening, L.; Harjulehto, P.; Häzstö, P., & Ruzicka, M. Lebesgue and Sobolev spaces with variable exponents. Springer (2011).

6. Fan, X. L.; Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl, **263**(2), 424-446 (2001)

7. Jourhmane H., Kassidi A., Elomari M. and Hilal K. *Existence of periodic solution for double-phase parabolic problems with strongly nonlinear source* Filomat 37:27 (2023), 9357–9370.

8. Jourhmane H., Kassidi A., Elomari M. and Hilal K. *Periodic solutions for a degenerate double-phase parabolic equation with variable growth.* Advances in Operator Theory, 8(4), 71.

9. Jourhmane H., Kassidi A., Elomari M. and Hilal K. *Existence of non-negative periodic solutions for a degenerate anisotropic parabolic equation with strongly nonlinear source.* Analysis, (2023)(0).

10. Harjulehto, P.; Häzstö, P.: Sobolev inequalities with variable exponent attaining the values 1 and n. Publicacions matemàtiques, 347-363(2008).

11. Mahoui, S.; Moulay, M.S.; Omrane, A.: Finite element approximation to optimal pointwise control of parabolic problems with incomplete data. on Mathematical modelling in applied sciences. (2017)

12. Pao, C.V.: Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, (1992)

13. Wang, Y.; Wu, Z.; Yin, J.: Periodic solutions of evolution p-Laplacian equations with weakly nonlinear sources, Int. J. Math. Game Theory Algebra. **10** (1), 67-77 (2000)

14. Wang, Y.; Yin, J.; Wu, Z.: Periodic solutions of evolution p-Laplacian equations with nonlinear sources, J. Math. Anal. Appl. **219** (1), 76-96 (1998)

15. Wu, Z.Q.; Zhao, Q.N.; Yin, J.X.; Li, H.L.: Nonlinear Diffusion Equations, World Scientific, Singapore. (2001)

Hamza Jourhmane,
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
Beni Mellal, Morocco.
E-mail address: hamzajourh@gmail.com

and

Ali El Mfadel (Corresponding author),
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
Beni Mellal, Morocco.
E-mail address: a.elmfadel@usms.ma

and

Abderrazak Kassidi,
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
Beni Mellal, Morocco.
E-mail address: abderrazakassidi@gmail.com