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Inverse Source Problem with a Variable Source Coefficient

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ABSTRACT: The current research investigates an inverse source problem with variable coefficients in a nonhomogeneous hyperbolic equation. This linear inverse force problem divides into a direct problem and an inverse problem. Although the solution's uniqueness is well established in the literature, the inverse problem remains ill-posed because small data changes can cause significant errors. To obtain stable numerical solutions, we use a finite difference method combined with zero-order Tikhonov regularization. We test different regularization parameter and to choose the optimal parameter based on the error norm. Numerical experiments demonstrate that our method produces accurate results with exact data and remains stable even when noise is present.

Keywords: Inverse source problem, nonhomogeneous hyperbolic equation, norm error, zero-order Tikhonov regularization.

Contents

1. Introduction

This paper investigates an inverse force problem related to a non-homogeneous hyperbolic wave equation. The forcing function is assumed to depend solely on the spatial variable "space," although its coefficient depends on the variable "such as time and sapce", to ensure the uniqueness of the solution. The issues of uniqueness and stability in inverse source problems like this have been extensively studied in existing literature, including works by Cannon and Dunninger [1], Engl et al. [2], Yamamoto [5], Tikhonov and Samarskii [6], and Hussein [8,10]. These studies offer sufficient conditions on the data to guarantee uniqueness and, in some cases, continuous dependence of the solution on the measured data. Despite a solid theoretical foundation, numerical analysis remains limited due to the problem's inherently ill-posed nature.

Furthermore, Inverse force problems with hyperbolic equations are often ill-posed because minor data variations can lead to significant errors in the source term. To mitigate this, approaches such as Carleman estimates and control strategies have been developed [4,5]; in this study, we use zero-order Tikhonov regularization. Numerical analyses of space-dependent sources in the wave equation have also been performed [3,8,10,11,12,14], employing additional data to enhance stability.

Previous studies addressed inverse wave problems using techniques such as separation of variables and boundary-based methods [8]. However, these approaches relied on restrictive assumptions about wave speed and source structure [3,10], which limited their effectiveness when wave speed varies or the forcing term deviates from typical forms. To overcome these limitations and expand applicability, this research divided the inverse source problem into two direct problems: a homogeneous hyperbolic equation and an inverse problem involving a non-homogeneous hyperbolic equation. A finite difference method (FDM)

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was used to discretize the hyperbolic equation numerically in [9,11,12,13,14], where the source coefficient was constant; in this work, the coefficient depends on variables such as space and time.

Discretization results in a set of ill-conditioned linear equations, emphasizing the ill-posed nature of the inverse problem. To ensure a stable solution, zero-order Tikhonov regularization is applied, as shown in [6,10]. The choice of the regularization parameter is vital for balancing stability and accuracy; in this work, it is determined using an error criterion [11,12,13,14]. The primary contribution is the development of an efficient numerical method for solving the inverse force problem with variable coefficients. A systematic analysis of stability and accuracy is conducted under both exact and noisy data conditions, confirming the effectiveness of the approach through numerical experiments.

The paper's organization is as follows: Section 2 briefly overviews inverse force problems related to the hyperbolic wave equation with a variable source coefficient, including the algorithm for this method. Section 3 discusses the finite difference method (FDM) for direct and inverse problems. Section 4 presents a numerical example with source coefficients like $\{t, 1+t, 1+x+t\}$.

2. Mathematical Formulation

This research aims to identify the source term $f(x)$ in a hyperbolic differential equation, where the source coefficient depends on $h(x,t)$. Unlike previous studies [9,11,12,13,14], which assumed a constant source coefficient, our approach differs. Additionally, whereas [8,10] directly solved the problem using the finite difference method, we first decompose the hyperbolic equation into direct and inverse problems, then solve them numerically with the finite difference method. The approach is outlined in the algorithm below:

Step 1: divide $u_{tt}(x,t) = c^2\nabla^2 u(x,t) + f(x)h(x,t), \quad (x,t) \in [0,L] \times [0,T]$, into

$$\text{a) } v_{tt}(x,t) = c^2 v_{xx}(x,t) \tag{2.1}$$

$$\text{b) } w_{tt}(x,t) = w_{xx}(x,t) + f(x)h(x,t), \tag{2.2}$$

for simplicity, we will take $c = 1$

Step 2: a) equation (2.1) is defined by the given initial and boundary conditions, as follows:

$$v(x,0) = u(x,0) = u_0(x), v_t(x,0) = u_t(x,0) = v_0(x), \quad x \in [0,L] \tag{2.3}$$

$$v(0,t) = u(0,t) =: p_0(t), \quad t \in (0,T] \tag{2.4}$$

$$v(L,t) = u(L,t) =: p_L(t), \quad t \in (0,T] \tag{2.5}$$

where $\{u(x,0), u_t(x,0), u(0,t), u(L,t)\}$ are given data.

b) the zero initial and boundary conditions consider for equation (2.2) as:

$$w(x,0) = 0, \quad w_t(x,0) = 0, \quad x \in [0,L] \tag{2.6}$$

$$w(0,t) = 0, \quad t \in (0,T] \tag{2.7}$$

$$w(L,t) = 0, \quad t \in (0,T] \tag{2.8}$$

Step 3: In this steps, we will find the function $f(x)$ using Step 2(b) and

$$-w_x(0,t) = q(0,t) - qq(0,t) =: q_0(t) - qq_0(t), \quad t \in (0,T] \tag{2.9}$$

where, $q(0,t)$ is the given data from the main problem, such that $-u_x(0,t) = q(0,t) =: q_0(t)$, and $qq(0,t)$ obtainable from Step 2(a) and equation (2.1), where $-v_x(0,t) = qq(0,t) =: qq_0(t)$. After calculating $f(x)$, you can derive the function $w(x,t)$ either through an iterative process or a matrix-based method. Once $w(x,t)$ is determined, the solution $u(x,t)$ can be recovered because $v(x,t)$ is obtained in Step 2(a) and $w(x,t)$ is computed in this step. Since $u(x,t) = v(x,t) + w(x,t)$, we can then determine the function $u(x,t)$.

3. Numerical Analysis

The explicit central finite difference method is used to solve the problem described in Section 2. This discretization technique, based on reference [9,11,12,13,14], is summarized briefly below:

- First, use a loop or linear system in MATLAB for the discretization described below, when it is step 2(a) and (2.1) in section 2, to determine $v_{i,j}$ and $qq_0(t_j)$:

$$v_{i,j}=r^2 v_{i+1,j}+2\left(1-r^2\right) v_{i,j}+r^2 v_{i-1,j}-v_{i,j-1}, \quad i=\overline{1,(M-1)}, j=\overline{1,(N-1)},$$
$$v_{i,1}=\frac{1}{2} r^2 v_0\left(x_{i+1}\right)+\left(1-r^2\right) v_0\left(x_i\right)+\frac{1}{2} r^2 v_0\left(x_{i-1}\right)+(\Delta t) v_0\left(x_i\right), \quad j=0, \quad i=\overline{1,(M-1)},$$
$$v_{i,0}=v_0\left(x_i\right), \quad i=\overline{0, M}, \quad \frac{v_{i,1}-v_{i,-1}}{2(\Delta t)}=v_0\left(x_i\right), \quad i=\overline{1,(M-1)}$$
$$v_{0,j}=p_0\left(t_j\right), \quad v_{M,j}=p_L\left(t_j\right), \quad j=\overline{0, N}$$

- Second, discretization step 2(b) and (2.2) in section 2 as follows:

$$w_{i,j+1}-(\Delta t)^2 f_i h_{i,j}=r^2 w_{i+1,j}+2\left(1-r^2\right) w_{i,j}+r^2 w_{i-1,j}-w_{i,j-1}, \quad i=\overline{1,(M-1)},$$
$$j=\overline{1,(N-1)}$$
$$w_{i,1}-\frac{1}{2}(\Delta t)^2 f_i h_{i,0}=\frac{1}{2} r^2 w_0\left(x_{i+1}\right)+\left(1-r^2\right) w_0\left(x_i\right)+\frac{1}{2} r^2 w_0\left(x_{i-1}\right)+(\Delta t) v_0\left(x_i\right),$$
$$j=0, i=\overline{1,(M-1)}$$
$$w\left(x_i, 0\right)=w_{i0}=0, \quad w_t\left(x_i, 0\right)=0, \quad w\left(0, t_j\right)=w\left(L, t_j\right)=0$$
$$w_x\left(0, t_j\right)=q_0\left(t_j\right)-qq_0\left(t_j\right),$$

to determine the values of $(w_{i,j}, f_i h_{i,j})$, utilize MATLAB to solve the linear system derived from the above equations;

$$B\binom{\underline{w}}{\underline{f}}=\underline{C}, \tag{3.1}$$

where $v_{i,j}=v\left(x_i, t_j\right), w_{i,j}=w\left(x_i, t_j\right), f_i=f\left(x_i\right), h\left(x_i, t_j\right)=h_{i,j}, x_i=i \Delta x, t_j=j \Delta t$ and $r=c \Delta t / \Delta x$, for $i=0,1, \ldots, M, j=0,1, \ldots N$, in addition $\Delta x=\frac{L}{M}$ and $\Delta t=\frac{T}{N}$, such that divide the domain $(0, L) \times(0, T)$ into M and N

- Third, the existence and uniqueness theorems are presented in Theorems 1 and 2 of reference [8], with further discussion in references [1,2,9,10]. Consequently, this study concentrates solely on stability analysis. To assess stability, Gaussian normal distribution with a mean of zero and standard deviation $\sigma=p \times \max_{t \in[0, T]}\left|q_0(t)\right|$, where p is the percentage of noise, is added to the additional conditions to simulate real-world data imprecision:

$$q_0^\epsilon\left(t_j\right)=q_0\left(t_j\right)+\epsilon_j, \quad j=\overline{1, N}$$

where $\left(\epsilon_j\right)_{j=\overline{1, N}}$ are N random noisy variables and using the MATLAB function "normrd"

- Four, to address potential instabilities in our solution, we use zero-order Tikhonov regularization. Prior to implementation, we reduce the size of the matrices involved. Due to linearity, we can eliminate the variable w from equation (3.1), simplifying the system from $(M-1) \times N+N$ equations with $(M-1) \times N+(M-1)$ unknowns to N equations with ($M-1$) unknowns, as:

$$A \underline{f}=\underline{b}^\epsilon \tag{3.2}$$

- Five, least squares "zero-order Tikhonov regularization" have been used to solve the matrices above, as follows:

$$\underline{f_\lambda}=\left(A^{tr} A+\lambda I\right)^{-1} A^{tr} \underline{b}^\epsilon, \tag{3.3}$$

where λ is a regularization parameter and I is an identity matrix.

- Six, to choose the best λ value through norm error minimization:

$$\left\|f_{\text{numerical }}-f_{\text{exact }}\right\|=\sqrt{\sum_{n=1}^N\left(f_{\text{numerical }}\left(t_n\right)-f_{\text{exact }}\left(t_n\right)\right)^2} \tag{3.4}$$

4. Numerical Results and Discussion

We examine the same examples in [8,10] to understand how splitting the main problem into two parts, "direct and inverse problem," and solving each with the FDM affects the results. The uniqueness results stated in (Theorems (1) and (2) in [8]) are expected to hold, the following are initial and boundary conditions

$$u(x,0) = \phi(x) = 0, \quad u_t(x,0) = \psi(x) = 0, \quad x \in [0,1] \tag{4.1}$$

$$u(0,t) = p_0(t) = 0, \quad u(1,t) = p_L(t) = 0, \quad t \in (0,1] \tag{4.2}$$

with additional condition

$$-\frac{\partial u}{\partial x}(0,t) = q_0(t) = -\pi. \quad t \in (0,1] \tag{4.3}$$

And the exact solutin $f(x)$ is

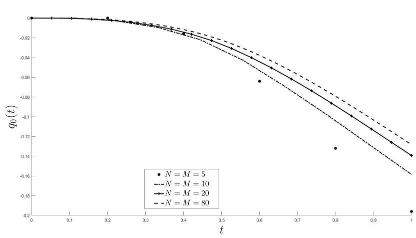
$$f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} < x \le 1 \end{cases} \tag{4.4}$$

We will analyze various coefficients, such as $h(x,t)$ including (a) t, (b) $1+t$, and (c) $1+x+t$, as shown in sections a, b, and c of figures 1-5. Although these values differ, they all use the same input data (4.1)-(4.3); the only variation is in the choice of the coefficient $h(x,t)$.

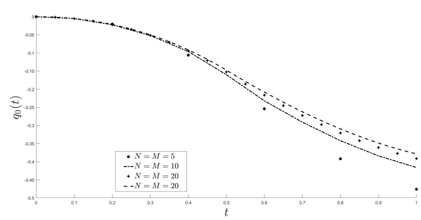
Since all coefficients of $h(x,t)$ are nonzero at (0,0), it follows from Theorem 2 and equation (3.12) in [8] that the solution is unique. Additionally, note that: first, the force $f(x)$ (4.1) in this example is triangular, continuous, and non-differentiable at $x = 1/2$. Second, the displacement $u(x,t)$ does not have an analytical solution.

First using step 3 to get numerical soultion $qq(0,t)$, which is from Step 2(a) and equation (2.1) "$-v_x(0,t) = qq(0,t) =: qq_0(t)$ ". Although the exact solution $(v(x,t), qq(0,t))$ unknown, the results suggest that the obtained solution is plausible and reliable, as shown in Figure 1, the curves converge when $N = M$ increase, confirming the solution's accuracy. Furthermore, Figures 1(a), 1(b) and 1(c) illustrate $qq(0,t)$ for $h(x,t) = t, h(x,t) = 1+t$, and $h(x,t) = 1+x+t$, respectively.

(a) $h(x,t) = t$



(b) $h(x,t) = 1+t$



(c) $h(x,t) = 1+x+t$

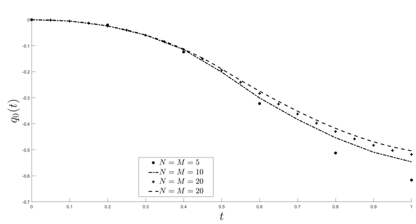
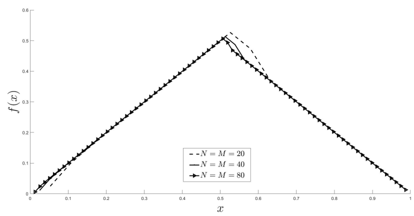


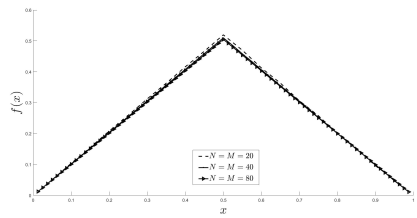
Figure 1: A numerical solution for the flux tension $qq(0,t) =: qq_0$ at $x = 0$ for different $N = M \in \{5, 10, 20, 80\}$, where (a) $h(x,t) = t$, (b) $h(x,t) = 1+t$, and (c) $h(x,t) = 1+x+t$.

To establish both existence and uniqueness, we form the linear system $Af = b$ from equations (2.6)-(2.9) to solve for $f(x)$. The numerical results, shown in Figure 2, indicate that as $M = N$ increases from 20, 40, and 80, the solutions increasingly approach the exact solution (4.4), confirming the method's accuracy.

(a) $h(x,t) = t$



(b) $h(x,t) = 1 + t$



(c) $h(x,t) = 1 + x + t$

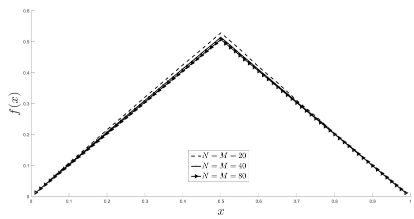
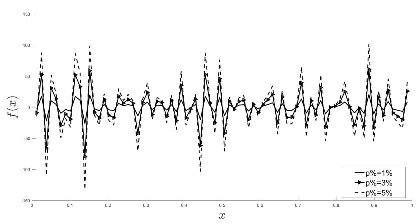


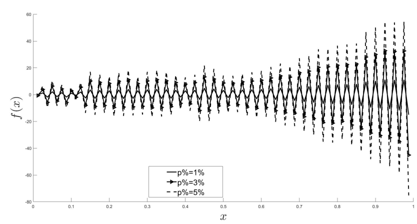
Figure 2: The numerical solution derived from (2.6)-(2.9) is compared with the exact solution (4.4) for $f(x)$, $N = M \in \{5, 10, 20, 80\}$, where (a) $h(x,t) = t$, (b) $h(x,t) = 1 + t$, and (c) $h(x,t) = 1 + x + t$.

To analyze stability, noise "$p = 1, 3, 5\%$" is introduced to equation (4.3), which is based on system (3.2). As illustrated in Figure 3, this results in the numerical solution $f(x)$ becoming unstable.

(a) $h(x,t) = t$



(b) $h(x,t) = 1 + t$



(c) $h(x,t) = 1 + x + t$

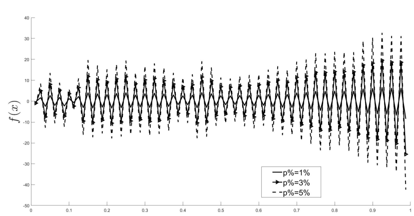


Figure 3: The numerical reconstruction from equation (3.2) for noisy data with noise levels $p = \{1, 3, 5\}\%$ is contrasted with the exact solution $f(x)$ provided by equation (4.4), where (a) $h(x,t) = t$, (b) $h(x,t) = 1 + t$, and (c) $h(x,t) = 1 + x + t$.

To stabilize the numerical solution $f(x)$, we apply equation (3.3), which corresponds to zero-order Tikhonov regularization. The optimal regularization parameter λ is selected using equation (3.4). After testing various λ values (such as $\{3 \times 10^{-5}, 5 \times 10^{-7}, 10^{-8}, \dots\}$), we choose the best λ based on the criterion in equation (3.4), as illustrated in Figure 4. The results for this optimal λ are presented in Figure 4: (a) $\lambda = 9 \times 10^{-8}$, (b) $\lambda = 5 \times 10^{-7}$, and (c) $\lambda = 4 \times 10^{-6}$. Using this λ, the stabilized numerical solution $f(x)$ is computed and displayed in Figure 5, confirming the solution's stabilization.

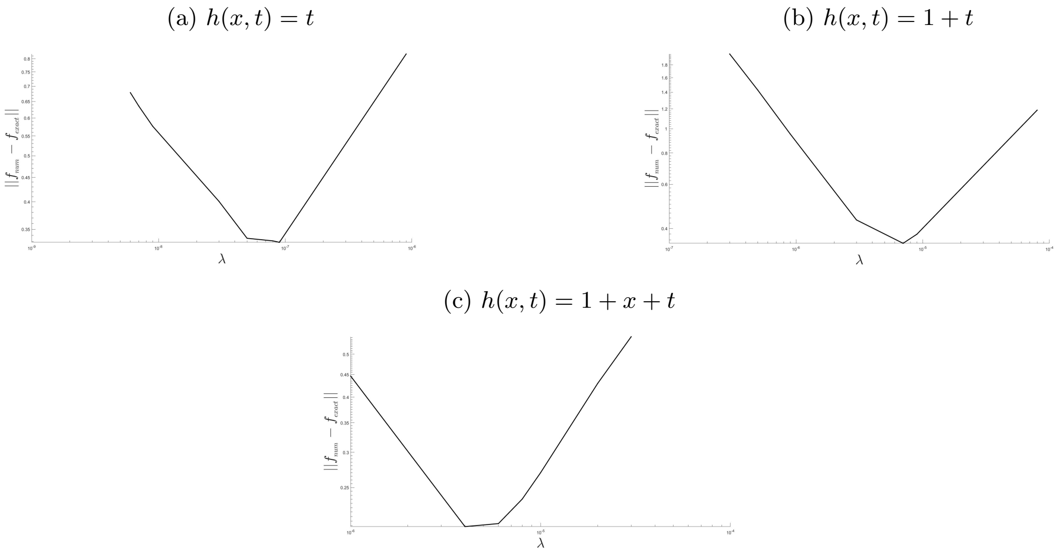


Figure 4: The norm error for $f(x)$ is given as a function of λ for $M = N = 80$ and noise $p\% = 1\%$, for the same grid size $\|f_{\text{numerical}} - f_{\text{exact}}\|$, where (a) $h(x,t) = t$, (b) $h(x,t) = 1 + t$, and (c) $h(x,t) = 1 + x + t$.

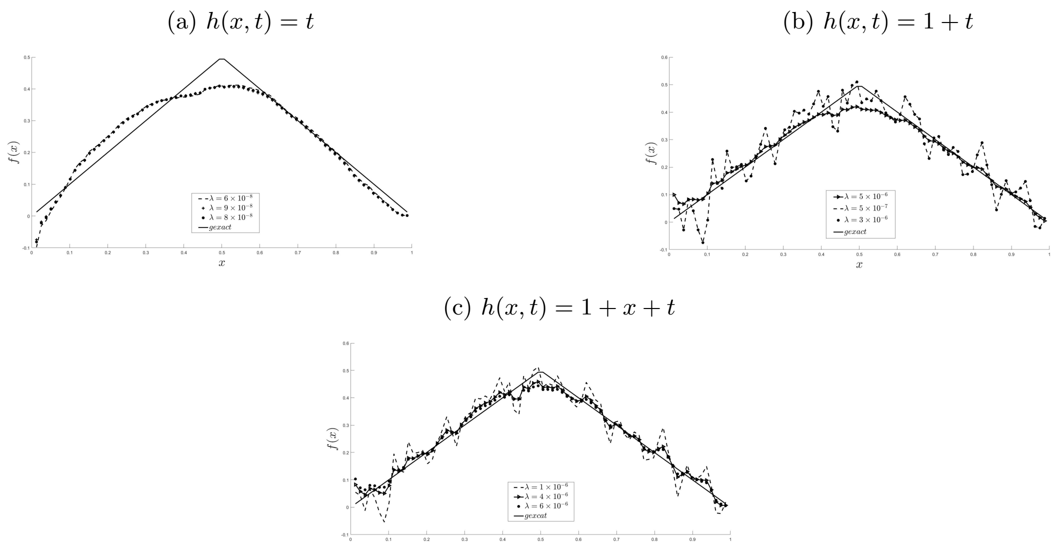


Figure 5: The exact solution $f(x)$ in (4.4) is compared with the numerical solution for $M = N = 80$ using $\lambda \in \{6 \times 10^{-8},\, 9 \times 10^{-8},\, 8 \times 10^{-8}\}$, where (a) $h(x,t) = t$, (b) $h(x,t) = 1 + t$, and (c) $h(x,t) = 1 + x + t$.

5. Conclusion

This study examined numerical solutions to a non-homogeneous hyperbolic equation where the coefficient of the non-homogeneous term depends on space, time, or the variable itself. The main problem is divided into direct (homogeneous hyperbolic equation) and inverse (non-homogeneous hyperbolic equation) problems, both of which are tackled using the Finite Difference Method (FDM) combined with zero-order Tikhonov regularization to improve stability. Numerical examples with corresponding figures demonstrate that these methods are both effective and accurate.

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