Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 3** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.81817

Generalized Derivative Inequalities and Their Extensions in L_p Normed Spaces

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ABSTRACT: In this work, we investigate Turán-type inequalities in the L_q-norm for generalized derivatives of polynomials. The study is further extended to encompass the generalized polar derivative. The findings not only broaden several established results but also yield additional inequalities as particular instances. The theorems presented herein provide a significant generalization of several established results in the theory of polynomial inequalities.

Keywords: Polynomial, inequalities, generalized polar derivative and L_P-norm.

Contents

1. Introduction and Preliminaries

Let P_n denote the space of all algebraic polynomials $P(z)$ of degree n of the form $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$, and let $P'(z)$ be its ordinary derivative. Let $\mathbb{R}_+^n$ be the set of all n-tuples $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ of positive real numbers with $\gamma_j \geq 1$ for $1 \leq j \leq n$. The generalized derivative $P^\gamma(z)$ of $P(z)$ is defined as $P^\gamma(z)=\sum_{j=1}^{n}\gamma_j\prod_{\substack{v=1\\ v\neq j}}^{n}(z-z_v)$, where $z_1,z_2,\ldots,z_n$ are the zeros of $P(z)$. When $\gamma=(1,1,\ldots,1)$, the generalized derivative $P^\gamma(z)$ reduces to $P'(z)$. Polynomial inequalities, particularly those involving derivatives, play a central role in approximation theory and applied mathematics. The origins of such problems can be traced to the work of chemist Dmitri Mendeleev [6], who investigated the relationship between the specific gravity of solutions and the percentage of dissolved substances. During his research, Mendeleev formulated a problem: for a quadratic polynomial $P(x)$ satisfying $|P(x)| \leq 1$ on $x \in [-1,1]$, what is the maximum possible value of its derivative $|P'(x)|$ over the same interval. He solved this problem himself, showing that for such polynomials, $|P'(x)| \leq 4$.

Later, A. A. Markov [5] extended this result to polynomials of arbitrary degree n with real coefficients. Specifically, he proved that if $P(x)=\sum_{j=1}^{n} a_j x^j$ is a real polynomial of degree n, then:

$$\max_{-1\leq x\leq 1}|P'(x)| \leq n^2 \max_{-1\leq x\leq 1}|P(x)|. \tag{1.1}$$

Two decades later, S. Bernstein [18] established a complex analogue of Markov's inequality for polynomials defined on the unit disc. He showed that for any $P \in \mathcal{P}_n$,

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1.2}$$

This bound on the derivative's modulus in terms of the polynomial's maximum modulus has motivated extensive research, leading to generalizations, refinements, and related results (see [16,7,4]. In a similar

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2020 *Mathematics Subject Classification:* 30A10, 30C15.
Submitted February 08, 2026. Published April 13, 2026

vein, Paul Turán [15] pioneered the study of lower bounds for the derivative's modulus. He proved that if $P \in \mathcal{P}_n$ has no zeros outside the unit disc ($|z| > 1$), then:

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

As a generalization of inequality (1.3), Malik [9] proved the following result.

Theorem 1.1 *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Turán's inequality is sharp, with equality attained when all zeros of $P(z)$ lie on the unit circle.

The following result was given by Rather *et al.* [11]

Theorem 1.2 *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, where $k \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$, we have*

$$\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}.\right)\right) \tag{1.5}$$

In this direction, Rather *et al.* [11] obtained the following result.

Theorem 1.3 *Let $P(z) = z^s\left(a_0 + a_1 z + \cdots + a_{n-s} z^{n-s}\right) \in \mathcal{P}_n$, where $0 \leq s \leq n$. If all the zeros of $P(z)$ lie in $|z| \leq k$ with $k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(\frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right) \max_{|z|=1} |P(z)| \tag{1.6}$$

The result is sharp, with equality holding for the polynomial $P(z) = z^s(z+k)^{n-s}, \quad s < n.$

S. Z. Nagy [17] pioneered the notationof the generalized derivative, which was formally defined as follows:

Definition 1.1 (Sz-Nagy Generalized Derivative) Given a polynomial $P(z) = c(z - z_1)(z - z_2)\cdots(z - z_n)$ of degree n and an n-tuple $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of non-negative real numbers (not all zero), Sz-Nagy [17] introduced a *generalized derivative* of $P(z)$ defined by

$$P^{\gamma}(z) := P(z) \sum_{j=1}^{n} \frac{\gamma_j}{z - z_j} = \sum_{j=1}^{n} \gamma_j P_j(z) \tag{1.7}$$

where $P_j(z) = c \prod_{\substack{i=1 \\ i \neq j}}^{n} (z - z_i)$ for $1 \leq j \leq n$. Note that the ordinary derivative $P'(z)$ of $P(z)$ can be obtained from $P^{\gamma}(z)$ by taking $\gamma_j = 1$ for $j = 1, 2, \ldots, n$, that is,

$$P^{\gamma}(z) = P'(z) \text{ for } \gamma = (1, 1, \ldots, 1). \tag{1.8}$$

Throughout the paper we shall use the following notations $S = \{\gamma : \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n), \gamma_i > 0 \;\forall\; i = 1, 2, \ldots, n\}$ and $\Lambda := \sum_{j=1}^{n} \gamma_j$.

Definition 1.2 (Polar Derivative) The polar derivative of a polynomial $P(z) \in \mathcal{P}_n$ with respect to $\alpha \in \mathbb{C}$ is defined as:

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) \tag{1.9}$$

This operator generalizes the ordinary derivative in the sense that:

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly for all z in compact sets $\{z : |z| \leq R\}$ with $R > 0$.

Aziz and Rather [2] obtained several estimates for the maximum modulus of $D_\alpha P(z)$ on $|z| = 1$ and, among other results, they [3] extended inequality (1.4) to the polar derivative. In fact, they proved the following result:

Theorem 1.4 *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n\,(|\alpha| - k)}{1+k} \max_{|z|=1} |P(z)|. \tag{1.10}$$

Definition 1.3 (Generalized Polar Derivative) For any $\alpha \in \mathbb{C}$ and $0 \neq \gamma \in S$, the generalized polar derivative of polynomial $P(z)$ is defined by,

$$D_\alpha^\gamma P(z) = \Lambda P(z) + (\alpha - z) P^\gamma(z).$$

(1.8) It generalizes the derivative $P^\gamma(z)$ in the sense that,

$$\lim_{\alpha \to \infty} \frac{D_\alpha^\gamma P(z)}{\alpha} = P^\gamma(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$. The polynomial $D_\alpha^\gamma P(z)$ is generalized polar derivative of $P(z)$ in the sense that if we take $\gamma = (1, 1, \ldots, 1)$ in equation (1.8), then we get

$$D_\alpha^{(1,1,\ldots,1)} P(z) = nP(z) + (\alpha - z)P'(z) = D_\alpha P(z),$$

which is the polar derivative of $P(z)$.

Irfan Ahmad Wani *et al.* [8] extend Theorem 1.2 to the class of generalized derivatives.

Theorem 1.5 *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$, and $\gamma \in R_+^n$ with $\gamma \geq 1$, $1 \leq j \leq n$*

$$\operatorname{Re}\left(\frac{zP^\gamma(z)}{P(z)}\right) \geq \frac{\sum_{j=1}^n \gamma_j}{1+k}\left(1 + \frac{k}{\sum_{j=1}^n \gamma_j}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}.\right)\right). \tag{1.11}$$

Next, Irfan Ahmad et al. [8] established the following extension of Theorem 1.3 to generalized derivatives.

Theorem 1.6 *Let $P(z) = z^s\big(a_0 + a_1 z + \cdots + a_{n-s} z^{n-s}\big)$ be a polynomial of degree n with $0 \leq s \leq n$, whose zeros all lie in $|z| \leq k$ for some $k \leq 1$. If $\gamma \in R_+^n$ with $\gamma_j \geq 1$, $1 \leq j \leq n$, then*

$$\max_{|z|=1} P^\gamma(z) \geq \left\{\sum_{j=1}^s \gamma_j + \frac{\sum_{j=1}^{n-s}\gamma_j}{1+k}\left(1 + \frac{k}{\sum_{j=1}^n \gamma_j}\left(\frac{k^{n-s}|a_n| - |a_0|}{k^{n-s}|a_n| + |a_0|}\right)\right) \max_{|z|=1} P(z)\right\} \tag{1.12}$$

Next, Irfan Ahmad et al. [8] established the following extension of Theorem 1.4 to generalized polar derivatives.

Theorem 1.7 *Let $P(z) = z^s\big(a_0 + a_1 z + \cdots + a_{n-s} z^{n-s}\big)$ be a polynomial of degree n with $0 \leq s \leq n$, whose zeros all lie in $|z| \leq k$ for some $k \leq 1$. If $\gamma \in R_+^n$ with $\gamma_j \geq 1$, $1 \leq j \leq n$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,*

$$\begin{aligned}\max_{|z|=1} |D_\alpha^\gamma P(z)| \geq (|\alpha| - k) &\left\{\sum_{j=1}^s \gamma_j + \frac{\sum_{j=1}^{n-s}\gamma_j}{1+k}\right.\\ &\left.\times\left(1 + \frac{k}{\sum_{j=1}^n \gamma_j}\left(\frac{k^{n-s}|a_n| - |a_0|}{k^{n-s}|a_n| + |a_0|}\right)\right)\right\} \max_{|z|=1} |P(z)|\end{aligned} \tag{1.13}$$

The result is sharp, with equality attained for the polynomial $P(z) = z^s(z+k)^{n-s}$ when $s < n$ and $|\alpha| \geq k$.

These inequalities, where the lower bound of the maximum modulus of a polynomial is estimated in terms of the maximum modulus of the polynomial itself, are usually referred to as Turán-type inequalities in the literature. Researchers have obtained various variants of these inequalities, providing improvements, generalizations, and extensions of the aforementioned results (for references, see [14,19]). One such generalization replaces the supremum norm with a quantity involving the integral mean.
For $P \in \mathcal{P}_n$ and $0 < q < \infty$, we consider

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right)^{\frac{1}{q}},$$

and in the case $q = \infty$, we use

$$\max_{|z|=1}|P(z)|.$$

The generalization of inequality (1.3) to the framework of L_q-norms has attracted considerable attention. Among the early contributions, Malik [10] established an L_q-norm version of Turán-type inequalities for polynomials $P \in \mathcal{P}_n$ having all their zeros in $|z| \le 1$. In fact, he proved that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \le 1$, then for each $q > 0$,

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right)^{1/q}\max_{|z|=1}|P'(z)| \;\ge\; n\left(\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right)^{1/q}. \tag{1.14}$$

2. Main results

In this section, we extend the results of Theorem 1.5, Theorem 1.6, and Theorem 1.7 to the setting of L_q-normed spaces, establishing sharper and more generalized inequalities. First, we present the following result, which generalizes Theorem 1.5 to the class of generalized derivatives in the L_q-norm.

Theorem 2.1 *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \le 1$, then for all z on $|z| = 1$ for which $P(z) \ne 0$, and $\gamma \in R_+^n$ with $\gamma \ge 1,\ 1 \le j \le n$*

$$\left(\int_0^{2\pi}\left|P^{\gamma}(e^{i\theta})\right|^q d\theta\right)^{\frac{1}{q}} \ge \frac{\sum_{j=1}^n\gamma_j}{1+k}\left(1+\frac{k}{\sum_{j=1}^n\gamma_j}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right)\left(\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right)^{\frac{1}{q}}. \tag{2.1}$$

If we take $\gamma_j = (1,1,..,1)$, then

Corollary 2.1 *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \le 1$, then for all z on $|z| = 1$ for which $P(z) \ne 0$, and $\gamma \in R_+^n$ with $\gamma \ge 1,\ 1 \le j \le n$*

$$\left(\int_0^{2\pi}\left|P'(e^{i\theta})\right|^q d\theta\right)^{\frac{1}{q}} \ge \frac{n}{1+k}\left(1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right)\left(\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right)^{\frac{1}{q}}. \tag{2.2}$$

Next, we present the following result, which generalizes Theorem 1.6 to the class of generalized derivatives in the L_q-norm.

Theorem 2.2 *Let $P(z) = z^s\left(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}\right)$ be a polynomial of degree n with $0 \le s \le n$, whose zeros all lie in $|z| \le k$ for some $k \le 1$. If $\gamma \in R_+^n$ with $\gamma_j \ge 1,\ 1 \le j \le n$, then*

$$\begin{aligned}\left(\int_0^{2\pi}|P^{\gamma}(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}} \ge &\left\{\sum_{j=1}^{s}\gamma_j + \frac{k}{1+k}\left(1+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\sum_{j=1}^{n-s}\gamma_j\right\}\\ &\times\left(\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}}.\end{aligned} \tag{2.3}$$

If we take $\gamma_j = (1,1,..,1)$, then

Corollary 2.2 *Let* $P(z) = z^s\big(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}\big)$ *be a polynomial of degree* n *with* $0 \le s \le n$, *whose zeros all lie in* $|z| \le k$ *for some* $k \le 1$. *If* $\gamma \in R_+^n$ *with* $\gamma_j \ge 1,\ 1 \le j \le n,$ *then ,*

$$\left(\int_0^{2\pi} |P'(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}} \ge \left\{s + \frac{k}{1+k}\left(1 + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)(n-s)\right\} \times \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}}. \tag{2.4}$$

Next, we present the following result, which generalizes Theorem 1.7 to the class of generalized polar derivatives in the L_q-norm.

Theorem 2.3 *Let* $P(z) = z^s\big(a_0+a_1 z+\cdots+a_{n-s}z^{n-s}\big)$ *be a polynomial of degree* n *with* $0 \le s \le n$, *whose zeros all lie in* $|z| \le k$ *for some* $k \le 1$. *If* $\gamma \in R_+^n$ *with* $\gamma_j \ge 1,\ 1 \le j \le n,$ *then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \ge k$,

$$\left(\int_0^{2\pi} |D_\alpha^\gamma P(z)|^q d\theta\right)^{\frac{1}{q}} \ge (|\alpha| - k)\left\{\sum_{j=1}^{s}\gamma_j + \frac{k}{1+k}\left(1 + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\sum_{j=1}^{n-s}\gamma_j\right\} \times \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}}. \tag{2.5}$$

The result is sharp, with equality attained for the polynomial $P(z) = z^s(z+k)^{n-s}$ *when* $s < n$ *and* $|\alpha| \ge k$.

If we take $\gamma_j = (1,1,..,1)$, then

Corollary 2.3 *Let* $P(z) = z^s\big(a_0+a_1 z+\cdots+a_{n-s}z^{n-s}\big)$ *be a polynomial of degree* n *with* $0 \le s \le n$, *whose zeros all lie in* $|z| \le k$ *for some* $k \le 1$. *If* $\gamma \in R_+^n$ *with* $\gamma_j \ge 1,\ 1 \le j \le n,$ *then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \ge k$,

$$\left(\int_0^{2\pi} |D_\alpha P(z)|^q d\theta\right)^{\frac{1}{q}} \ge (|\alpha| - k)\left\{s + \frac{k}{1+k}\left(1 + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)n - s\right\} \times \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}}. \tag{2.6}$$

3. Lemmas

In this section, we mention the following lemmas that will be used to prove our main results in the next section.

Lemma 3.1 (see [12]) *For* $0 \le x_j \le 1$ *and* $\gamma_j \in \mathbb{R}^+$ $(j = 1,\ldots,n)$, *we have the inequality:*

$$\sum_{j=1}^{n}\gamma_j\frac{1-x_j}{1+x_j} \ge \frac{1-\sum_{j=1}^{n}x_j}{1+\sum_{j=1}^{n}x_j}, \quad \forall\ n \in \mathbb{N}. \tag{3.1}$$

Lemma 3.2 *Let* $P(z) \in \mathcal{P}_n$ *be a polynomial of degree* n *with all zeros satisfying* $|z| \le k$, $k \le 1$. *For parameters* $\gamma_j \in \mathbb{R}^+$ *with* $\gamma_j \ge 1$, $1 \le j \le n$, *the inequality*

$$\left(\int_0^{2\pi} |Q^\gamma(z)|^q\, d\theta\right)^{\frac{1}{q}} \le k\left(\int_0^{2\pi} \left|P^\gamma(e^{i\theta})\right|^q d\theta\right)^{\frac{1}{q}} \quad \text{for} \quad |z| = 1 \tag{3.2}$$

holds for all $|z| = 1$, *where:*

- $Q(z) = z^n P(1/\overline{z})$ *is the reciprocal polynomial*

- *$P^\gamma(z)$ denotes the generalized derivative of a polynomial*

Proof of Lemma 3.2. *Since all the zeros of a polynomial $P(z)$ lie in $|z| \leq k$, where $k \leq 1$, we can write*

$$P(z) = a_n \prod_{j=1}^{n}(z - z_j), \tag{3.3}$$

where $|z_j| \leq k$, $j = 1, 2, \dots, n$. Since all the zeros of $P(z)$ are in $|z| \leq k$, then $F(z) = P(kz)$ has all its zeros in $|z| \leq 1$. For $\gamma \in \mathbb{R}_n^+$, this gives:

$$\frac{zF^\gamma(z)}{F(z)} = \sum_{j=1}^{n} \frac{\gamma_j z}{z - \zeta_j}, \tag{3.4}$$

where $\zeta_j = \frac{z_j}{k}$ and $|\zeta_j| \leq 1$, $1 \leq j \leq n$.

For points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, other than the zeros of $F(z)$, we have

$$\operatorname{Re}\left\{\frac{e^{i\theta}F^\gamma(e^{i\theta})}{F(e^{i\theta})}\right\} = \operatorname{Re}\left\{\sum_{j=1}^{n} \frac{\gamma_j e^{i\theta}}{e^{i\theta} - \zeta_j}\right\} = \sum_{j=1}^{n} \gamma_j \operatorname{Re}\left\{\frac{e^{i\theta}}{e^{i\theta} - \zeta_j}\right\} \geq \frac{1}{2}\sum_{j=1}^{n} \gamma_j. \tag{3.5}$$

Which implies,

$$\operatorname{Re}\left\{\frac{e^{i\theta}F^\gamma(e^{i\theta})}{\left(\sum_{j=1}^{n} \gamma_j\right) F(e^{i\theta})}\right\} \geq \frac{1}{2}. \tag{3.6}$$

For points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, which are not the zeros of $F(z)$, we have

$$\left|1 - \frac{e^{i\theta}F^\gamma(e^{i\theta})}{\left(\sum_{j=1}^{n} \gamma_j\right) F(e^{i\theta})}\right| \leq \left|\frac{e^{i\theta}F^\gamma(e^{i\theta})}{\left(\sum_{j=1}^{n} \gamma_j\right) F(e^{i\theta})}\right|. \tag{3.7}$$

Equivalently,

$$\left|\left(\sum_{j=1}^{n} \gamma_j\right) F(e^{i\theta}) - e^{i\theta}F^\gamma(e^{i\theta})\right| \leq \left|F^\gamma(e^{i\theta})\right|. \tag{3.8}$$

Since this inequality is trivially true for points $e^{i\theta}$ which are zeros of $F(z)$, it follows that

$$\left|\sum_{j=1}^{n} \gamma_j F(z) - zF^\gamma(z)\right| \leq |F^\gamma(z)| \quad \text{for} \quad |z| = 1. \tag{3.9}$$

Since $F(z) = P(kz)$, we have

$$F^\gamma(z) = F(z)\sum_{j=1}^{n} \frac{\gamma_j}{z - \frac{z_j}{k}} = kP'(kz)\sum_{j=1}^{n} \frac{\gamma_j}{kz - z_j} = kP^\gamma(kz). \tag{3.10}$$

Replacing $F(z)$ by $P(kz)$ and $F^\gamma(z)$ by $kP^\gamma(kz)$, we obtain

$$\left|\sum_{j=1}^{n} \gamma_j P(kz) - zkP^\gamma(kz)\right| \leq k\,|P^\gamma(kz)| \quad \text{for} \quad |z| = 1. \tag{3.11}$$

Since $k \leq 1$, taking $z = \frac{e^{i\theta}}{k}$, $0 \leq \theta \leq 2\pi$ gives:

$$\left|\sum_{j=1}^{n} \gamma_j P(e^{i\theta}) - e^{i\theta}P^\gamma(e^{i\theta})\right| \leq k\left|P^\gamma(e^{i\theta})\right|. \tag{3.12}$$

This shows that,

$$\left|\sum_{j=1}^{n}\gamma_j P(z) - zP^{\gamma}(z)\right| \leq k\left|P^{\gamma}(z)\right|, \quad \textit{for } |z| = 1. \tag{3.13}$$

Since

$$\begin{aligned}\sum_{j=1}^{n}\gamma_j P(z) - zP^{\gamma}(z) &= P(z)\sum_{j=1}^{n}\left(\gamma_j - \frac{z\gamma_j}{z-z_j}\right)\\ &= -P(z)\sum_{j=1}^{n}\frac{\gamma_j z_j}{z-z_j}.\end{aligned} \tag{3.14}$$

Also,

$$\begin{aligned}z^{n-1}Q^{\gamma}\left(\frac{1}{z}\right) &= -z^{n}Q\left(\frac{1}{z}\right)\sum_{j=1}^{n}\frac{\gamma_j z_j}{z-z_j}\\ &= -P(z)\sum_{j=1}^{n}\frac{\gamma_j z_j}{z-z_j}.\end{aligned} \tag{3.15}$$

Combining (3.14) *and* (3.15), *for* $|z| = 1$ *we have*

$$Q^{\gamma}(z) = \sum_{j=1}^{n}\gamma_j P(z) - zP^{\gamma}(z). \tag{3.16}$$

From (3.13) *and* (3.16), *we obtain*

$$\left|Q^{\gamma}(z)\right| \leq k\left|P^{\gamma}(z)\right|, \quad \textit{for } |z| = 1, \tag{3.17}$$

the above inequality is equivalently expressed as

$$\left|Q^{\gamma}(z)\right|^{q} \leq k\left|P^{\gamma}(e^{i\theta})\right|^{q} \quad \text{for} \quad |z| = 1 \tag{3.18}$$

Integrating both side w.r.t θ from 0 to 2π

$$\left(\int_0^{2\pi}\left|Q^{\gamma}(z)\right|^{q}d\theta\right)^{\frac{1}{q}} \leq k\left(\int_0^{2\pi}\left|P^{\gamma}(e^{i\theta})\right|^{q}d\theta\right)^{\frac{1}{q}} \quad \text{for} \quad |z| = 1 \tag{3.19}$$

Lemma 3.3 (see in [13]) *If $P(z)$ is a polynomial of degree n and $P^{\gamma}(z)$ is its generalized derivative as defined in (1.7), then all the zeros of $P^{\gamma}(z)$ lie in the convex hull of the zeros of $P(z)$.*

4. Proof of theorems

Proof of Theorem 2.1. Since all the zeros of $P(z)$ lie in $|z| \leq k$, where $k \leq 1$, we can write

$$P(z) = a_n\prod_{j=1}^{n}(z - z_j), \tag{4.1}$$

where $|z_j| \leq k$, $j = 1, 2, \ldots, n$. Then, for $\gamma \in \mathbb{R}_n^{+}$ and $P(z) \neq 0$ we have

$$\frac{P^{\gamma}(z)}{P(z)} = \sum_{j=1}^{n}\frac{\gamma_j}{z - z_j}. \tag{4.2}$$

Since for the points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, other than the zeros of $P(z)$, we have

$$\operatorname{Re}\left\{\frac{e^{i\theta}P^{\gamma}(e^{i\theta})}{P(e^{i\theta})}\right\} = \sum_{j=1}^{n}\gamma_j \operatorname{Re}\left\{\frac{e^{i\theta}}{e^{i\theta}-z_j}\right\}. \tag{4.3}$$

Now, we write

$$\begin{aligned}\operatorname{Re}\left\{\frac{e^{i\theta}P^{\gamma}(e^{i\theta})}{P(e^{i\theta})}\right\} &= \operatorname{Re}\left\{\sum_{j=1}^{n}\frac{\gamma_j e^{i\theta}}{e^{i\theta}-z_j}\right\}\\ &= \sum_{j=1}^{n}\gamma_j \operatorname{Re}\left\{\frac{e^{i\theta}}{e^{i\theta}-z_j}\right\}\\ &\geq \sum_{j=1}^{n}\frac{\gamma_j(1-|z_j|)}{1+|z_j|}\\ &= \frac{\sum_{j=1}^{n}\gamma_j}{1+k} + \frac{k}{1+k}\sum_{j=1}^{n}\gamma_j\frac{k-|z_j|}{k+k|z_j|}.\end{aligned}$$

Since $k \leq 1$, thus $k+k|z_j| \leq k+|z_j|$ and $\gamma \in \mathbb{R}_n^+$ with $\gamma_j \geq 1$, $1 \leq j \leq n$, for the points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, other than the zeros of $P(z)$, the above inequality yields

$$\operatorname{Re}\left\{\frac{e^{i\theta}P^{\gamma}(e^{i\theta})}{P(e^{i\theta})}\right\} \geq \frac{\sum_{j=1}^{n}\gamma_j}{1+k} + \frac{k}{1+k}\sum_{j=1}^{n}\gamma_j\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}. \tag{4.4}$$

By Lemma 3.1 and noting that $\frac{|z_j|}{k} \leq 1$, $1 \leq j \leq n$, we get for all z on $|z|=1$, for which $P(z) \neq 0$:

$$\begin{aligned}\operatorname{Re}\left\{\frac{zP^{\gamma}(z)}{P(z)}\right\} &\geq \frac{\sum_{j=1}^{n}\gamma_j}{1+k} + \frac{k}{1+k}\frac{1-\prod_{j=1}^{n}\frac{|z_j|}{k}}{1+\prod_{j=1}^{n}\frac{|z_j|}{k}}.\\ &\geq \frac{\sum_{j=1}^{n}\gamma_j}{1+k}\left(1+\frac{k}{\sum_{j=1}^{n}\gamma_j}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right),\end{aligned}$$

now, we can write above inequality

$$|P^{\gamma}(z)| \geq \frac{\sum_{j=1}^{n}\gamma_j}{1+k}\left(1+\frac{k}{\sum_{j=1}^{n}\gamma_j}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right)|P(z)|$$

the above inequality is equivalently expressed as

$$\left|P^{\gamma}(e^{i\theta})\right|^q \geq \left[\frac{\sum_{j=1}^{n}\gamma_j}{1+k}\left(1+\frac{k}{\sum_{j=1}^{n}\gamma_j}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right)\right]^q \left|P(e^{i\theta})\right|^q$$

Integrating both side w.r.t θ from 0 to 2π

$$\left(\int_0^{2\pi}\left|P^{\gamma}(e^{i\theta})\right|^q d\theta\right)^{\frac{1}{q}} \geq \frac{\sum_{j=1}^{n}\gamma_j}{1+k}\left(1+\frac{k}{\sum_{j=1}^{n}\gamma_j}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right)\left(\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right)^{\frac{1}{q}}.$$

This proves Theorem 2.1.

Proof of Theorem 2.2. Let $P(z) = z^sF(z) \in \mathcal{P}_n$, where $F(z) = a_0 + a_1z + \cdots + a_{n-s}z^{n-s}$, $0 \leq s \leq n$, then on $|z|=1$,

$$\operatorname{Re}\left\{\frac{zP^{\gamma}(z)}{P(z)}\right\} = \sum_{j=1}^{s}\gamma_j + \operatorname{Re}\left\{\frac{zF^{\gamma}(z)}{F(z)}\right\}. \tag{4.5}$$

Since $F(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \leq k$, $k \leq 1$, therefore by Theorem 2.1 applied to $F(z)$, for all points on $|z|=1$, other than the zeros of $P(z)$, we have

$$\operatorname{Re}\left\{\frac{zP^{\gamma}(z)}{P(z)}\right\} \geq \sum_{j=1}^{s}\gamma_j + \frac{k}{1+k}\left(1+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\sum_{j=1}^{n-s}\gamma_j. \tag{4.6}$$

Which implies for all points z on $|z|=1$, other than the zeros of $P(z)$, we have

$$\left|\frac{zP^{\gamma}(z)}{P(z)}\right| \geq \operatorname{Re}\left\{\frac{zP^{\gamma}(z)}{P(z)}\right\} \tag{4.7}$$

$$\geq \sum_{j=1}^{s}\gamma_j + \frac{k}{1+k}\left(1+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\sum_{j=1}^{n-s}\gamma_j. \tag{4.8}$$

This gives, for $|z|=1$:

$$|P^{\gamma}(z)| \geq \left\{\sum_{j=1}^{s}\gamma_j + \frac{k}{1+k}\left(1+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\sum_{j=1}^{n-s}\gamma_j\right\}|P(z)|. \tag{4.9}$$

the above inequality is equivalently expressed as

$$|P^{\gamma}(e^{i\theta})|^q \geq \left\{\sum_{j=1}^{s}\gamma_j + \frac{k}{1+k}\left(1+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\sum_{j=1}^{n-s}\gamma_j\right\}^q |P(e^{i\theta})|^q. \tag{4.10}$$

Integrating both side w.r.t θ from 0 to 2π

$$\left(\int_0^{2\pi}|P^{\gamma}(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}} \geq \left\{\sum_{j=1}^{s}\gamma_j + \frac{k}{1+k}\left(1+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\sum_{j=1}^{n-s}\gamma_j\right\}$$

$$\times\left(\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}}. \tag{4.11}$$

This proves Theorem 2.2 □

Proof of Theorem 2.3. Since $Q(z)=z^nP\left(\frac{1}{\bar{z}}\right)$, then from inequality (3.16), we have

$$Q^{\gamma}(z) = \sum_{j=1}^{n}\gamma_j P(z) - zP^{\gamma}(z) \quad \text{for} \quad |z|=1. \tag{4.12}$$

Since all the zeros of a polynomial $P(z)$ lie in $|z| \leq k$, where $k \leq 1$, therefore by Lemma 3.2, we have

$$k|P^{\gamma}(z)| \geq \left|\sum_{j=1}^{n}\gamma_j P(z) - zP^{\gamma}(z)\right| \quad \text{for} |z|=1. \tag{4.13}$$

therefore by Lemma 3.2, we have

$$\left(\int_0^{2\pi}|Q^{\gamma}(z)|^q\, d\theta\right)^{\frac{1}{q}} \leq k\left(\int_0^{2\pi}\left|P^{\gamma}(e^{i\theta})\right|^q\, d\theta\right)^{\frac{1}{q}} \quad \text{for} \quad |z|=1 \tag{4.14}$$

Now for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, we have for $|z|=1$,

$$D_{\alpha}^{\gamma}P(z) = \sum_{k=1}^{n}\gamma_k P(z) + (\alpha - z)P^{\gamma}(z) \tag{4.15}$$

$$\geq |\alpha||P^{\gamma}(z) - \left|\sum_{j=1}^{n}\gamma_j P(z) - zP^{\gamma}(z)\right| \tag{4.16}$$

Using (4.14) the above inequality is equivalently expressed as

$$|D_\alpha^\gamma P(e^{i\theta})|^q \geq (|\alpha| - k)^q \, |P^\gamma(e^{i\theta})|^q \tag{4.17}$$

Integrating both side w.r.t θ from 0 to 2π

$$\left(\int_0^{2\pi} |D_\alpha^\gamma P(z)| d\theta\right)^{\frac{1}{q}} \geq (|\alpha| - k) \left(\int_0^{2\pi} |P^\gamma(z)| d\theta\right)^{\frac{1}{q}} \tag{4.18}$$

Combining (4.11) and (4.18), we obtain for $|z| = 1$:

$$\left(\int_0^{2\pi} |D_\alpha^\gamma P(e^{i\theta})| d\theta\right)^{\frac{1}{q}} \geq (|\alpha| - k) \left\{\sum_{j=1}^{s} \gamma_j + \frac{k}{1+k}\left(1 + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right) \sum_{j=1}^{n-s} \gamma_j\right\}$$
$$\times \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}}. \tag{4.19}$$

This proves Theorem 2.3 ☐

5. Conclusion

The results obtained in this study contribute to the advancement of polynomial inequality theory by extending classical Turán-type inequalities to the framework of generalized derivatives and generalized polar derivatives in the L_q-norm. These generalizations not only unify several known inequalities but also produce new results as direct corollaries. Consequently, the work enhances the scope of existing theory and provides a broader foundation for further exploration of extremal properties of polynomials.

Acknowledgements

The authors are highly grateful to the referee for his valuable suggestions and comments.

Declaration

Conflict of interest The authors state that they have no conflicts of interest.

Data availability statement

Data Avalaibility. Data sharing is not applicable to this article, as no datasets were generated or analyzed during the course of this study.

Funding

The authors declare that no funding was received for this research.

Author contributions

All authors contributed equally to this work.

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