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An Efficient Numerical Simulation Technique for a System of 1D Burgers Equations

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ABSTRACT: The present paper aims to study the behavior of one-dimensional system of the famous non-linear convective–diffusive partial differential equations known as Burgers equations that occur in various fields of physics and applied mathematics. In this paper, efficient numerical experiments are being performed to study the behavior of one dimensional Burgers equations with discontinuous and non-differentiable initial conditions. A system of three Burgers equations is solved using a modified cubic B-spline collocation method and the results are compared with single and coupled Burgers equations. It is observed that the diffusion term significantly smoothens the solutions. The numerical results are illustrated through two- and three-dimensional graphical representations.

Key Words: Partial differential equations, Burgers equation, System of Burgers equations, modified cubic B-spline functions, collocation method, RK4 method.

Contents

1. Introduction

Burger's equations are the famous non-linear convective-diffusive partial differential equations that occur in various areas of physics and applied mathematics such as fluid dynamics, gas dynamics, non-linear acoustics and traffic flow. Burger's equation was originally introduced by Bateman [3,32] in 1915. J.M. Burgers [6] further studied Burger's equations in 1948. Mathematical models of Burgers equations describe various physical phenomena of great importance [8,27,2,33]. The numerical simulations of Burgers equations [16] are one of the most significant research interests for many decades, both in heat transfer and in fluid dynamics. A brief review [5] is also conducted to explore the past and the recent developments and to emphasize the importance of Burgers equations in the modern engineering scenario.

Many numerical simulation approaches to study the coupled linear and non-linear Burgers initial-boundary value problems have been proposed in recent years. Harmonic DQ Finite differences coupled approach [7], Adomian and Variational methods, Conjugate Filter approach [31] are some of the available schemes to approximate the solutions of these equations. A difference scheme was proposed by Jain et al. [13] for numerical solutions of two-dimensional Navier-Stokes equations. A difference scheme is proposed by Mohanty [26] to numerically approximate the solutions of a system of non-linear parabolic partial differential equations with mixed derivatives and variable coefficients. A pseudo-spectral method is also proposed by Rashid and Ismail [28] to approximate the solutions of coupled Burger's equations. Kaya

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[14] and Soliman [30] obtained the exact solutions of coupled Burger's equations using the Adomian Decomposition method and modified extended tanh-function method respectively. Bhatt and Khaliq [4] introduces two new modified fourth order compact exponential time differencing Runge-Kutta scheme in combination with a global fourth order compact finite difference scheme for the coupled nonlinear viscous Burgers equations. Khater et al. [15] and Mittal and Arora [18] approximated the solutions of coupled system by using a cubic B-spline collocation method. Mittal and Jiwari [19] obtained the solutions using the Differential quadrature method and found the results in good agreement with those already available in the literature. Mittal and Tripathi [21] proposed a collocation method for numerical solutions of coupled Burgers equations. Most recently, Zhang et al. [34] produced the improved Backward substitution method for the simulation of time dependent nonlinear coupled Burgers equations. However, no significant work has been done to numerically approximate the solutions of a system of three one-dimensional Burgers equations. Abada et al. [1] used finite difference methods to solve the onedimensional unsteady Burgers equation and analysed the shock waves in aircraft dynamics. Asymptotic behaviors of the special solutions of Burgers equation are analysed by Samanta et al. [29]. As an introduction to a new approach, Gaillard P. [9] represents the solutions of the Burgers equation as Wronskian. Malwana et al. [17] presented solutions of nonlinear Burger's equation arising in longitudinal dispersion phenomena.

In the present paper, an approach is proposed to simulate such systems numerically and the results are compared with those available in the literature for the independent equations. The effects of copulation are also analysed. The advantages to simulate numerically by the proposed modified cubic B-splines method are also discussed. The ability of the method to handle the nonlinearity of the system with fewer efforts is the key benefit. In the proposed numerical scheme, the modified cubic B-spline basis functions are collocated over finite elements for spatial variables and its derivatives. This produces a system of first order ordinary differential equations which in turn is solved by RK4 method [12]. The proposed scheme is straightforward and quite simple to implement. The numerical results obtained for the systems of Burgers equations are further analysed with those for the corresponding single and the coupled Burgers equations. The numerical solutions of the following system of three Burgers equation will be discussed in this paper. These equations are described by non-linear partial differential equations of the form:

$$u_t = u_{xx} - \eta u u_x - p(uvw)_x \; ; \qquad a < x < b,\ 0 \le t \le T \tag{1.1}$$

$$v_t = v_{xx} - \xi v v_x - q(uvw)_x \; ; \qquad a < x < b,\ 0 \le t \le T \tag{1.2}$$

$$w_t = w_{xx} - \mu w w_x - r(uvw)_x \; ; \qquad a < x < b,\ 0 \le t \le T \tag{1.3}$$

with the initial conditions;

$$u(x,0) = f_1(x) \; ; \qquad a < x < b \tag{1.4}$$

$$v(x,0) = f_2(x) \; ; \qquad a < x < b \tag{1.5}$$

$$w(x,0) = f_3(x) \; ; \qquad a < x < b \tag{1.6}$$

and the boundary conditions;

$$u(a,t) = g_0(t), \quad u(b,t) = g_1(t) \; ; \qquad 0 \le t \le T \tag{1.7}$$

$$v(a,t) = g_2(t), \quad v(b,t) = g_3(t) \; ; \qquad 0 \le t \le T \tag{1.8}$$

$$w(a,t) = g_4(t), \quad w(b,t) = g_5(t) \; ; \qquad 0 \le t \le T \tag{1.9}$$

where η, ξ and μ are real constants and p, q and r are arbitrary constants.

This paper is organized as: In Sections 2 and 3, the description of the proposed method is given. In Section 4, the proposed method is implemented to the given problem. In Section 5, the initial vector is discussed. In Section 6, three different examples for (1.1)–(1.9) are given and the results are discussed in Section 7. The overall conclusions of the present study are discussed in Section 8.

2. Description of the Method

In the present method, the approximate solution over the concerned approximation space is expressed as a linear combination of the cubic B-spline basis functions.

Let us consider an interval $a \leq x \leq b$ a one-dimensional solution domain of interest. Let this solution domain be uniformly partitioned by the knots x_j with a mesh

$$a = x_0 < x_1 \ldots\ldots\ldots\ldots < x_N = b$$

where

$$h = x_j - x_{j-1} = \frac{(b-a)}{N}; j = 1, 2 \ldots\ldots, N$$

is the uniform step size of the mesh considered.

The numerical approach to solve the considered coupled Burger's equations using the collocation of cubic B-spline basis functions is to find the approximate solutions $U^N(x,t), V^N(x,t)$ and $W^N(x,t)$ to the exact solutions $u(x,t), v(x,t)$ and $w(x,t)$ respectively, in the form:

$$U^N(x,t) = \sum_{j=-1}^{N+1} \alpha_j(t) B_j(x), \qquad a \leq x \leq b,\ t > 0 \tag{2.1}$$

$$V^N(x,t) = \sum_{j=-1}^{N+1} \beta_j(t) B_j(x), \qquad a \leq x \leq b,\ t > 0 \tag{2.2}$$

$$W^N(x,t) = \sum_{j=-1}^{N+1} \gamma_j(t) B_j(x), \qquad a \leq x \leq b,\ t > 0 \tag{2.3}$$

where $\alpha_j(t), \beta_j(t)$ and $\gamma_j(t)$ are unknown time dependent coefficients of the basis spline functions $B_j(x)$ in the approximate solution. These time dependent coefficients are determined from the boundary conditions and the collocation from the differential equations.

The cubic B-spline basis functions $B_j(x)$ at the knots are given by:

$$B_j(x) = \frac{1}{h^3} \begin{cases} (x - x_{j-2})^3, & x \in [x_{j-2},\, x_{j-1}) \\ (x - x_{j-2})^3 - 4\,(x - x_{j-1})^3, & x \in [x_{j-1},\, x_j) \\ (x_{j+2} - x)^3 - 4\,(x_{j+1} - x)^3, & x \in [x_j,\, x_{j+1}) \\ (x_{j+2} - x)^3, & x \in [x_{j+1},\, x_{j+2}) \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

where the functions $B_{-1}, B_0, B_1, \ldots\ldots\ldots B_{N-1}, B_N, B_{N+1}$ form a basis over the domain $a \leq x \leq b$. Each cubic B-spline function covers four elements so that each element is covered by four cubic B-spline functions. At a particular knot x_j, there exist only three cubic B-splines, namely B_{j-1}, B_j, B_{j+1}.

Using (2.4), the values of $B_j(x)$ and its two successive derivatives $B_j'(x), B_j''(x)$ over the prescribed set of knots are given in Table 1 below.

Table 1: Table 1: Cubic B-spline functions and their derivatives at the knots

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$B_j(x)$	0	1	4	1	0
$B_j'(x)$	0	−3/h	0	3/h	0
$B''_j(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

Using the approximate solution function (2.1) and B-spline functions (2.4), the approximate value $U^N(x,t)$ and its first two successive derivatives at any time t and at a particular knot x_j can be expressed in terms of time-dependent parameters $\alpha_j(t)$ as:

$$\begin{cases} U_j = \alpha_{j-1} + 4\alpha_j + \alpha_{j+1}, \\ h\,U'_j = 3\left(\alpha_{j+1} - \alpha_{j-1}\right), \\ h^2 U''_j = 6\left(\alpha_{j-1} - 2\alpha_j + \alpha_{j+1}\right) \end{cases} \tag{2.5}$$

In the similar way, the corresponding values for the approximate solution $V^N(x,t)$ and $W^N(x,t)$ and their derivatives can be obtained.

3. Modified Cubic B-Spline Collocation Method

In the present paper, the Dirichlet type boundary conditions are taken. So, the modified cubic B-spline basis functions are used to have a diagonally dominant system of differential equations. The modified cubic B -spline functions [20] $\tilde{B}_{-1}, \tilde{B}_0, \tilde{B}_1 \dots\dots\dots \tilde{B}_{N-1}, \tilde{B}_N, \tilde{B}_{N+1}$ are given by:

$$\begin{array}{rlc} \tilde{B}_0(x) = B_0(x) + 2B_{-1}(x), & & \text{for } j = 0 \\ \tilde{B}_1(x) = B_1 - B_{-1}(x), & & \text{for } j = 1 \\ \tilde{B}_j(x) = B_j(x), & & \text{for } j = 2, 3, \dots\dots, (N-2) \\ \tilde{B}_{N-1}(x) = B_{N-1} - B_{N+1}(x), & & \text{for } j = N-1 \\ \tilde{B}_N(x) = B_N(x) + 2B_{N+1}(x), & & \text{for } j = N \end{array} \tag{3.1}$$

Now, using the collocation of these modified cubic B-spline basis functions, the approximate solutions can be expressed as;

$$U^N(x,t) = \sum_{j=0}^{N} \alpha_j(t)\,\tilde{B}_j(x), \qquad a \le x \le b,\ t > 0 \tag{3.2}$$

$$V^N(x,t) = \sum_{j=0}^{N} \beta_j(t)\,\tilde{B}_j(x), \qquad a \le x \le b,\ t > 0 \tag{3.3}$$

$$W^N(x,t) = \sum_{j=0}^{N} \gamma_j(t)\,\tilde{B}_j(x), \qquad a \le x \le b,\ t > 0 \tag{3.4}$$

4. Implementation of the Proposed Scheme

Clustering the given system of partial differential equations (1.1)–(1.3) at the internal knots $x_j; j = 1, 2 \dots\dots\dots, N-1$, by using the assumed approximate solutions (3.2)–(3.4), we have

$$
\begin{aligned}
\left[\sum_{j=0}^{N}\dot{\alpha}_j(t)\tilde{B}_j(x)\right]_{x=x_j} &= \left[\sum_{j=0}^{N}\alpha_j(t)\widetilde{B}''_j(x)\right]_{x=x_j} \\
&- \eta\left[\left(\sum_{j=0}^{N}\alpha_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\alpha_j(t)\widetilde{B}'_j(x)\right)\right]_{x=x_j} \\
&- p\Bigg[\left(\sum_{j=0}^{N}\alpha_j(t)\widetilde{B}'_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\widetilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\tilde{B}_j(x)\right) \\
&\quad + \left(\sum_{j=0}^{N}\alpha_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\widetilde{B}'_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\tilde{B}_j(x)\right) \\
&\quad + \left(\sum_{j=0}^{N}\alpha_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\widetilde{B}'_j(x)\right)\Bigg]_{x=x_j}
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
\left[\sum_{j=0}^{N}\dot{\beta}_j(t)\tilde{B}_j(x)\right]_{x=x_j} &= \left[\sum_{j=0}^{N}\beta_j(t)\widetilde{B}''_j(x)\right]_{x=x_j} \\
&- \xi\left[\left(\sum_{j=0}^{N}\beta_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\widetilde{B}'_j(x)\right)\right]_{x=x_j} \\
&- q\Bigg[\left(\sum_{j=0}^{N}\alpha_j(t)\widetilde{B}'_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\tilde{B}_j(x)\right) \\
&\quad + \left(\sum_{j=0}^{N}\alpha_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\widetilde{B}'_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\tilde{B}_j(x)\right) \\
&\quad + \left(\sum_{j=0}^{N}\alpha_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\widetilde{B}'_j(x)\right)\Bigg]_{x=x_j}
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
\left[\sum_{j=0}^{N}\dot{\gamma}_j(t)\tilde{B}_j(x)\right]_{x=x_j} &= \left[\sum_{j=0}^{N}\gamma_j(t)\widetilde{B}''_j(x)\right]_{x=x_j} \\
&- \mu\left[\left(\sum_{j=0}^{N}\gamma_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\widetilde{B}'_j(x)\right)\right]_{x=x_j} \\
&- r\Bigg[\left(\sum_{j=0}^{N}\alpha_j(t)\widetilde{B}'_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\tilde{B}_j(x)\right) \\
&\quad + \left(\sum_{j=0}^{N}\alpha_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\widetilde{B}'_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\tilde{B}_j(x)\right) \\
&\quad + \left(\sum_{j=0}^{N}\alpha_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\beta_j(t)\tilde{B}_j(x)\right)\left(\sum_{j=0}^{N}\gamma_j(t)\widetilde{B}'_j(x)\right)\Bigg]_{x=x_j}
\end{aligned} \tag{4.3}
$$

where $j = 1, 2, \ldots\ldots\ldots, (N-1), \quad a < x < b, \quad t > 0$. Using (3.1), equation (4.1) is reduced to a system of ordinary differential equations of the form

$$A_u \, \dot{\alpha}_u = \varphi_u \tag{4.4}$$

where

$$\dot{\alpha}_u = \left[\dot{\alpha}_0 \; \dot{\alpha}_1 \; \cdots \; \dot{\alpha}_{N-1} \; \dot{\alpha}_N\right]^{\mathrm{T}},$$

$$\varphi_u = \left[\varphi_1 \; \varphi_2 \; \cdots \; \varphi_{N-2} \; \varphi_{N-1}\right]^{\mathrm{T}}.$$

with

$$\begin{aligned}\varphi_j = {} & \frac{6}{h^2}\left(\alpha_{j-1} + 2\alpha_j + \alpha_{j+1}\right) - \eta\left(\alpha_{j-1} + 4\alpha_j + \alpha_{j+1}\right)\frac{3}{h}\left(\alpha_{j+1} - \alpha_{j-1}\right) \\ & - p\Bigg[\frac{3}{h}\left(\alpha_{j+1} - \alpha_{j-1}\right)\left(\beta_{j-1} + 4\beta_j + \beta_{j+1}\right)\left(\gamma_{j-1} + 4\gamma_j + \gamma_{j+1}\right) \\ & + \left(\alpha_{j-1} + 4\alpha_j + \alpha_{j+1}\right)\frac{3}{h}\left(\beta_{j+1} - \beta_{j-1}\right)\left(\gamma_{j-1} + 4\gamma_j + \gamma_{j+1}\right) \\ & + \left(\alpha_{j-1} + 4\alpha_j + \alpha_{j+1}\right)\left(\beta_{j-1} + 4\beta_j + \beta_{j+1}\right)\frac{3}{h}\left(\gamma_{j+1} - \gamma_{j-1}\right)\Bigg]\end{aligned} \tag{4.5}$$

and

$$A_u = \begin{bmatrix} 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & & \cdots & \cdots & \cdots & & & \\ & & & \cdots & \cdots & \cdots & & & \\ & & & & & 1 & 4 & 1 & \\ & & & & & & 1 & 4 & 1 \end{bmatrix}_{(N-1)\times(N+1)} \tag{4.6}$$

Here, it is to be observed that φ_u is an $(N+1)\times 1$ known column vector, depending upon the boundary conditions. Whereas, A_u is a diagonally dominant tri-diagonal matrix.
Similarly, the other two following systems of equations will be obtained by reducing (4.2) and (4.3).

$$A_u \dot{\beta}_v = \psi_v \tag{4.7}$$

$$A_u \dot{\gamma}_w = \sigma_w \tag{4.8}$$

where

$$\dot{\beta}_v = \left[\dot{\beta}_0 \; \dot{\beta}_1 \; \cdots \; \dot{\beta}_{N-1} \; \dot{\beta}_N\right]^T, \qquad \psi_u = \left[\psi_1 \; \psi_2 \; \cdots \; \psi_{N-2} \; \psi_{N-1}\right]^T$$

and

$$\dot{\gamma}_w = \left[\dot{\gamma}_0 \; \dot{\gamma}_1 \; \cdots \; \dot{\gamma}_{N-1} \; \dot{\gamma}_N\right]^T, \qquad \sigma_w = \left[\sigma_1 \; \sigma_2 \; \cdots \; \sigma_{N-2} \; \sigma_{N-1}\right]^T.$$

with

$$\begin{aligned}\psi_j = {} & \frac{6}{h^2}\left(\beta_{j-1} + 2\beta_j + \beta_{j+1}\right) - \xi\left(\beta_{j-1} + 4\beta_j + \beta_{j+1}\right)\frac{3}{h}\left(\beta_{j+1} - \beta_{j-1}\right) \\ & - q\Bigg[\frac{3}{h}\left(\alpha_{j+1} - \alpha_{j-1}\right)\left(\beta_{j-1} + 4\beta_j + \beta_{j+1}\right)\left(\gamma_{j-1} + 4\gamma_j + \gamma_{j+1}\right) \\ & + \left(\alpha_{j-1} + 4\alpha_j + \alpha_{j+1}\right)\frac{3}{h}\left(\beta_{j+1} - \beta_{j-1}\right)\left(\gamma_{j-1} + 4\gamma_j + \gamma_{j+1}\right) \\ & + \left(\alpha_{j-1} + 4\alpha_j + \alpha_{j+1}\right)\left(\beta_{j-1} + 4\beta_j + \beta_{j+1}\right)\frac{3}{h}\left(\gamma_{j+1} - \gamma_{j-1}\right)\Bigg]\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
\sigma_j = {} & \frac{6}{h^2}\left(\gamma_{j-1}+2\gamma_j+\gamma_{j+1}\right)-\mu\left(\gamma_{j-1}+4\gamma_j+\gamma_{j+1}\right)\frac{3}{h}\left(\gamma_{j+1}-\gamma_{j-1}\right) \\
& -r\Bigg[\frac{3}{h}\left(\alpha_{j+1}-\alpha_{j-1}\right)\left(\beta_{j-1}+4\beta_j+\beta_{j+1}\right)\left(\gamma_{j-1}+4\gamma_j+\gamma_{j+1}\right) \\
& +\left(\alpha_{j-1}+4\alpha_j+\alpha_{j+1}\right)\frac{3}{h}\left(\beta_{j+1}-\beta_{j-1}\right)\left(\gamma_{j-1}+4\gamma_j+\gamma_{j+1}\right) \\
& +\left(\alpha_{j-1}+4\alpha_j+\alpha_{j+1}\right)\left(\beta_{j-1}+4\beta_j+\beta_{j+1}\right)\frac{3}{h}\left(\gamma_{j+1}-\gamma_{j-1}\right)\Bigg]
\end{aligned}
\tag{4.10}
$$

The system of differential equations (4.4), (4.7), (4.8) will be solved simultaneously.
Now applying the boundary conditions (1.7)–(1.9), at the boundary knots x_0 and x_N, we have

$$
\begin{aligned}
6\alpha_0 &= g_0(t), \quad 6\alpha_N = g_1(t) \\
6\beta_0 &= g_2(t), \quad 6\beta_N = g_3(t) \\
6\gamma_0 &= g_4(t), \quad 6\gamma_N = g_5(t)
\end{aligned}
$$

Since, equations in the systems (4.4), (4.7) and (4.8) arise from the same time derivatives, we therefore have the combined system of ordinary differential equations of the form

$$
\begin{aligned}
A\dot{\alpha}_u &= \begin{bmatrix}\dot{g}_0(t) & \varphi_1 & \varphi_2 & \dots & \varphi_{N-2} & \varphi_{N-1} & \dot{g}_1(t)\end{bmatrix}^{\mathrm{T}} \\
A\dot{\beta}_v &= \begin{bmatrix}\dot{g}_2(t) & \psi_1 & \psi_2 & \dots & \psi_{N-2} & \psi_{N-1} & \dot{g}_3(t)\end{bmatrix}^{\mathrm{T}} \\
A\dot{\gamma}_w &= \begin{bmatrix}\dot{g}_4(t) & \sigma_1 & \sigma_2 & \dots & \sigma_{N-2} & \sigma_{N-1} & \dot{g}_5(t)\end{bmatrix}^{\mathrm{T}}
\end{aligned}
\tag{4.11}
$$

where all the symbols have their usual meaning as described above and the matrix A is given by

$$
A = \begin{bmatrix}
6 & 0 & & & & & & & & & \\
1 & 4 & 1 & & & & & & & & \\
 & 1 & 4 & 1 & & & & & & & \\
 & & & & \dots & \dots & \dots & & & & \\
 & & & & \dots & \dots & \dots & & & & \\
 & & & & & & 1 & 4 & 1 & & \\
 & & & & & & & 1 & 4 & 1 & \\
 & & & & & & & & 0 & 6 &
\end{bmatrix}_{(N+1)\times(N+1)}
$$

Now, the systems of ordinary differential equations given by (4.11) are solved step by step by using RK4 method.
First, the system is solved by using a variant of Thomas algorithm only once at each time level $t > 0$ and a system of first order differential equations is obtained that can be solved by SSP-RK54 scheme.

5. The Initial Vectors

Now using the conditions (1.4) and (1.7) for $u(x,t)$ at the time $t = 0$, we have

$$
\begin{aligned}
u\left(x_0, 0\right) &= g_0(0) \\
u\left(x_1, 0\right) &= f_1\left(x_1\right) \\
&\dots\dots\dots \\
&\dots\dots\dots \\
u\left(x_{N-1}, 0\right) &= f_1\left(x_{N-1}\right) \\
u\left(x_N, 0\right) &= g_1(0)
\end{aligned}
$$

Using the approximate solution (3.2) and the Table 1 , the above conditions reduced as:

$$\begin{gathered}6\alpha_0(0) = g_0(0)\\ \alpha_0(0) + 4\alpha_1(0) + \alpha_2(0) = f_1(x_1)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \alpha_{N-2}(0) + 4\alpha_{N-1}(0) + \alpha_N(0) = f_1(x_{N-1})\\ 6\alpha_N(0) = g_1(0)\end{gathered}$$

This system of equations can equivalently be written in the following matrix equation

$$A\alpha^0 = [g_0(0) \quad f_1(x_1) \ldots\ldots\ldots\ldots f_1(x_{N-1}) \quad g_1(0)]^T \tag{5.1}$$

where

$$\alpha^0 = [\alpha_0(0) \quad \alpha_1(0) \ldots\ldots \alpha_{N-1}(0) \quad \alpha_N(0)]^T = [\ \alpha_0^0 \quad \alpha_1^0 \ldots\ldots \alpha_{N-1}^0 \quad \alpha_N^0\ ]^T \tag{5.2}$$

and

$$A = \begin{bmatrix} 6 & 0 & & & & & & & & \\ 1 & 4 & 1 & & & & & & & \\ & 1 & 4 & 1 & & & & & & \\ & & & & \cdots & \cdots & \cdots & & & \\ & & & & \cdots & \cdots & \cdots & & & \\ & & & & & & 1 & 4 & 1 & \\ & & & & & & & 1 & 4 & 1 \\ & & & & & & & & 0 & 6 \end{bmatrix}_{(N+1)\times(N+1)} \tag{5.3}$$

The matrix A is a diagonally dominant tri-diagonal matrix and the system (5.1) can be solved for the initial vector α^0 by Thomas algorithm. In the similar fashion, the initial vectors β^0 and γ^0 for $v(x,t)$ and $w(x,t)$ can be obtained. Once the vectors $\alpha(t), \beta(t)$ and $\gamma(t)$ have been determined at a specified time ($t = 0$), the solutions at any point in the solution range can be determined then. Consequently, the approximate solutions $U^N(x,t), V^N(x,t)$ and $W^N(x,t)$ are completely known.

6. Numerical Experiments

In this section, three different test problems are solved by the proposed scheme. The adaptability and the efficiency of the proposed method will be tested for different values of the parameters in different possible cases. The computed numerical solutions are analysed and compared with the solutions already available in the literature.

Each of the considered test problems of the system of Burgers equations are first solved independently by the same proposed scheme as single independent Burger equations, secondly as three pairs of Coupled Burger's equations and then as a system of three Burger's equation. The computed results are then compared and the effects of copulation and the system are analysed.

Firstly, the three independent single Burger's equations are considered for each of the test problems:

$$u_t = u_{xx} - \eta u u_x \quad ; \quad a < x < b, \quad 0 \le t \le T \tag{6.1}$$

with the initial conditions;

$$u(x,0) = f_1(x) \quad ; \quad a < x < b \tag{6.2}$$

and the boundary conditions;

$$u(a,t) = g_0(t), \quad u(b,t) = g_1(t) \quad ; \quad 0 \le t \le T \tag{6.3}$$

And

$$v_t = v_{xx} - \xi v v_x \quad ; \quad a < x < b, \quad 0 \le t \le T \tag{6.4}$$

with the initial conditions;

$$v(x,0) = f_2(x) \quad ; \quad a < x < b \tag{6.5}$$

and the boundary conditions;

$$v(a,t) = g_2(t), \quad v(b,t) = g_3(t) \quad ; \quad 0 \le t \le T \tag{6.6}$$

And

$$w_t = w_{xx} - \mu w w_x \quad ; \quad a < x < b, 0 \le t \le T \tag{6.7}$$

with the initial conditions;

$$w(x,0) = f_3(x) \quad ; \quad a < x < b \tag{6.8}$$

and the boundary conditions;

$$w(a,t) = g_4(t), \quad w(b,t) = g_5(t) \quad ; \quad 0 \le t \le T \tag{6.9}$$

Secondly, the three pairs of Coupled Burger's equation are considered for each of the test problems:

$$u_t = u_{xx} - \eta u u_x - p(uv)_x, \qquad a < x < b,\ 0 \le t \le T, \tag{6.10}$$
$$v_t = v_{xx} - \xi v v_x - q(uv)_x, \qquad a < x < b,\ 0 \le t \le T. \tag{6.11}$$

$$u(x,0) = f_1(x), \qquad a < x < b, \tag{6.12}$$
$$v(x,0) = f_2(x), \qquad a < x < b. \tag{6.13}$$

$$u(a,t) = g_0(t), \qquad u(b,t) = g_1(t), \qquad 0 \le t \le T, \tag{6.14}$$
$$v(a,t) = g_2(t), \qquad v(b,t) = g_3(t), \qquad 0 \le t \le T. \tag{6.15}$$

$$v_t = v_{xx} - \xi v v_x - q(vw)_x, \qquad a < x < b,\ 0 \le t \le T, \tag{6.16}$$
$$w_t = w_{xx} - \mu w w_x - r(vw)_x, \qquad a < x < b,\ 0 \le t \le T. \tag{6.17}$$

$$v(x,0) = f_2(x), \qquad a < x < b, \tag{6.18}$$
$$w(x,0) = f_3(x), \qquad a < x < b. \tag{6.19}$$

$$v(a,t) = g_2(t), \qquad v(b,t) = g_3(t), \qquad 0 \le t \le T, \tag{6.20}$$
$$w(a,t) = g_4(t), \qquad w(b,t) = g_5(t), \qquad 0 \le t \le T. \tag{6.21}$$

$$u_t = u_{xx} - \eta u u_x - p(uw)_x, \qquad a < x < b,\ 0 \le t \le T, \tag{6.22}$$
$$w_t = w_{xx} - \mu w w_x - r(uw)_x, \qquad a < x < b,\ 0 \le t \le T, \tag{6.23}$$

$$u(x,0) = f_1(x), \qquad a < x < b, \tag{6.24}$$
$$w(x,0) = f_3(x), \qquad a < x < b, \tag{6.25}$$

$$u(a,t) = g_0(t), \qquad u(b,t) = g_1(t), \qquad 0 \le t \le T, \tag{6.26}$$
$$w(a,t) = g_4(t), \qquad w(b,t) = g_5(t), \qquad 0 \le t \le T. \tag{6.27}$$

Finally, each of the three test problems are solved for the system of Burgers equation (1.1)–(1.9) are solved and analysed by the proposed scheme.

Test Problem 1:

In this problem the numerical solutions are obtained for the following parameters

$$\begin{array}{ccc} p = 1, & q = 1, & r = 1 \\ \eta = -2, & \xi = -2, & \mu = -2 \end{array}$$

Computations are made for $a = 0, b = 12$ with $\Delta x = h = 0.1$ and the time step length of $k = 0.0001$. For initial conditions, we have taken

$$f_1(x) = \begin{cases} 1, & x \in (0,6), \\ 0, & x \in [6,12). \end{cases} \tag{6.28}$$

$$f_2(x) = \begin{cases} 0, & x \in (0,6], \\ 1, & x \in (6,12). \end{cases} \tag{6.29}$$

$$f_3(x) = \begin{cases} 1, & x \in (0,5), \\ 6-x, & x \in [5,6], \\ 0, & x \in [6,12). \end{cases} \tag{6.30}$$

And the boundary conditions are taken as

$$\begin{array}{cc} g_0(t) = 1, & g_1(t) = 0 \\ g_2(t) = 0, & g_3(t) = 1 \\ g_4(t) = 1, & g_5(t) = 0 \end{array}$$

The results for (6.1)–(6.3), (6.4)–(6.6) and (6.7)–(6.9) are summarized and illustrated in the Figure 1(a), 1(b) and 1(c) respectively. The plots of the corresponding sets of Coupled Burger's equations (6.10)–(6.15), (6.16)–(6.21) and (6.22)–(6.27) are given in Figure 2(a), 2(b) and 2(c) respectively. Finally, the results for the system of Burger's equation (1.1)–(1.9) are illustrated in Figure 3 and Figure 4.

Test Problem 2:

In this problem the numerical solutions are obtained for the following parameters

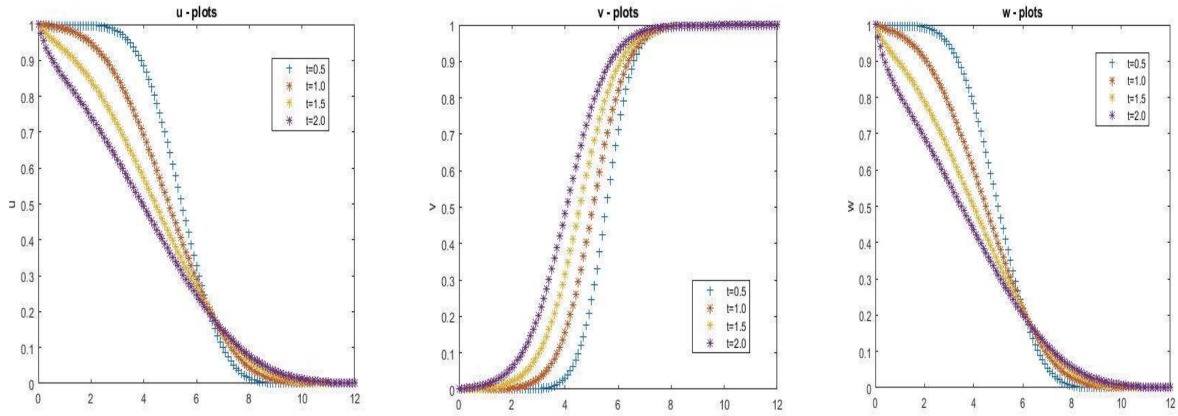
$$\begin{array}{ccc} p = 10, & q = 10, & r = 10 \\ \eta = 2, & \xi = 2, & \mu = 2 \end{array}$$

Computations are made for $a = -1, b = 1$ with $\Delta x = h = 0.1$ and the time step length of $k = 0.0001$. For initial conditions, we have taken

$$f_1(x) = \begin{cases} 0 & ; \quad x \in (-1,0] \\ 1 & ; \quad x \in (0,1) \end{cases}$$
$$f_2(x) = \begin{cases} 1 & ; \quad x \in (-1,0) \\ 0 & ; \quad x \in [0,1) \end{cases}$$
$$f_3(x) = \begin{cases} 0 & ; \quad x \in (-1,0.1) \\ 1 & ; \quad x \in [0.1,1) \end{cases}$$

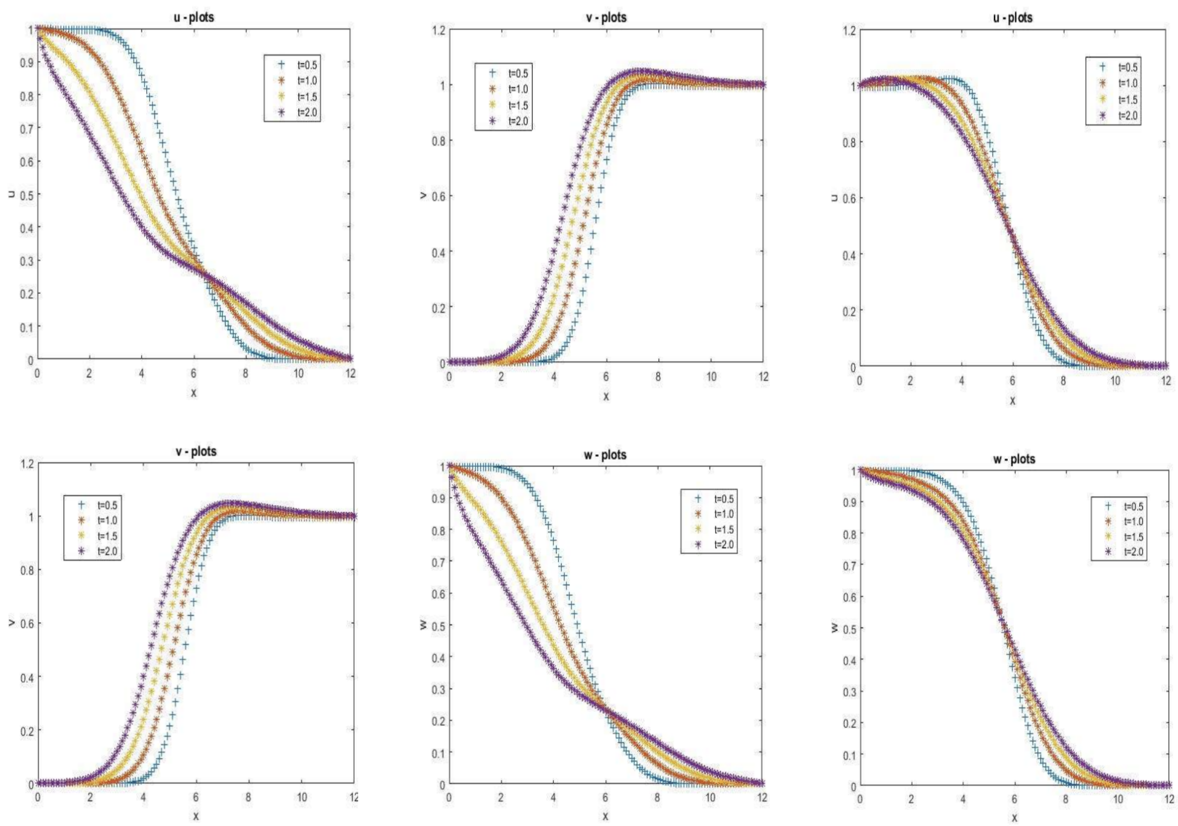
And the boundary conditions are taken as

$$\begin{array}{cc} g_0(t) = 0, & g_1(t) = 1 \\ g_2(t) = 1, & g_3(t) = 0 \\ g_4(t) = 0, & g_5(t) = 1 \end{array}$$



1(a) 1(b) 1(c)

Figure 1: Computed solutions of u, v, w for the single independent Burger's equations in Example 1.



2(a) Copulation of u & v 2(b) Copulation of v & w 2(c) Copulation of u & w

Figure 2: Computed solutions of u, v, w for the three pairs of Coupled Burgers equations in Example 1.

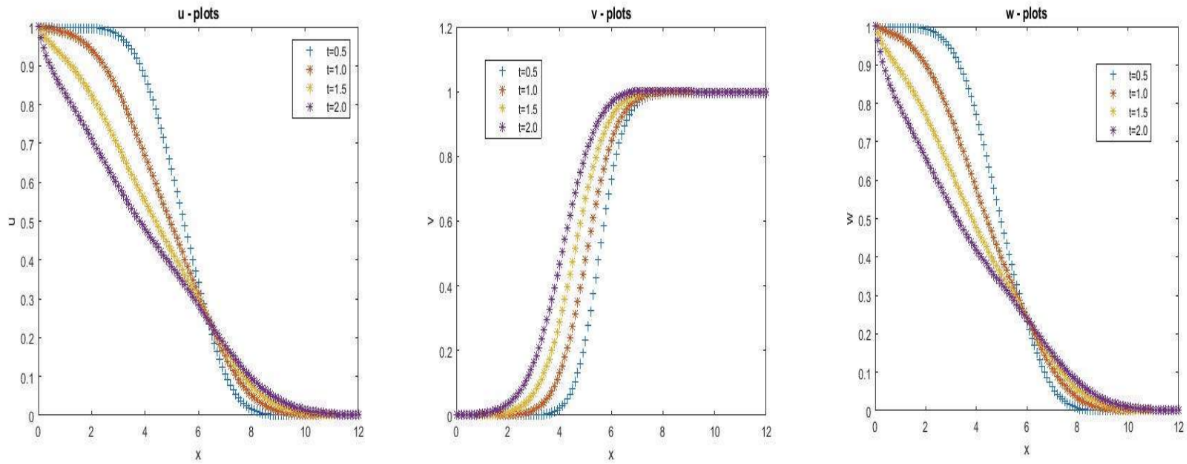


Figure 3: Computed solutions of u, v, w respectively for the system of Burger's equation for in Example 1

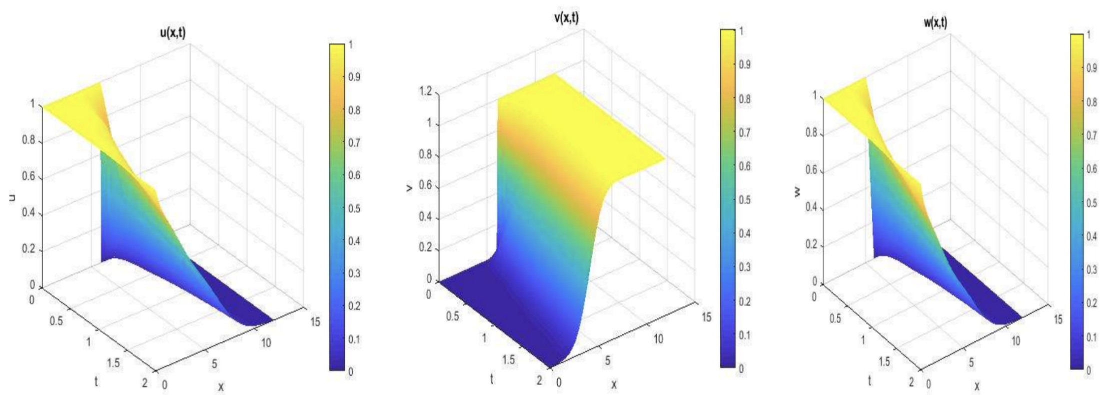


Figure 4: Computed solutions of u, v, w for the system of Burger's equation in Example 1. .

The results for (6.1)–(6.3), (6.4)–(6.6) and (6.7)–(6.9) are summarized and illustrated in the Figure 5(a), 5(b) and 5(c) respectively. For better comparison of the results obtained, the values of u, v and w for (6.1)–(6.3), (6.4)–(6.6) and (6.7)–(6.9) are given in Table 1. The plots of the corresponding sets of coupled Burgers equations (6.10)–(6.15)–(6.16)–(6.21) and (6.22)–(6.27) are given in Figure 6(a), Figure 6(b) and Figure 6(c), respectively. Finally, the results for the system of Burgers equation (1.1)–(1.9) are illustrated in Figure 7 and Figure 8.

Test Problem 3:

In this problem, the numerical solutions are obtained for the following parameters

$$p = 10, \qquad q = 10, \qquad r = 10,$$

$$\eta = 2, \qquad \xi = 2, \qquad \mu = 2.$$

Computations are made for $a = 0$, $b = 1$ with $\Delta x = h = 0.04$ and the time step length $k = 0.00001$. For the initial conditions, we have taken

$$f_1(x) = \begin{cases} \sin(2\pi x) & x \in (0, 0.5], \\ 0 & x \in (0.5, 1), \end{cases}$$

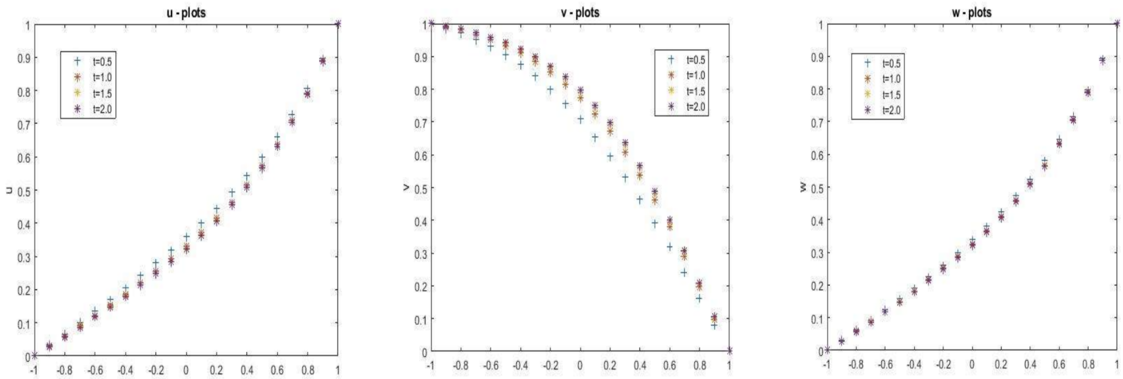
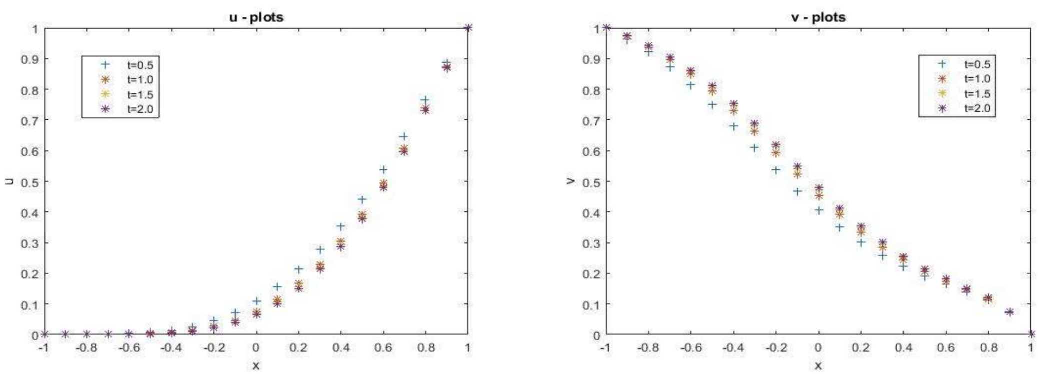
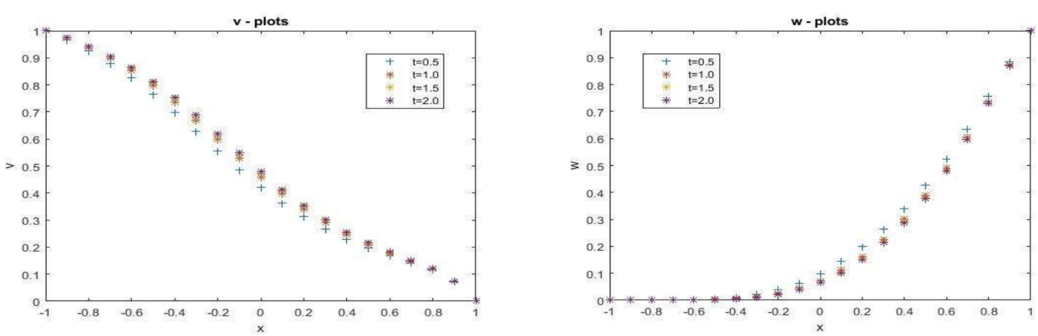


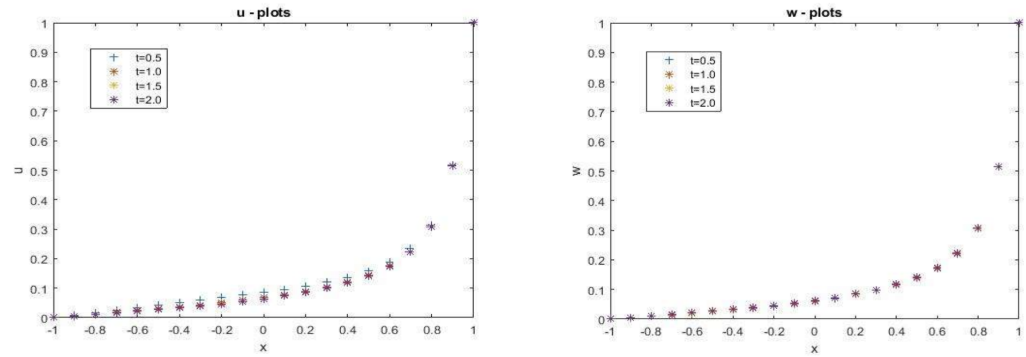
Figure 5: Computed solutions of u, v, w for the single independent equations in Example 2.



6 (a) Copulation of u and v



6 (b) Copulation of v and w



6 (c) Copulation of u and w

Figure 6: Computed solutions of u, v and w for the three pairs of coupled Burgers equations in Example 2.

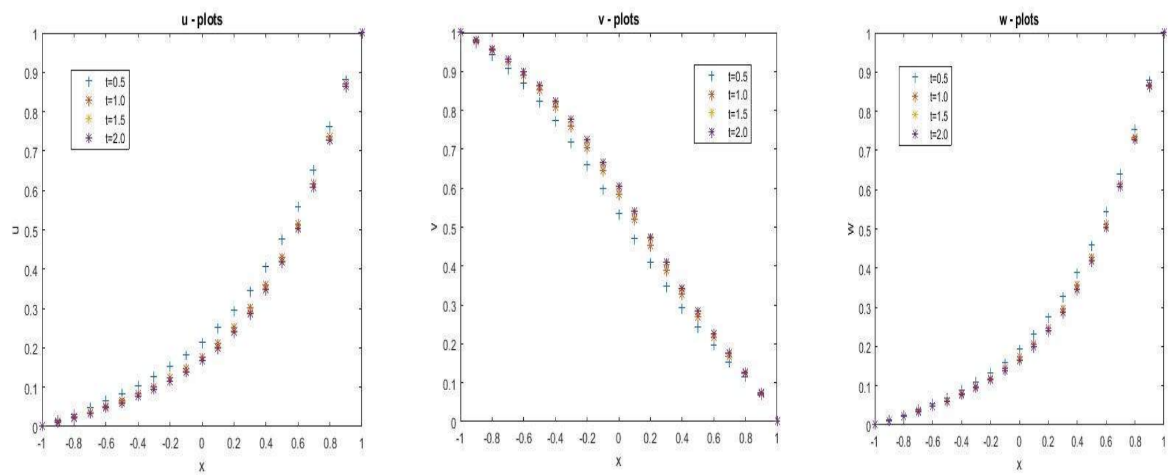


Figure 7: Computed solutions of u, v, w for the system of Burger's equation in Example 2

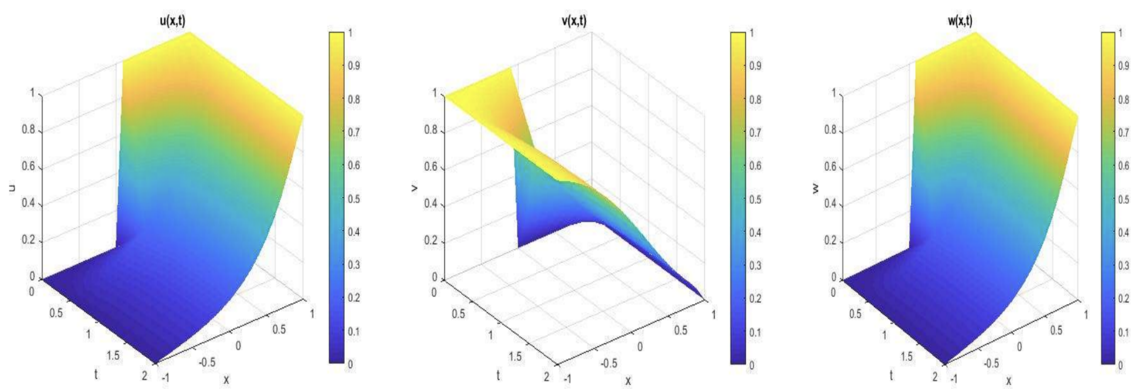


Figure 8: Computed solutions of u, v, w respectively for the system of Burger's equation for $0 \leq t \leq 2$ in Example 2.

$$f_2(x) = \begin{cases} 0 & x \in (0, 0.5], \\ -\sin(2\pi x) & x \in (0.5, 1), \end{cases}$$

$$f_3(x) = \begin{cases} 0 & x \in (0, 0.5], \\ \sin(2\pi x) & x \in (0.5, 1). \end{cases}$$

And the boundary conditions are taken as

$$g_0(t) = 0 \qquad g_1(t) = 0,$$

$$g_2(t) = 0 \qquad g_3(t) = 0,$$

$$g_4(t) = 0 \qquad g_5(t) = 0.$$

The results for (6.1)–(6.3), (6.4)–(6.6) and (6.7)–(6.9) are summarized and illustrated in Figure 9(a), Figure 9(b) and Figure 9(c), respectively. The plots of the corresponding sets of coupled Burgers equations (6.10)–(6.15), (6.16)–(6.21) and (6.22)–(6.27) are given in Figure 10(a), Figure 10(b) and Figure 10(c), respectively. Finally, the results for the system of Burgers equation (1.1)–(1.9) are illustrated in Figure 11 and Figure 12.

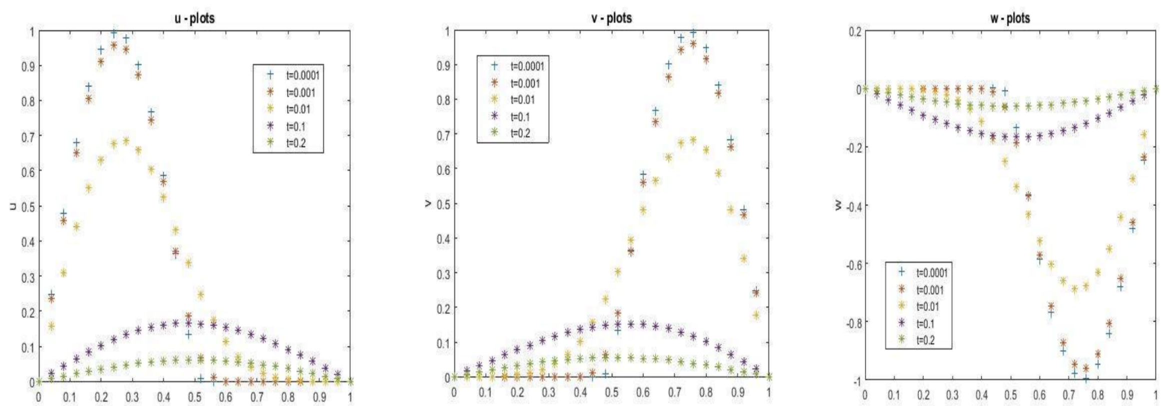
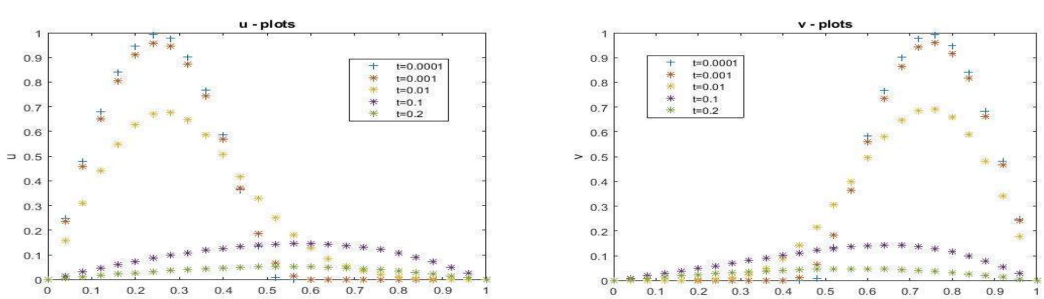
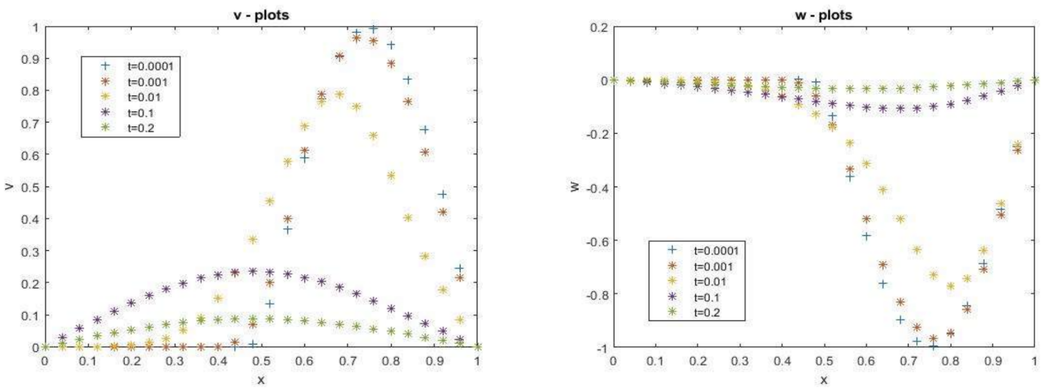


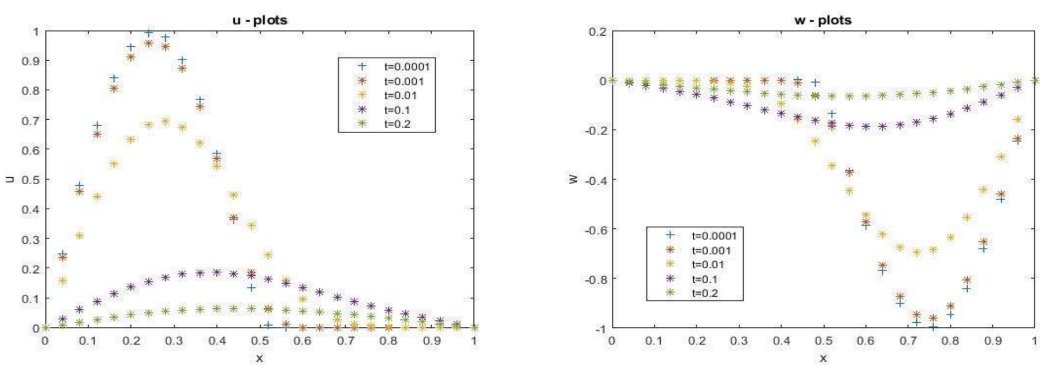
Figure 9: Computed solutions of u, v, w for the single independent Burger's equations in Example 3



10 (a) Copulation of u and v



10 (b) Copulation of v and w



10 (c) Copulation of u and w

Figure 10: Computed solutions of u, v and w for the three pairs of coupled Burgers equations in Example 3.

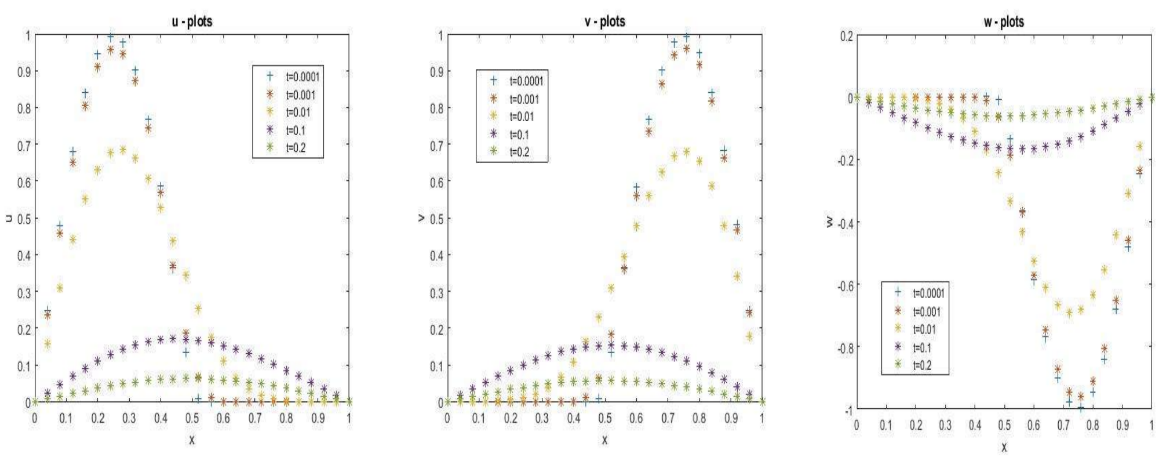


Figure 11: Computed solutions of u, v, w for the system of Burger's equation in Example 3

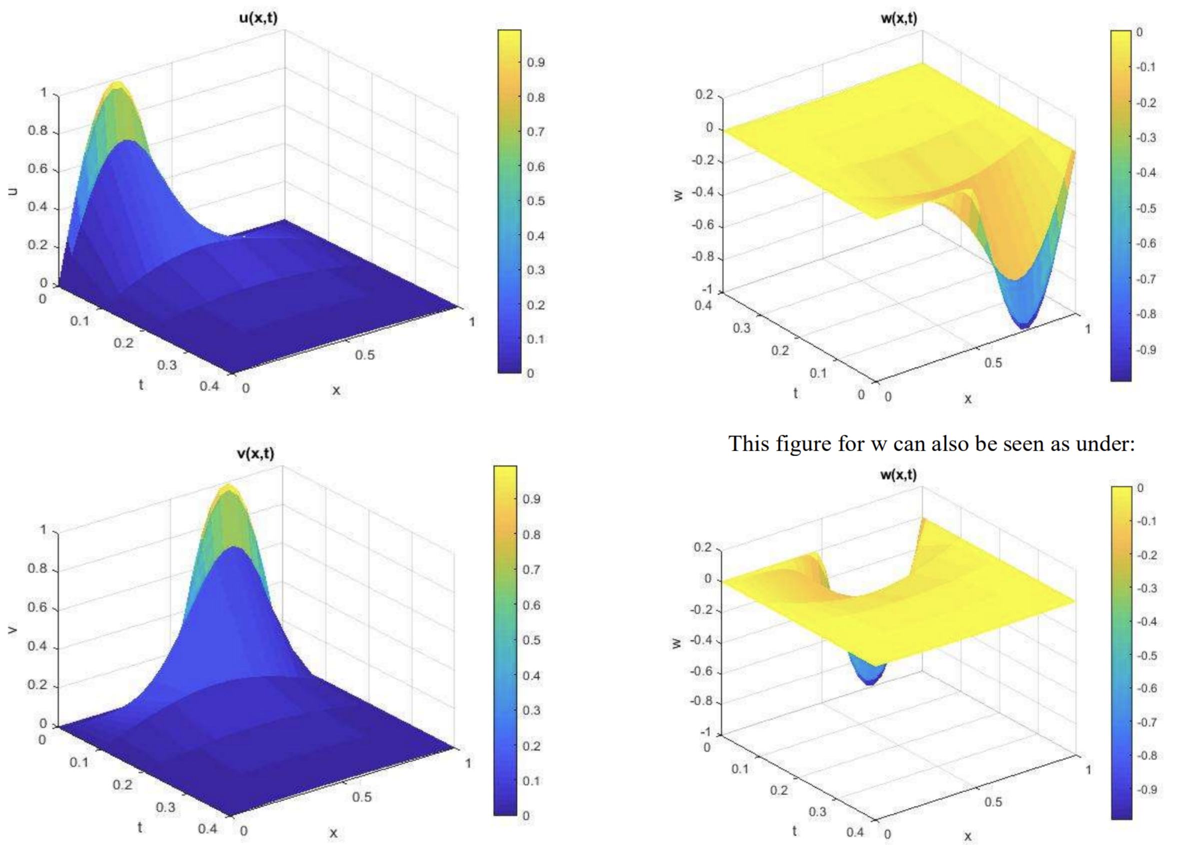


Figure 12: Computed solutions of u, v, w respectively for the system of Burger's equation for $0 \leq t \leq 2$ in Example 3.

7. Results and Discussions

The numerical results are computed for the single, coupled and the system of Burgers equations for the three different test problems using the collocation of cubic B-spline basis functions. In Problem 1, u and v have discontinuous initial conditions at $x = 6$ while w is continuous in the domain of interest but is not differentiable at $x = 5$ and $x = 6$. It is observed that u and w have almost similar behaviors in all the three cases of the single, coupled and the system of equations considered. It is also seen that v does not change significantly with time as compared to u and w. The copulation of Burgers equations gives rise to slight deviations of the solutions with time. However, the solutions for the coupled and the system of Burgers equations do not change significantly and behave alike to a great extent.

In Problem 2, it is again observed that the behaviors of u and w are quite similar. When the problem is solved as a single independent Burgers equation, it is seen that there is negligible variation in the numerical solutions with time. However, variation in solutions with time is observed when the same problem is solved for the coupled equations. The solutions for the coupled case and the system of equations are found to be almost similar.

In Problem 3, the initial solutions of u, v and w are non-differentiable at $t = 0.5$. In the case of single Burgers equations, it is observed that the solution curves become flat with time at a faster rate as compared to that in the case of copulation.

For better understanding of the numerical solutions obtained by the proposed scheme for Problems 1, 2 and 3, three-dimensional plots are also given in Figures 4, 8 and 12, respectively.

8. Conclusions

The proposed numerical scheme to solve the system of three Burgers equations is successfully implemented and the results obtained are in good agreement with those expected. It can also be concluded that the discontinuity and the non-differentiability in the initial solutions do not affect significantly the solutions of the system of Burgers equations in the longer run of time. The simulated results are found to be comparatively better and smoother for the problems with large diffusion coefficients. Upon successful implementation of the proposed numerical scheme for a system of three Burgers equations, it is now recommended that the cubic B-spline collocation method [10,11,22,23,24,25] can further be effectively applied to solve larger systems of nonlinear Burgers equations arising in physical and mathematical fields.

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