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Some New Congruences for Andrews' Singular Over partitions and 9-Regular Over Partitions

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ABSTRACT: In this paper, we establish several new congruence relations for Andrews' singular over partitions $C_{k,i}$ (n) and l-regular overpartitions A_l (n). Specifically, we prove infinite families of congruences modulo 8 and 9 for $C_{12,5}$(n), complementing earlier work on these partition functions. Additionally, we establish new congruences modulo 16 forA_9(n), extending recent results by Barman and Ray. Our proofs utilize Ramanujan's theta functions, cubic theta functions, and dissection formulas involving q-series identities. These results contribute to the growing body of arithmetic properties of singular and regular overpartition functions [1].

Keywords: Andrews' singular over partitions, l-regular overpartitions, congruences, q-series, partition functions.

Contents

1. Introduction

The theory of partition functions and their congruence properties have been a central topic in combinatorics and number theory since Ramanujan's foundational work.Corteel and Lovejoy [13] introduced the notion of overpartitions.An overpartition of a nonnegative integer n is a partitionof n in which the first occurrence of a part may be over-lined. For example 14 overpartitions of 4 are 4, $\overline{4}$, 3+1, $\overline{3}$+1, 3+$\overline{1}$, $\overline{3}$+$\overline{1}$, 2+2, $\overline{2}$+2, 2+1+1,$\overline{2}$ +1+1 2+$\overline{1}$+1, $\overline{2}$+$\overline{1}$+1,1+1+1+1, $\overline{1}$+1+1+1.Let $\overline{p}$(n) denote the number of overpartitions of n. The generating function for $\overline{p}$(n) is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{f_2}{{f_1}^2} \tag{1.1}$$

where here and the sequel, we assume $|q| < 1$ and use the customary notation

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i).$$

and and f_k stands for $(q^k;q^k)_\infty$.
Andrews [9] defined the partition function $\overline{C}_{k,i}$ called singlar overpartition which counts the number of overpartitions of n in which no part is divisible by k and only parts ≡ ± i (mod k) may be overlined.For

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example $\overline{C}_{3,1}(3) = 6$.The six singular overpartitions of 3 are 2+1, $\overline{2}$ +1, 2+$\overline{1}$, $\overline{2}$+$\overline{1}$,1+1+1, $\overline{1}$+1+1. For $k \geq 3$ and 1≤ i≤ $\lfloor \frac{k}{2} \rfloor$ the generating function for $\overline{C}_{k,i}$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty} \tag{1.2}$$

As a part of his work, Andrews proved that for all $n \geq 0$

$$\overline{C}_{3,1}(n) \equiv 0 \pmod 3 \tag{1.3}$$

Baruah and Ahmed [6] found congruences modulo 4, 18 and 36 for $\overline{C}_{3,1}(n)$. Barman and Ray [2] proved congruences modulo 144 for$\overline{C}_{3,1}(n)$ as conjecture by Naika and Gireesh [14]. Li and Yao [1] proved some infinite family of congruences modulo 4 and 8 for $\overline{C}_{6,2}(n)$, $\overline{C}_{12,3}(n)$ and $\overline{C}_{28,7}(n)$.Fthima and Kathiravan [3] obtained some congruences modulo 2 for $\overline{C}_{92,23}(n)$ and $\overline{C}_{60,15}(n)$. Liu and Du [4] proved some congruences modulo 2 for $\overline{C}_{48,6}(n)$ and $\overline{C}_{48,18}(n)$.Here in this paper we have come up with some new congruences for $\overline{C}_{12,1}(n)$ and $\overline{C}_{12,5}(n)$.

The general theta function as defined by Ramanujan is [21, p. 34, Eq. 18.1]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, |ab| < 1. \tag{1.4}$$

The theta function $f(a,b)$ satisfy the following simple properties [21, p. 34, Entry 18] :

$$f(a,b) = f(b,a), \tag{1.5}$$
$$f(1,a) = 2f(a,a^3), \tag{1.6}$$
$$f(-1,a) = 0, \tag{1.7}$$
$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{1.8}$$

These properties are used frequently in this paper without citing them. Setting $n = 1, a = q^r, b = q^s$ in (1.8), we get

$$f(q^r,q^s) = q^r f(q^{2r+s}, q^{-r}). \tag{1.9}$$

This identity is often used whenever negative powers of argument occur in $f(a,b)$. The Jacobi triple product identity [21, p. 35, Entry 19] in Ramanujan's notation takes the following form:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The three special cases of (1.4) as recorded by Ramanujan are [21, p. 36, Entry 22]

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.10}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.11}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.12}$$

Ramanujan also defines another important function $\chi(q) := (-q;q^2)_\infty$. For sake of convenience, define $f_n := f(-q^n) = (q^n;q^n)_\infty$, for $n \geq 1$. Setting $n = 2$ in [21, p. 48, Entry 31], we get

$$f(a,b) = f(ab^3, a^3b) + af(b/a, a^5b^3). \tag{1.13}$$

In [15] Lovejoy investigated the function $\overline{A}_l$(n), which counts the number of overpartitions of n into parts not divisible by l . In a very recent paper Shen [16] calls the overpartitions enumerated by the function $\overline{A}_l$(n) as l-regular overpartitions. The generating function for $\overline{A}_l$(n) is

$$\sum_{n=0}^{\infty} \overline{A}_l(n) q^n = \frac{(-q;q)_\infty (q^l;q^l)_\infty}{(q;q)_\infty (-q^l;q^l)_\infty} = \frac{\phi(-q^l)}{\phi(-q)} \tag{1.14}$$

Shen [16] finds some congruences modulo 3, 6, and 24 for $\overline{A}_3$(n) and $\overline{A}_4$(n).Recently Barman and Ray [2] prove some congruences modulo 4 and 16 for $\overline{A}_9$(n). Here in this paper we will prove more new congruences modulo 16. These are stated as theorems below.

2. Definition and Preliminary Results

We will require the following identities

Lemma 2.1 *We recall the definition of cubic theta function $a(q)$, $b(q)$ and $c(q)$ due to Borwein et al. [10], namely,*

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{2.1}$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \quad \omega = exp(2\pi i/3), \tag{2.2}$$

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} = 3q^{1/3} \frac{{f_3}^3}{f_1}. \tag{2.3}$$

$$a(q) = \phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6) \tag{2.4}$$

Lemma 2.2 *From [11, p.109] we note that*

$$b(q) = \frac{{f_1}^3}{f_3}. \tag{2.5}$$

Lemma 2.3 *From [11, p.110], we have*

$$\frac{c(q)}{c(q^4)} = 1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)}. \tag{2.6}$$

Lemma 2.4 *From [11, p.111], we have*

$$a(q) = \frac{\psi^3(q)}{\psi(q^3)} + 3q\frac{\psi^3(q^3)}{\psi(q)} \tag{2.7}$$

Lemma 2.5 *From [12, p.345], we have*

$$1 + \frac{1}{v^3} = \frac{1}{q}\frac{\psi^4(q)}{\psi^4(q^3)} \tag{2.8}$$

$$\frac{\psi(q^{1/3})}{q\psi(q)} = 1 + \frac{1}{v} \tag{2.9}$$

where

$$v = q^{1/3} w(q) = \frac{f_1 {f_6}^3}{f_2 {f_3}^3} \tag{2.10}$$

Lemma 2.6 *The following 2-dissections formula is true:*

$$\frac{f_3}{{f_1}^3} = \frac{{f_4}^6{f_6}^3}{{f_2}^9{f_{12}}^2} + 3q\frac{{f_4}^2f_6{f_{12}}^2}{{f_2}^7} \tag{2.11}$$

$$\frac{{f_1}^3}{f_3} = \frac{{f_4}^3}{f_{12}} - 3q\frac{{f_2}^2{f_{12}}^3}{{f_6}^2f_4} \tag{2.12}$$

$${f_1}^2 = \frac{f_2{f_8}^5}{{f_4}^2{f_{16}}^2} - 2q\frac{f_2{f_{16}}^2}{f_8} \tag{2.13}$$

$$\frac{{f_3}^3}{f_1} = \frac{{f_4}^3{f_6}^2}{{f_2}^2f_{12}} + q\frac{{f_{12}}^3}{f_4} \tag{2.14}$$

$$f_3f_1 = \frac{f_2{f_8}^2{f_{12}}^4}{{f_4}^2f_6{f_{24}}^2} - q\frac{{f_4}^4f_6{f_{24}}^2}{f_2{f_8}^2{f_{12}}^2} \tag{2.15}$$

$$\frac{{f_3}^2}{{f_1}^2} = \frac{{f_4}^4f_6{f_{12}}^2}{{f_2}^5f_8f_{24}} + 2q\frac{f_4{f_6}^2f_8f_{24}}{{f_2}^4f_{12}} \tag{2.16}$$

Proof:

$$(-q:-q)_\infty = \frac{{f_2}^3}{f_1f_4} \tag{2.17}$$

(36), (38), (39) and (40) are 22.1.13, 22.1.14, 30.12.1 and 30.10.4 in [**?**].For (35) and (37) replace q by -q 22.1.13 and 1.94 of [**?**] then apply (41) □

Lemma 2.7 *Following 2-dissections is due to Xia and Yao [17]*

$$\frac{f_3}{f_1} = \frac{f_4f_6f_{16}{f_{24}}^2}{{f_2}^2f_8f_{12}f_{48}} + q\frac{f_6{f_8}^2f_{48}}{{f_2}^2f_{16}f_{24}} \tag{2.18}$$

Lemma 2.8 *Following 2-dissections is due to Wang [20]*

$$\frac{1}{f_3{f_1}^5} = (\frac{{f_4}^{14}}{{f_2}^{17}f_6{f_{12}}^2} + 3q^2\frac{{f_4}^6{f_{12}}^6}{{f_6}^5{f_2}^{13}}) + q(5\frac{{f_4}^{10}{f_{12}}^2}{{f_2}^{15}{f_6}^3} - 9q^2\frac{{f_4}^2{f_{12}}^{10}}{{f_6}^7{f_2}^{11}}) \tag{2.19}$$

Lemma 2.9 *Following 2-dissections is due to Xia and yao [19]*

$$\frac{f_9}{f_1} = \frac{{f_{12}}^3f_{18}}{{f_2}^2f_6f_{36}} + q\frac{{f_4}^2f_6f_{36}}{{f_2}^3f_{12}} \tag{2.20}$$

Lemma 2.10 *The following 3-dissections formula is true:*

(2.21)

$${f_1}^3 = f_3a(q^3) - 3q{f_9}^3 \tag{2.22}$$

$$\frac{1}{{f_1}^3} = \frac{{f_9}^3}{{f_3}^{10}}(a^2(q^3) + 3qa(q^3)\frac{{f_9}^3}{f_3} + 9q^2\frac{{f_9}^6}{{f_3}^2}) \tag{2.23}$$

(2.24)

Proof: (46) and (47) are 39.2.8 and 21.3.1 in [**?**] respectively □

3. Main Theorem

Theorem 3.1 *For any nonnegative integer n we have*

$$\overline{C}_{12,5}(3^{2k}(144n+48r+42)) \equiv 0 \pmod 8, k \geq 0, r = 0, 2 \tag{3.1}$$

$$\overline{C}_{12,5}(3^{4k}(144n+102)) \equiv 0 \pmod 8, k \geq 0 \tag{3.2}$$

$$\overline{C}_{12,5}(24n+8r+6) \equiv 0 \pmod 9, r = 0, 1 \tag{3.3}$$

$$\sum_{n=0}^{\infty} \overline{C}_{12,5}(24n+6)q^n \equiv 3\psi^2(q) \pmod 9 \tag{3.4}$$

$$\sum_{n=0}^{\infty} \overline{C}_{12,5}(48n+10)q^n \equiv 4f_1f_4 \pmod 8 \tag{3.5}$$

Theorem 3.2 *For any nonnegative integer n we have*

$$\overline{C}_{12,5}(3^{2k}(144n+48r+42)-1) \equiv 0 \pmod 8, k \geq 0, r = 0, 2 \tag{3.6}$$

$$\overline{C}_{12,5}(3^{4k}(144n+102)-1) \equiv 0 \pmod 8, k \geq 0 \tag{3.7}$$

$$\overline{C}_{12,5}(24n+8r+5) \equiv 0 \pmod 9, r = 0, 1 \tag{3.8}$$

$$\sum_{n=0}^{\infty} \overline{C}_{12,5}(24n+5)q^n \equiv 3\psi^2(q) \pmod 9 \tag{3.9}$$

$$\sum_{n=0}^{\infty} \overline{C}_{12,5}(48n+9)q^n \equiv 4f_1f_4 \pmod 8 \tag{3.10}$$

Theorem 3.3 *For any nonnegative integer n we have*

$$\overline{A_9}((24n+20) \equiv 0 \pmod{16} \tag{3.11}$$

$$\overline{A_9}((48n+2) \equiv 0 \pmod{16} \tag{3.12}$$

$$\overline{A_9}(3^k(36n+30)) \equiv 0 \pmod{16}, k \geq 0 \tag{3.13}$$

$$\overline{A_9}(2.3^{k+1}(36n+6r+1)) \equiv 0 \pmod{16}, k \geq 0, r = 1(1)5 \tag{3.14}$$

$$\sum_{n=0}^{\infty} \overline{A_9}(216n+6)q^n \equiv 8{f_1}^2 \pmod{16} \tag{3.15}$$

4. Proof of the Main Theorem

4.1. Congruence for $\overline{C}_{12,5}$(n) and $\overline{C}_{12,1}$(n)

Proof: From equation no 28 and 29 of [14] we get

$$Y_1 + qX_1 = \frac{{f_2}^3}{{f_1}^2 f_4} \tag{4.1}$$

$$Y_1 - qX_1 = \frac{f_4 {f_6}^5}{{f_2}^2 {f_3}^2 {f_{12}}^2} \tag{4.2}$$

where X_1andY_1 are defined as

$$X(q) = \frac{(q;q^{12})_\infty (q^{11};q^{12})_\infty f_{12}}{f_1} = X_1 \tag{4.3}$$

$$Y(q)=\frac{(q^5;q^{12})_\infty(q^7;q^{12})_\infty f_{12}}{f_1}=Y_1 \tag{4.4}$$

From above equations (49) and (50) we get

$$Y(-q)=\frac{1}{2}(\frac{{f_1}^2f_4}{{f_2}^3}+\frac{{f_3}^2f_4}{{f_2}^2f_6}) \tag{4.5}$$

Now putting k=12 and i= 5 in (1) we get

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(n)q^n=\frac{(-q^5;q^{12})_\infty(-q^7;q^{12})_\infty f_{12}}{f_1}=\frac{{f_2}^3}{{f_1}^2f_4}Y(-q) \tag{4.6}$$

$$=\frac{1}{2}\left(1+\frac{f_2}{f_6}\left(\frac{f_3}{f_1}\right)^2\right) \tag{4.7}$$

Now applying (42) in (55) we extract

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(2n)q^n=\frac{1}{2}\frac{f_3}{{f_1}^3}\left[\frac{{f_1}^3}{f_3}+\frac{{f_2}^2{f_8}^2{f_{12}}^4}{{f_4}^2{f_6}^2{f_{24}}^2}+q\frac{{f_4}^4{f_{24}}^2}{{f_8}^2{f_{12}}^2}\right] \tag{4.8}$$

Applying (35) and (36) in (56) we extract

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(4n+2)q^n=\frac{1}{2}\left[\frac{{f_2}^{10}{f_{12}}^2}{{f_6}^4{f_4}^2}\frac{{f_3}^3}{{f_1}^9}+3\frac{{f_6}^6{f_4}^2}{{f_{12}}^2}\frac{1}{f_3{f_1}^5}\right] \tag{4.9}$$

Applying (43) and (35) in (57) we extract

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(8n+2)q^n=2\frac{{f_2}^8{f_3}^5}{{f_1}^{13}{f_6}^4}\left(\psi^4(q)+9q\psi^4(q^3)\right) \tag{4.10}$$

Taking congruence modulo 8 we get

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(8n+2)q^n\equiv 2\frac{{f_2}^8{f_3}^5}{{f_1}^{13}{f_6}^4}\left(\psi^2(q^2)+q\psi^2(q^6)\right) \pmod 8 \tag{4.11}$$

Equation (29) can written in the form

$$\psi^2(q^2)+q\psi^2(q^6)=\frac{f_4f_{12}{f_3}^3}{f_1{f_6}^2} \tag{4.12}$$

Applying (60) in (59) we get

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(8n+2)q^n\equiv 2\frac{f_4f_{12}}{{f_6}^2}{f_1}^2 \pmod 8 \tag{4.13}$$

Applying (37) in (61) we extract

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(16n+10)q^n\equiv 4{f_1}^3{f_4}^3 \pmod 8 \tag{4.14}$$

Applying (46) in (62) we extract

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(48n+42)q^n\equiv 4q{f_3}^3{f_{12}}^3 \pmod 8 \tag{4.15}$$

Equating coefficients of q^{3n+r}, r=0,2 from both side of (63) we get

$$\overline{C}_{12,5}(144n+48r+42)) \equiv 0 \pmod{8}, r=0,2 \tag{4.16}$$

Equating coefficients of q^{3n+1}, from both side of (63) we get

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(3^2(16n+10))q^n \equiv 4{f_1}^3{f_4}^3 \pmod{8} \tag{4.17}$$

From (62), (64) and (65) it follows that

$$\overline{C}_{12,5}(3^{2k}(144n+48r+42)) \equiv 0 \pmod{8}, k\geq 0, r=0,2 \tag{4.18}$$

Equating coefficients of q^{3n+1} from both side of (62) we get

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(48n+10)q^n \equiv 4f_1f_4 \pmod{8} \tag{4.19}$$

Again from (57) we extract

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(8n+6)q^n = 12\frac{{f_6}^3{f_4}^{12}}{{f_2}^{15}} \tag{4.20}$$

Taking congruence modulo 9 from both side of (68) we get

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(8n+6)q^n \equiv 3\frac{{f_6}^4}{{f_3}^2} \pmod{9} \tag{4.21}$$

Equating coefficients of q^{3n+r}, r=1,2 from both side of (69) we get

$$\overline{C}_{12,5}(48n+8r+6) \equiv 0 \pmod{9}, r=1,2 \tag{4.22}$$

Equating coefficients of q^{3n+1} from both side of (64) we get

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(24n+6)q^n \equiv 3\psi^2(q) \pmod{9} \tag{4.23}$$

Again taking congruence modulo 8 from both side of (68) we get

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(8n+6)q^n \equiv 4\frac{{f_3}^3}{f_1}{f_2}^5 \pmod{8} \tag{4.24}$$

Applying (35) in (72) we extract

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(16n+6)q^n \equiv 4{f_1}^3{f_2}^3 \pmod{8} \tag{4.25}$$

Applying () in (68) we extract

$$\sum_{n=0}^{\infty}\overline{C}_{12,5}(48n+6)q^n \equiv 4({f_1}^3+q{f_3}^3{f_6}^3) \pmod{8} \tag{4.26}$$

Again applying () in (69) and equating coefficients of q^{3n+2} we get

$$\overline{C}_{12,5}(144n+102) \equiv 0 \pmod{8} \tag{4.27}$$

Similarly equating coefficients of q^{3n+1} we get

$$\sum_{n=0}^{\infty} \overline{C}_{12,5}(144n+54)q^n \equiv 4({f_3}^3 + {f_1}^3{f_2}^3) \pmod 8 \tag{4.28}$$

Now applying () in (71) we extract

$$\sum_{n=0}^{\infty} \overline{C}_{12,5}(432n+54)q^n \equiv 4q{f_3}^3{f_6}^3 \pmod 8 \tag{4.29}$$

From (72) equating coefficients of q^{3n+1} we get

$$\sum_{n=0}^{\infty} \overline{C}_{12,5}(3^4(16n+6))q^n \equiv 4{f_1}^3{f_2}^3 \pmod 8 \tag{4.30}$$

Now from (68), (70) and (73) we get

$$\overline{C}_{12,5}(3^{4k}(144n+102)) \equiv 0 \pmod 8, k \geq 0 \tag{4.31}$$

Congruences for $\overline{C}_{12,1}$:-

Now putting k=12 and i= 1 in (1) we get

$$\sum_{n=0}^{\infty} \overline{C}_{12,1}(n)q^n = \frac{(-q;q^{12})_\infty(-q^{11};q^{12})_\infty f_{12}}{f_1} = \frac{{f_2}^3}{{f_1}^2 f_4} X(-q) \tag{4.32}$$

$$= \frac{1}{2q}\left(1 - \frac{f_2{f_3}^2}{f_6{f_1}^2}\right) \tag{4.33}$$

So all the theorems of $\overline{c}_{12,1}(n)$ can be proved similarly. □

4.2. Congruence for 9-regular partition

Proof: Putting l=9 in (2) we get

$$\sum_{n=0}^{\infty} A_9(n)q^n = \frac{f_2}{f_{18}}\left(\frac{f_9}{f_1}\right)^2 \tag{4.34}$$

Applying (44) in (77) we extract

$$\sum_{n=0}^{\infty} A_9(2n)q^n = \frac{{f_3}^2}{{f_6}^2}\psi(q^9)\frac{1}{f_3}(\frac{\psi^4(q^3)}{\psi^2(q^9)} + q\psi^2(q)) \tag{4.35}$$

Equation (78) can be witten in the following form by applying (33) and (34)

$$\sum_{n=0}^{\infty} A_9(2n)q^n = \frac{{f_3}^2}{{f_6}^2}\psi^3(q^9)\frac{1}{{f_1}^3}(2q^3 + \frac{1}{w^3(q^3)} + \frac{q}{w^2(q^3)} + \frac{2q^2}{w(q^3)}) \tag{4.36}$$

Applying (47) and equating coefficients of q^{3n+1} from both side of (79) we get

$$\sum_{n=0}^{\infty} A_9(6n+2)q^n = \frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)(6qa(q)\frac{{f_3}^3}{f_1} + 3a(q)\frac{{f_3}^3}{f_1}\frac{1}{w^3(q^3)} + \frac{a^2(q)}{w^2(q)} + 18q\frac{{f_3}^6}{f_1}\frac{1}{w(q)}) \tag{4.37}$$

Applying (26) equation (32) can be written as

$$w(q)=\frac{1}{q^{1/3}}\frac{c(q^2)}{c(q)} \tag{4.38}$$

Applying (86) and (26) in (81) we get

$$\sum_{n=0}^{\infty}A_9(6n+2)q^n=\frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)q^{2/3}\left(a(q)(2+\frac{c^3(q)}{c^3(q^2)})+\frac{a^2(q)c(q)}{c^2(q^2)}+2\frac{c^{(}q)}{c(q^2)}\right)c(q) \tag{4.39}$$

Applying (26) and (28) equations (30) and (31) can be written as

$$1+\frac{c^3(q)}{c^3(q^2)}=3\frac{c(q)}{c^2(q^2)}\frac{b^2(q^2)}{b(q)} \tag{4.40}$$

$$a(q)=\frac{c^2(q^2)}{c(q)}+\frac{b^2(q^2)}{b(q)} \tag{4.41}$$

Taking $y=\frac{c^2(q^2)}{c(q)}$, and $x=\frac{b^2(q^2)}{b(q)}$ in equation (87) and applying (88) and (89)

$$\sum_{n=0}^{\infty}A_9(6n+2)q^n=\frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)q^{2/3}c(q)\left((x+y)(1+\frac{3x}{y})+\frac{(x+y)^2}{y}+2y(\frac{3x}{y}-1)\right) \tag{4.42}$$

$$=\frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)q^{2/3}c(q)4x\left(3+\frac{x}{y}\right) \tag{4.43}$$

$$=\frac{4}{3}\frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)q^{2/3}c(q)x\left(4+\frac{c^3(q)}{c^3(q^2)}\right) \tag{4.44}$$

$$=4q\frac{{f_3}^4{f_2}^4{f_6}^4}{{f_1}^{12}}\left(4+\frac{1}{v^3}\right) \tag{4.45}$$

Applying (31) in (93) we have

$$\sum_{n=0}^{\infty}A_9(6n+2)q^n=4q\frac{{f_3}^4{f_2}^4{f_6}^4}{{f_1}^{12}}(3+\frac{1}{q}\frac{\psi^4(q)}{\psi^4(q^3)}) \tag{4.46}$$

$$=4\frac{{f_3}^8{f_2}^4}{{f_1}^{12}}(\psi^4(q)+3q\psi^4(q^3)) \tag{4.47}$$

Taking modulo 16 from both side of (95)

$$\equiv 4\frac{{f_3}^8{f_2}^4}{{f_1}^{12}}(\psi^2(q^2)+3q\psi^2(q^6))\ (\text{mod}16) \tag{4.48}$$

Equation (35) can be written in the form

$$\frac{f_3}{{f_1}^3}=\frac{{f_6}^3{f_4}^2}{{f_2}^7{f_{12}}^2}(\psi^2(q^2)+3q\psi^2(q^6)) \tag{4.49}$$

Applying (97) in (96) we get

$$\sum_{n=0}^{\infty}A_9(6n+2)q^n\equiv 4qf_1f_3\frac{f_6}{f_2}\ (\text{mod}16) \tag{4.50}$$

Applying (39) in (98) we extract

$$\sum_{n=0}^{\infty} A_9(12n+2)q^n \equiv 4\frac{{f_4}^2}{{f_2}^2} \text{ (mod16)} \tag{4.51}$$

Equating coefficients of q^{2n+1} from both side of (99) we get

$$\overline{A_9}(24n+14) \equiv 0 \text{ (mod 16)} \tag{4.52}$$

Equating coefficients of q^{2n} from both side of (99) we get

$$\sum_{n=0}^{\infty} A_9(24n+2)q^n \equiv 4{f_1}^2 \text{ (mod16)} \tag{4.53}$$

Again from (98) equating coefficients of q^{2n+1} from both side we get

$$\sum_{n=0}^{\infty} A_9(12n+8)q^n \equiv -4\frac{{f_3}^2{f_2}^4{f_{12}}^2}{{f_1}^2{f_4}^2{f_6}^2} \text{ (mod16)} \tag{4.54}$$

$$\equiv 12\frac{{f_3}^2}{{f_1}^2}{f_6}^2 \text{ (mod16)} \tag{4.55}$$

Applying (40) in (103) and equating coefficients of q^{2n+1} we get

$$\sum_{n=0}^{\infty} A_9(24n+20)q^n \equiv 24\frac{f_2{f_3}^4f_4f_{12}}{{f_1}^4f_{12}} \text{ (mod16)} \tag{4.56}$$

$$\equiv 8\frac{f_6f_4f_{12}}{f_2} \text{ (mod16)} \tag{4.57}$$

Equating coefficients of q^{2n+1} from both side of (105) we get

$$\overline{A_9}(48n+44) \equiv 0 \text{ (mod 16)} \tag{4.58}$$

Equating coefficients of q^{3n+1} from both side of (79) we get

$$\sum_{n=0}^{\infty} A_9(6n)q^n = \frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)\left(qa^2(q)+\frac{a^2(q)}{w^3(q)}+q\left(a(q)+3\frac{{f_3}^3}{f_1}\frac{1}{w(q)}\right)^2\right) \tag{4.59}$$

$$= q\frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)\left(a^2(q)\left(1+\frac{c^3(q)}{c^3(q^2)}\right)+\left(a(q)+\frac{c^2(q)}{c(q^2)}\right)^2\right) \tag{4.60}$$

It can be seen that

$$\frac{c^2(q)}{c(q^2)} = \frac{c^3(q)}{c^3(q^2)}.y \tag{4.61}$$

Taking x and y as earlier we get

$$\sum_{n=0}^{\infty} A_9(6n)q^n = q\frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)\left((x+y)^2\frac{3x}{y}+\left(x+y+y\left(\frac{3x}{y}-1\right)\right)^2\right) \tag{4.62}$$

$$= q\frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)\frac{x}{y}(3x^2+22xy+3y^2) \tag{4.63}$$

Taking congruence modulo 16 from both side of (111)

$$\equiv 3q\frac{{f_3}^3}{{f_1}^8{f_2}^2}\psi^3(q^3)\frac{x}{y}(x+y)^2 \pmod{16} \tag{4.64}$$

Applying (88) in (112) we get

$$\equiv q\frac{{f_6}^6}{{f_1}^8{f_2}^2}\left(1+\frac{c^3(q)}{c^3(q^2)}\right)a^2(q) \pmod{16} \tag{4.65}$$

Applying (86) we get

$$\equiv q\frac{{f_6}^6}{{f_1}^8{f_2}^2}\left(1+\frac{1}{v^3}\right)a^2(q) \pmod{16} \tag{4.66}$$

Applying (31) in (114) we get

$$\equiv \frac{{f_3}^4{f_2}^6}{{f_1}^{12}{f_6}^2}a^2(q) \pmod{16} \tag{4.67}$$

Applying (27) in (115) we get

$$\sum_{n=0}^{\infty} A_9(6n)q^n \equiv \frac{{f_3}^4{f_2}^6}{{f_1}^{12}{f_6}^2}(\phi^2(q)\phi^2(q^3)+8q\phi(q)\phi(q^3)\psi(q^2)\psi(q^6)) \pmod{16} \tag{4.68}$$

$$\equiv \frac{{f_2}^{16}{f_6}^8}{{f_1}^{16}{f_4}^4{f_{12}}^4}+8q\frac{{f_3}^4{f_2}^5{f_4}^2{f_{12}}^2}{{f_1}^{12}{f_6}^3} \pmod{16} \tag{4.69}$$

$$\equiv \frac{{f_4}^4{f_6}^8}{{f_2}^8 f_{12}^4}+8q\frac{{f_6}^3{f_4}^2}{f_2} \pmod{16} \tag{4.70}$$

$$\equiv \frac{{f_4}^4{f_6}^8}{{f_2}^8 f_{12}^4}+8q{f_6}^3{f_2}^3 \pmod{16} \tag{4.71}$$

Equating coefficients of q^{2n+1} from both side of (119) we get

$$\sum_{n=0}^{\infty} A_9(12n+6)q^n \equiv 8{f_1}^3{f_3}^3 \tag{4.72}$$

Applying (46) we get

$$\equiv 8{f_3}^3(f_3+q{f_9}^3) \tag{4.73}$$

Equating coefficients of q^{3n+2} from both side of (122) we get

$$\overline{A_9}(36n+30) \equiv 0 \pmod{16} \tag{4.74}$$

Also equating coefficients of q^{3n+1} from both side of (122) we get

$$\sum_{n=0}^{\infty} A_9(3(12n+6))q^n \equiv 8{f_1}^3{f_3}^3 (\text{mod}16) \tag{4.75}$$

From (121),(124) and (123) we get

$$\overline{A_9}(3^k(36n+30)) \equiv 0 \ (\text{mod}16), k \geq 0 \tag{4.76}$$

Equating coefficients of q^{3n} from both side of (122) we get

$$\sum_{n=0}^{\infty} A_9(36n+6)q^n \equiv 8{f_3}^4 (\text{mod}16) \tag{4.77}$$

$$\equiv 8{f_6}^2(\text{mod}16) \tag{4.78}$$

Equating coefficients of q^{6n+r} from both side of (127) we get

$$\overline{A_9}(216n+36r+6) \equiv 0 \ (\text{mod}16), r = 1(1)5 \tag{4.79}$$

From (128) ,(121) and (123) we get

$$\overline{A_9}(2.3^{k+1}(36n+6r+1)) \equiv 0 \ (\text{mod}16), k \geq 0, r = 1(1)5 \tag{4.80}$$

Equating coefficients of q^{6n} from both side of (122) we get

$$\sum_{n=0}^{\infty} A_9(216n+6)q^n \equiv 8{f_1}^2(\text{mod}16) \tag{4.81}$$

Hence proved all the theorems.

□

5. Conclusion and Future Directions

This paper establishes new infinite families of congruences for Andrews' singular over partitions $C_{12,1}$(n) and $C_{12,5}$(n), as well as enhanced congruences modulo 16 for 9-regular overpartitions A_9(n). The methodology relies heavily on Ramanujan's theta functions, cubic theta functions, and dissection formulas—techniques that have proven powerful in partition theory and will likely continue to yield fruitful results in related areas.

6. Future Research Directions

1) Investigating congruences for $C_{k,i}$ (n) with larger values of k and i
2) Exploring connections to l-regular overpartitions for larger values of l
3) Searching for congruence properties modulo higher prime powers
4)Applying recent developments in modular forms and Hecke operators to singular over partitions
5) Establishing algebraic relationships between different singular over partition families
6) Investigating connections to vector partitions and colored partitions
These results contribute to the systematic understanding of arithmetic properties of partition functions, a topic central to combinatorics and number theory.

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