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Approximation Degree of Bivariate Riemann-Liouville type fractional Stancu-Kantorovich operators

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ABSTRACT: In this article, we develop bivariate extension of the operators introduced by Sehrawat and Kajla [37]. The convergence behaviour of the proposed operators is analyzed using various moduli of continuity, and Voronovskaja-type asymptotic theorems. The approximation behaviour is further investigated in Lipschitz-type spaces. In addition, the rate of convergence of the operators is illustrated through graphical representations, which are generated using the Maple computational software.

Key Words: Kantorovich operators, modulus of continuity, Cauchy-Schwarz inequality.

Contents

1. Introduction

The classical Bernstein polynomials introduced by Bernstein [15] form the foundation of constructive approximation theory. Over the time, several extensions-such as the Kantorovich, Durrmeyer, Chlodowsky and Szász-Mirakyan operators were developed to enhance their flexibility and applicability. In 1930, Kantorovich [25] introduced the integral modification of Bernstein polynomials for the class of Lebesgue integrable functions on $[0,1]$. Stancu [40] introduced two dimensional Bernstein polynomials on the triangle. Gupta [21] defined Durrmeyer modification of classical Bernstein operators. Within this progression, Stancu [41] introduced the Bernstein operators with dual parameters $z, r \in \mathbb{N} \cup \{0\}$($\mathbb{N}$ is the set of natural numbers), and $j > 2zr$, as

$$B_j(\varphi;\varkappa) = \sum_{\xi=0}^{j-zr} p_{(j-zr,\xi)}(\varkappa) \sum_{m=0}^{z} p_{z,m}(\varkappa)\varphi\left(\frac{\xi+mr}{j}\right), \tag{1.1}$$

where

$$p_{z,m}(\varkappa) = \binom{z}{m}\varkappa^m(1-\varkappa)^{z-m}.$$

Abel et al. [1] introduced the Durrmeyer modification of (1.1), described as

$$\mathcal{S}_j(\varphi;\varkappa) = \sum_{\xi=0}^{j-zr} p_{(j-zr,\xi)}(\varkappa) \sum_{m=0}^{z} p_{z,m}(\varkappa)(j+1)\int_0^1 p_{j,\xi+mr}(\iota)\varphi(\iota)d\iota.$$

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Then, Kajla [23] presented the Kantorovich modification of (1.1), defined as follows

$$\mathcal{K}_{\mathsf{j}}(\varphi;\varkappa)=\sum_{\xi=0}^{\mathsf{j-zr}}\mathsf{p}_{(\mathsf{j-zr},\xi)}(\varkappa)\sum_{\mathsf{m}=0}^{\mathsf{z}}\mathsf{p}_{\mathsf{z,m}}(\varkappa)\int_0^1\varphi\left(\frac{\xi+\mathsf{mr}+\iota}{\mathsf{j}}\right)d\iota. \tag{1.2}$$

Agrawal et al. [5] constructed bivariate case of the operators (1.2) and examine convergence properties. For $\mathcal{I}^2=\mathcal{I}\times\mathcal{I}$, where $\mathcal{I}=[0,1]$ and let $C(\mathcal{I}^2)$ represent the family of all real valued continuous functions on $\mathcal{I}^2$, under the norm $||\varphi||_{C(\mathcal{I}^2)}=\sup\limits_{(\varkappa,\varrho)\in\mathcal{I}^2}|\varphi(\varkappa,\varrho)|$. The operators defined in [5] are described as follows

$$\begin{aligned}\mathcal{W}_{\mathsf{j_1,j_2}}^{\mathsf{z_1,r_1,z_2,r_2}}(\varphi;\varkappa,\varrho)=&\sum_{\xi_1=0}^{\mathsf{j_1-z_1r_1}}\mathsf{p}(\mathsf{j_1-z_1r_1},\xi_1)(\varkappa)\sum_{\mathsf{m_1}=0}^{\mathsf{z_1}}\mathsf{p}_{\mathsf{z_1,m_1}}(\varkappa)\sum_{\xi_2=0}^{\mathsf{j_2-z_2r_2}}\mathsf{p}(\mathsf{j_2-z_2r_2},\xi_2)(\varrho)\\&\times\sum_{\mathsf{m_2}=0}^{\mathsf{z_2}}\mathsf{p}_{\mathsf{z_2,m_2}}(\varrho)\times\int_0^1\int_0^1\varphi\left(\frac{\xi_1+\mathsf{m_1r_1}+\iota}{\mathsf{j_1}},\frac{\xi_2+\mathsf{m_2r_2}+\theta}{\mathsf{j_2}}\right)d\iota d\theta,\forall\,(\varkappa,\varrho)\in\mathcal{I}^2.\end{aligned} \tag{1.3}$$

Mahhmudov and Kara [27] defined the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators. Baytunç et al. [13] developed Riemann-Liouville type fractional Bernstein-Kantorovich operators parameterized by α and highlighted their approximation properties. Nasiruzzaman [31] studied the bivariate and GBS associated properties of the Szász-Mirakjan-Jakimovski-Leviatan-Kantorovich operators. Berwal et al. [16] constructed a new sequence of Riemann-Liouville type fractional α-Bernstein-Kantorovich operators and studied their convergence behaviour. Kursun [26] introduced a new class of sampling Kantorovich-type operators defined via fractional-type integrals and studied approximation properties of newly constructed operators and derive the convergence rate through modulus of continuity. Aslan [11] examined the approximation characteristics of Riemann-Liouville type fractional Bernstein-Stancu-Kantorovich operators of order α. Sehrawat and Kajla [37] constructed Riemann-Liouville type fractional Stancu-Kantorovich operators of order $\zeta>0$, which are defined as

$$\mathcal{G}_{\mathsf{j}}^{\zeta}(\varphi;\varkappa)=\Gamma(\zeta+1)\sum_{\xi=0}^{\mathsf{j-zr}}\mathsf{p}_{(\mathsf{j-zr},\xi)}(\varkappa)\sum_{\mathsf{m}=0}^{\mathsf{z}}\mathsf{p}_{\mathsf{z,m}}(\varkappa)\int_0^1\frac{(1-\iota)^{\zeta-1}}{\Gamma(\zeta)}\varphi\left(\frac{\xi+\mathsf{mr}+\iota}{\mathsf{j}}\right)d\iota, \tag{1.4}$$

where $\mathsf{p}_{\mathsf{z,m}}(\varkappa)$ is defined above.

For more information about bivariate, Kantorovich and fractional type operators, readers are referred to prior study [2,3,6,7,8,9,14,17,19,20,22,24,28,29,30,32,33,34,35,36,38,39].

2. Construction of operators

In this part, we develop the bivariate extension of the operators described in (1.4) and examine the convergence rate. Let $C^2(\mathcal{I}^2)$ represent the set of all functions $\varphi\in C(\mathcal{I}^2)$ provided that $\frac{\partial^{\mathsf{i}}\varphi}{\partial\varkappa^{\mathsf{i}}},\frac{\partial^{\mathsf{i}}\varphi}{\partial\varrho^{\mathsf{i}}}$ for $\mathsf{i}=1,2$ belong to $C(\mathcal{I}^2)$. The norm on the space $C^2(\mathcal{I}^2)$ is defined as

$$||\varphi||_{C^2(\mathcal{I}^2)}=||\varphi||_{C(\mathcal{I}^2)}+\sum_{\mathsf{i}=1}^{2}\left(\left\|\frac{\partial^{\mathsf{i}}\varphi}{\partial\varkappa^{\mathsf{i}}}\right\|_{C(\mathcal{I}^2)}+\left\|\frac{\partial^{\mathsf{i}}\varphi}{\partial\varrho^{\mathsf{i}}}\right\|_{C(\mathcal{I}^2)}\right).$$

The Peetre's $K-$functional of $\varphi\in C(\mathcal{I}^2)$ is described as

$$\mathcal{K}(\varphi;\delta)=\inf_{\mathsf{g}\in C^2(\mathcal{I}^2)}\{||\varphi-\mathsf{g}||_{C(\mathcal{I}^2)}+\delta||\mathsf{g}||_{C^2(\mathcal{I}^2)},\delta>0\}.$$

It is also known that the following inequality

$$\mathcal{K}(\varphi;\delta)\leq M_1\{\overline{\omega_2}(\varphi;\sqrt{\delta})+\min(1,\delta)||\varphi||_{C(\mathcal{I}^2)}\}, \tag{2.1}$$

applicable $\forall\ \delta > 0$ ([18], page 192). M_1 is a constant that does not depend on δ, φ, or $\overline{\omega_2}(\varphi; \sqrt{\delta})$.
For $\varphi : C(\mathcal{I}^2) \to C(\mathcal{I}^2)$ and $\zeta_1, \zeta_2 > 0$, we describe the bivariate form of the operators (1.4) as follows

$$\begin{aligned}\mathcal{Q}^{\zeta_1,\zeta_2}_{\mathsf{j}_1,\mathsf{j}_2}(\varphi; \varkappa, \varrho) &= \Gamma(\zeta_1+1)\Gamma(\zeta_2+1) \sum_{\xi_1=0}^{\mathsf{j}_1-\mathsf{z}_1\mathsf{r}_1} \mathsf{p}(\mathsf{j}_1 - \mathsf{z}_1\mathsf{r}_1, \xi_1)(\varkappa) \sum_{\mathsf{m}_1=0}^{\mathsf{z}_1} \mathsf{p}_{\mathsf{z}_1,\mathsf{m}_1}(\varkappa) \\ &\times \sum_{\xi_2=0}^{\mathsf{j}_2-\mathsf{z}_2\mathsf{r}_2} \mathsf{p}(\mathsf{j}_2 - \mathsf{z}_2\mathsf{r}_2, \xi_2)(\varrho) \sum_{\mathsf{m}_2=0}^{\mathsf{z}_2} \mathsf{p}_{\mathsf{z}_2,\mathsf{m}_2}(\varrho) \\ &\times \int_0^1\int_0^1 \frac{(1-\iota)^{\zeta_1-1}}{\Gamma(\zeta_1)}\frac{(1-\theta)^{\zeta_2-1}}{\Gamma(\zeta_2)}\varphi\left(\frac{\xi_1+\mathsf{m}_1\mathsf{r}_1+\iota}{\mathsf{j}_1}, \frac{\xi_2+\mathsf{m}_2\mathsf{r}_2+\theta}{\mathsf{j}_2}\right)d\iota d\theta, \forall\ (\varkappa,\varrho) \in \mathcal{I}^2.\end{aligned} \tag{2.2}$$

When $\zeta_1 = \zeta_2 = 1$, we get back the bivariate Kantorovich Stancu operators [5]. This paper aims to examine the approximation properties of the bivariate operators (2.2), including the uniform convergence theorem, the rate of convergence regarding the modulus of continuity, the Voronovskaja-type asymptotic theorem, and the Grüss-Voronovskaja-type theorem.

Lemma 2.1 *([37]) For $\zeta > 0$ and the test functions $e_i(\mathsf{t}) = \mathsf{t}^i$, i=0-4, we have*

(i) $\mathcal{G}^{\zeta}_{\mathsf{j}}(e_0; \varkappa) = 1$;

(ii) $\mathcal{G}^{\zeta}_{\mathsf{j}}(e_1; \varkappa) = \varkappa + \dfrac{1}{\mathsf{j}(\zeta+1)}$;

(iii) $\mathcal{G}^{\zeta}_{\mathsf{j}}(e_2; \varkappa) = \dfrac{\varkappa^2}{\mathsf{j}^2}\Big(-\mathsf{j}+\mathsf{j}^2+\mathsf{zr}-\mathsf{zr}^2\Big) + \dfrac{\varkappa}{\mathsf{j}^2(\zeta+1)}\Big((-1+\mathsf{r})(\zeta+1)\mathsf{zr}+\mathsf{j}(\zeta+3)\Big) + \dfrac{2}{\mathsf{j}^2(\zeta+1)(\zeta+2)}$;

(iv)
$$\begin{aligned}\mathcal{G}^{\zeta}_{\mathsf{j}}(e_3; \varkappa) &= \frac{\varkappa^3}{\mathsf{j}^3}\Big(2\mathsf{zr}(-1+\mathsf{r}^2) - 3\mathsf{j}^2 + \mathsf{j}^3 + \mathsf{j}(2 - 3\mathsf{z}(-1+\mathsf{r})\mathsf{r})\Big) \\ &+ \frac{\varkappa^2}{\mathsf{j}^3(\zeta+1)}\Big(3(\mathsf{j}^2(\zeta+2) - \mathsf{z}(-1+\mathsf{r})\mathsf{r}(2+\mathsf{r}+\zeta+\mathsf{r}\zeta) + \mathsf{j}(-2-\zeta+\mathsf{z}(-1+\mathsf{r})\mathsf{r}(\zeta+1)))\Big) \\ &+ \frac{\varkappa}{\mathsf{j}^3(\zeta+1)(\zeta+2)}\Big(\mathsf{j}(14+6\zeta+\zeta^2) + \mathsf{z}(-1+\mathsf{r})\mathsf{r}(\zeta+2)(4+\mathsf{r}+\zeta+\mathsf{r}\zeta)\Big) \\ &+ \frac{6}{\mathsf{j}^3(\zeta+1)(\zeta+2)(\zeta+3)};\end{aligned}$$

(v)
$$\begin{aligned}\mathcal{G}^{\zeta}_{\mathsf{j}}(e_4; \varkappa) &= \frac{\varkappa^4}{\mathsf{j}^4}\Big(-6\mathsf{j}^3 + \mathsf{j}^4 + \mathsf{j}^2(11 - 6\mathsf{z}(-1+\mathsf{r})\mathsf{r}) + 3\mathsf{zr}\left(2 + \mathsf{z}(-1+\mathsf{r})^2\mathsf{r} - 2\mathsf{r}^3\right) \\ &+ 2\mathsf{j}(-3+\mathsf{zr}(-7+3\mathsf{r}+4\mathsf{r}^2))\Big) + \frac{\varkappa^3}{\mathsf{j}^4(\zeta+1)}\Big(2(\mathsf{j}^3(5+3\zeta) \\ &+ 3\mathsf{j}^2(-5-3\zeta+\mathsf{z}(-1+\mathsf{r})\mathsf{r}(\zeta+1)) - 2\mathsf{j}(-5-3\zeta+3\mathsf{z}(-1+\mathsf{r})\mathsf{r}(3+\mathsf{r} \\ &+ 2\zeta+\mathsf{r}\zeta)) - \mathsf{z}(-1+\mathsf{r})\mathsf{r}(3(-2+\mathsf{z})\mathsf{r}^2(\zeta+1) - 2(5+3\zeta) - \mathsf{r}(10+6\zeta+3\mathsf{z}(\zeta+1))))\Big) \\ &+ \frac{\varkappa^2}{\mathsf{j}^4(\zeta+1)(\zeta+2)}\Big(\mathsf{j}^2(50+33\zeta+7\zeta^2) + \mathsf{z}(-1+\mathsf{r})\mathsf{r}(-50-33\zeta-7\zeta^2 \\ &+ (-7+3\mathsf{z})\mathsf{r}^2(2+3\zeta+\zeta^2) - \mathsf{r}(\zeta+2)(19+7\zeta+3\mathsf{z}(\zeta+1))) + \mathsf{j}(-50-33\zeta-7\zeta^2 \\ &+ 2\mathsf{z}(-1+\mathsf{r})\mathsf{r}(\zeta+2)(11+5\zeta+2\mathsf{r}(\zeta+1)))\Big) + \frac{\varkappa}{\mathsf{j}^4(\zeta+1)(\zeta+2)(\zeta+3)}\Big(\mathsf{j}(90+43\zeta \\ &+ 10\zeta^{\zeta}+2^3) + \mathsf{z}(-1+\mathsf{r})\mathsf{r}(\zeta+3)(22+7\zeta+\zeta^2+\mathsf{r}^2(2+3\zeta+\zeta^2) \\ &+ \mathsf{r}(10+7\zeta+\zeta^2))\Big) + \frac{24}{\mathsf{j}^4(\zeta+1)(\zeta+2)(\zeta+3)(\zeta+4)}.\end{aligned}$$

Lemma 2.2 *Let $e_{cd}(\varkappa,\varrho)=\varkappa^c\varrho^d,(c,d)$ in $\mathbb{N}\times\mathbb{N}$, with $c+d\leq 4$ be the bivariate test functions. Then we have*

(i) $\mathcal{Q}^{\zeta_1,\zeta_2}_{j_1,j_2}(e_{00};\varkappa,\varrho)=1;$

(ii) $\mathcal{Q}^{\zeta_1,\zeta_2}_{j_1,j_2}(e_{10};\varkappa,\varrho)=\varkappa+\dfrac{1}{j_1(\zeta_1+1)};$

(iii) $\mathcal{Q}^{\zeta_1,\zeta_2}_{j_1,j_2}(e_{01};\varkappa,\varrho)=\varrho+\dfrac{1}{j_2(\zeta_2+1)};$

(iv)
$$\mathcal{Q}^{\zeta_1,\zeta_2}_{j_1,j_2}(e_{20};\varkappa,\varrho)=\frac{\varkappa^2}{j_1^2}\Big(-j_1+j_1^2+z_1r_1-z_1r_1^2\Big)+\frac{\varkappa}{j_1^2(\zeta_1+1)}\Big((-1+r_1)(\zeta_1+1)z_1r_1+j_1(\zeta_1+3)\Big)+\frac{2}{j_1^2(\zeta_1+1)(\zeta_1+2)};$$

(v)
$$\mathcal{Q}^{\zeta_1,\zeta_2}_{j_1,j_2}(e_{02};\varkappa,\varrho)=\frac{\varrho^2}{j_2^2}\Big(-j_2+j_2^2+z_2r_2-z_2r_2^2\Big)+\frac{\varrho}{j_2^2(\zeta_2+1)}\Big((-1+r_2)(\zeta_2+1)z_2r_2+j_2(\zeta_2+3)\Big)+\frac{2}{j_2^2(\zeta_2+1)(\zeta_2+2)};$$

(vi)
$$\begin{aligned}\mathcal{Q}^{\zeta_1,\zeta_2}_{j_1,j_2}(e_{30};\varkappa,\varrho)&=\frac{\varkappa^3}{j_1^3}\Big(-3j_1^2+j_1^3+j_1(2-3z_1(-1+r_1)r_1)+2z_1r_1(-1+r_1^2)\Big)\\&+\frac{6}{j_1^3(\zeta_1+1)(\zeta_1+2)(\zeta_1+3)}+\frac{\varkappa^2}{j_1^3(\zeta_1+1)}\Big(3(j_1^2(\zeta_1+2)-z_1(-1+r_1)r_1(2+r_1+\zeta_1\\&+r_1\zeta_1)+j_1(-2-\zeta_1+z_1(-1+r_1)r_1(\zeta_1+1)))\Big)\\&+\frac{\varkappa}{j_1^3(\zeta_1+1)(\zeta_1+2)}\Big(z_1(-1+r_1)r_1(\zeta_1+2)(4+r_1+\zeta_1+r_1\zeta_1)+j_1(14+6\zeta_1+\zeta_1^2)\Big);\end{aligned}$$

(vii)
$$\begin{aligned}\mathcal{Q}^{\zeta_1,\zeta_2}_{j_1,j_2}(e_{03};\varkappa,\varrho)&=\frac{\varrho^3}{j_2^3}\Big(-3j_2^2+j_2^3+j_2(2-3z_2(-1+r_2)r_2)+2z_2r_2(-1+r_2^2)\Big)\\&+\frac{6}{j_2^3(\zeta_2+1)(\zeta_2+2)(\zeta_2+3)}+\frac{\varrho^2}{j_2^3(\zeta_2+1)}\Big(3(j_2^2(\zeta_2+2)-z_2(-1+r_2)r_2(2+r_2+\zeta_2\\&+r_2\zeta_2)+j_2(-2-\zeta_2+z_2(-1+r_2)r_2(\zeta_2+1)))\Big)\\&+\frac{\varrho}{j_2^3(\zeta_2+1)(\zeta_2+2)}\Big(z_2(-1+r_2)r_2(\zeta_2+2)(4+r_2+\zeta_2+r_2\zeta_2)+j_2(14+6\zeta_2+\zeta_2^2)\Big);\end{aligned}$$

(viii)
$$\begin{aligned}\mathcal{Q}^{\zeta_1,\zeta_2}_{j_1,j_2}(e_{40};\varkappa,\varrho)&=\frac{\varkappa^4}{j_1^4}\Big(-6j_1^3+j_1^4+j_1^2(11-6z_1(-1+r_1)r_1)+3z_1r_1\left(2+z_1(-1+r_1)^2r_1-2r_1^3\right)\\&+2j_1(-3+z_1r_1(-7+3r_1+4r_1^2))\Big)+\frac{\varkappa^3}{j_1^4(\zeta_1+1)}\Big(2(j_1^3(5+3\zeta_1)+3j_1^2(-5-3\zeta_1\\&+z_1(-1+r_1)r_1(\zeta_1+1))-2j_1(-5-3\zeta_1+3z_1(-1+r_1)r_1(3+r_1+2\zeta_1+r_1\zeta_1))\\&-z_1(-1+r_1)r_1(3(-2+z_1)r_1^2(\zeta_1+1)-2(5+3\zeta_1)-r_1(10+6\zeta_1+3z_1(\zeta_1+1))))\Big)\\&+\frac{\varkappa^2}{j_1^4(\zeta_1+1)(\zeta_1+2)}\Big(j_1^2(50+33\zeta_1+7\zeta_1^2)+z_1(-1+r_1)r_1(-50-33\zeta_1-7\zeta_1^2\\&+(-7+3z_1)r_1^2(2+3\zeta_1+\zeta_1^2)-r_1(\zeta_1+2)(19+7\zeta_1+3z_1(\zeta_1+1)))\\&+j_1(-50-33\zeta_1-7\zeta_1^2+2z_1(-1+r_1)r_1(\zeta_1+2)(11+5\zeta_1+2r_1(\zeta_1+1)))\Big)\\&+\frac{\varkappa}{j_1^4(\zeta_1+1)(\zeta_2+2)(\zeta_3+3)}\Big(j_1(90+43\zeta_1+10\zeta_1^{\zeta_1}+2^3)\\&+z_1(-1+r_1)r_1(\zeta_1+3)(22+7\zeta_1+\zeta_1^2+r_1^2(2+3\zeta_1+\zeta_1^2)+r_1(10+7\zeta_1+\zeta_1^2))\Big)\\&+\frac{24}{j_1^4(\zeta_1+1)(\zeta_1+2)(\zeta_1+3)(\zeta_1+4)};\end{aligned}$$

(ix) $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(e_{04};\varkappa,\varrho)=\dfrac{\varrho^4}{j_2^4}\bigg(-6j_2^3+j_2^4+j_2^2(11-6z_2(-1+r_2)r_2)+3z_2r_2\left(2+z_2(-1+r_2)^2r_2-2r_2^3\right)$
$+2j_2(-3+z_2r_2(-7+3r_2+4r_2^2))\bigg)+\dfrac{\varrho^3}{j_2^4(\zeta_2+1)}\bigg(2(j_2^3(5+3\zeta_2)+3j_2^2(-5-3\zeta_2$
$+z_2(-1+r_2)r_2(\zeta_2+1))-2j_2(-5-3\zeta_2+3z_2(-1+r_2)r_2(3+r_2+2\zeta_2+r_2\zeta_2))$
$-z_2(-1+r_2)r_2(3(-2+z_2)r_2^2(\zeta_2+1)-2(5+3\zeta_2)-r_2(10+6\zeta_2+3z_2(\zeta_2+1))))\bigg)$
$+\dfrac{\varrho^2}{j_2^4(\zeta_2+1)(\zeta_2+2)}\bigg(j_2^2(50+33\zeta_2+7\zeta_2^2)+z_2(-1+r_2)r_2(-50-33\zeta_2-7\zeta_2^2$
$+(-7+3z_2)r_2^2(2+3\zeta_2+\zeta_2^2)-r_2(\zeta_2+2)(19+7\zeta_2+3z_2(\zeta_2+1)))$
$+j_2(-50-33\zeta_2-7\zeta_2^2+2z_2(-1+r_2)r_2(\zeta_1+2)(11+5\zeta_2+2r_2(\zeta_2+1)))\bigg)$
$+\dfrac{\varrho}{j_2^4(\zeta_2+1)(\zeta_2+2)(\zeta_2+3)}\bigg(j_2\left(90+43\zeta_2+10\zeta_2^{\zeta_2}+2^3\right)$
$+z_2(-1+r_2)r_2(\zeta_2+3)(22+7\zeta_2+\zeta_2^2+r_2^2(2+3\zeta_2+\zeta_2^2)+r_2(10+7\zeta_2+\zeta_2^2))\bigg)$
$+\dfrac{24}{j_2^4(\zeta_2+1)(\zeta_2+2)(\zeta_2+3)(\zeta_2+4)}.$

Lemma 2.3 *For $\varkappa,\varrho\in\mathcal{I}^2$ and using previous lemma as foundation, we have*

(i) $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\iota-\varkappa);\varkappa,\varrho)=\dfrac{1}{j_1(\zeta_1+1)}=\Lambda_{j_1,\zeta_1}(\varkappa);$

(ii) $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\theta-\varrho);\varkappa,\varrho)=\dfrac{1}{j_2(\zeta_2+1)}=\Lambda_{j_2,\zeta_2}(\varrho);$

(iii) $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\iota-\varkappa)(\theta-\varrho);\varkappa,\varrho)=\dfrac{1}{j_1j_2(\zeta_1+1)(\zeta_2+1)}=\Lambda_{j_1,j_2,\zeta_1,\zeta_2}(\varkappa,\varrho);$

(iv) $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\iota-\varkappa)^2;\varkappa,\varrho)=\dfrac{\varkappa^2}{j_1^2}\bigg(-j_1-z_1(-1+r_1)r_1\bigg)+\dfrac{\varkappa}{j_1^2(\zeta_1+1)}((j_1+z_1(-1+r_1)r_1)(\zeta_1+1)$
$+2j_1)+\dfrac{2}{j_1^2(\zeta_1+1)(\zeta_1+2)}=\lambda_{j_1,\zeta_1}(\varkappa);$

(v) $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\theta-\varrho)^2;\varkappa,\varrho)=\dfrac{\varrho^2}{j_2^2}\bigg(-j_2-z_2(-1+r_2)r_2\bigg)+\dfrac{\varrho}{j_2^2(\zeta_2+1)}((j_2+z_2(-1+r_2)r_2)(\zeta_2+1)$
$+2j_2)+\dfrac{2}{j_2^2(\zeta_2+1)(\zeta_2+2)}=\lambda_{j_2,\zeta_2}(\varrho);$

(vi) $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\iota-\varkappa)^3;\varkappa,\varrho)=\dfrac{\varkappa^3}{j_1^3}\bigg(2\left(j_1+z_1r_1\left(-1+r_1^2\right)\right)\bigg)+\dfrac{\varkappa^2}{j_1^3(\zeta_1+1)}\bigg(-3(j_1^2+j_1(\zeta_1+2)$
$+z_1(-1+r_1)r_1(2+r_1+\zeta_1+r_1\zeta_1))\bigg)+\dfrac{\varkappa}{j_1^3(\zeta_1+1)(\zeta_1+2)}\bigg((\zeta_1+2)(j_1(\zeta_1$
$+4)+z_1(-1+r_1)r_1(4+r_1+\zeta_1+r_1\zeta_1)\bigg)+\dfrac{6}{j_1^3(\zeta_1+1)(\zeta_1+2)(\zeta_1+3)};$

(vii) $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\theta-\varrho)^3;\varkappa,\varrho)=\dfrac{\varrho^3}{j_2^3}\bigg(2\left(j_2+z_2r_2\left(-1+r_2^2\right)\right)\bigg)+\dfrac{\varrho^2}{j_2^3(\zeta_2+1)}\bigg(-3(j_2^2+j_2(\zeta_2+2)$
$+z_2(-1+r_2)r_2(2+r_2+\zeta_2+r_2\zeta_2))\bigg)+\dfrac{\varrho}{j_2^3(\zeta_2+1)(\zeta_2+2)}\bigg((\zeta_2+2)(j_2(\zeta_2$
$+4)+z_2(-1+r_2)r_2(4+r_2+\zeta_1+r_2\zeta_2)\bigg)+\dfrac{6}{j_2^3(\zeta_2+1)(\zeta_2+2)(\zeta_2+3)};$

(viii) $$\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\iota-\varkappa)^4;\varkappa,\varrho) = \frac{\varkappa^4}{j_1^4}\Big(3(j_1^2+2j_1(-1+z_1(-1+r_1)r_1)+z_1r_1(2+z_1(-1+r_1)^2r_1-2r_1^3))\Big) + \frac{\varkappa^3}{j_1^4(\zeta_1+1)}\Big(2(2j_1^3-3j_1^2(\zeta_1+1)+j_1(10+6\zeta_1-6z_1(-1+r_1)r_1(\zeta_1+1)) - z_1(-1+r_1)r_1(3(-2+z_1)r_1^2(\zeta_1+1)-2(5+3\zeta_1)-r_1(10+6\zeta_1 + 3z_1(\zeta_1+1)))\Big) + \frac{\varkappa^2}{j_1^4(\zeta_1+1)(\zeta_1+2)}\Big(3j_1^2(2+3\zeta_1+\zeta_1^2)+j_1(-50-33\zeta_1 - 7\zeta_1^2+6z_1(-1+r_1)r_1(2+3\zeta_1+\zeta_1^2))+z_1(-1+r_1)r_1(-50-33\zeta_1-7\zeta_1^2 + (-7+3z_1)r_1^2\left(2+3\zeta_1+\zeta_1^2\right)-r_1(\zeta_1+2)(19+7\zeta_1+3z_1(\zeta_1+1)))\Big) + \frac{\varkappa}{j_1^4(\zeta_1+1)(\zeta_1+2)}\Big(j_1(22+7\zeta_1+\zeta_1^2)+z_1(-1+r_1)r_1(22+7\zeta_1+\zeta_1^2+r_1^2(2 + 3\zeta_1+\zeta_1^2)+r_1(10+7\zeta_1+\zeta_1^2))\Big) + \frac{24}{j_1^4(\zeta_1+1)(\zeta_1 a+2)(\zeta_1+3)(\zeta_1+4)};$$

(ix) $$\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\theta-\varrho)^4;\varkappa,\varrho) = \frac{\varrho^4}{j_2^4}\Big(3(j_2^2+2j_2(-1+z_2(-1+r_2)r_2)+z_2r_2(2+z_2(-1+r_2)^2r_2-2r_2^3))\Big) + \frac{\varrho^3}{j_2^4(\zeta_2+1)}\Big(2(2j_2^3-3j_2^2(\zeta_2+1)+j_2(10+6\zeta_2-6z_2(-1+r_2)r_2(\zeta_2+1)) - z_2(-1+r_2)r_2(3(-2+z_2)r_2^2(\zeta_2+1)-2(5+3\zeta_2)-r_2(10+6\zeta_2 + 3z_2(\zeta_2+1)))\Big) + \frac{\varrho^2}{j_2^4(\zeta_2+1)(\zeta_2+2)}\Big(3j_2^2(2+3\zeta_2+\zeta_2^2)+j_2(-50-33\zeta_2 - 7\zeta_2^2+6z_2(-1+r_2)r_2(2+3\zeta_2+\zeta_2^2))+z_2(-1+r_2)r_2(-50-33\zeta_2-7\zeta_2^2 + (-7+3z_2)r_2^2\left(2+3\zeta_2+\zeta_2^2\right)-r_2(\zeta_2+2)(19+7\zeta_2+3z_2(\zeta_2+1)))\Big) + \frac{\varrho}{j_2^4(\zeta_1+1)(\zeta_2+2)}\Big(j_2(22+7\zeta_2+\zeta_2^2)+z_2(-1+r_2)r_2(22+7\zeta_2+\zeta_2^2 + r_2^2(2+3\zeta_2+\zeta_2^2)+r_2(10+7\zeta_2+\zeta_2^2))\Big) + \frac{24}{j_2^4(\zeta_2+1)(\zeta_2+2)(\zeta_2+3)(\zeta_2+4)}.$$

Lemma 2.4 *We have the following results*

(1) $$\lim_{j_1\to\infty} j_1\ \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\iota-\varkappa);\varkappa,\varrho) = \frac{1}{(\zeta_1+1)};$$

(2) $$\lim_{j_2\to\infty} j_2\ \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\theta-\varrho);\varkappa,\varrho) = \frac{1}{(\zeta_2+1)};$$

(3) $$\lim_{j\to\infty} j\ \mathcal{Q}_{j,j}^{\zeta_1,\zeta_2}((\iota-\varkappa)(\theta-\varrho);\varkappa,\varrho) = 0;$$

(4) $$\lim_{j_1\to\infty} j_1\ \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\iota-\varkappa)^2;\varkappa,\varrho) = -\varkappa^2+\varkappa(1+\frac{2}{\zeta_1+1});$$

(5) $$\lim_{j_2\to\infty} j_2\ \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}((\theta-\varrho)^2;\varkappa,\varrho) = -\varrho^2+\varrho(1+\frac{2}{\zeta_2+1}).$$

3. Approximation properties of the operators $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}$

Theorem 3.1 *For any $\varphi \in C(\mathcal{I}^2)$, we have*

$$\lim_{j_1,j_2\to\infty} ||\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(\varphi) - \varphi|| = 0.$$

Proof: Since

$$\lim_{j_1,j_2\to\infty} \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(e_{ik}) = e_{ik}, (i,k) \in \{(0,0),(0,1),(1,0)\}$$

and

$$\lim_{j_1,j_2\to\infty} \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(e_{20}+e_{02}) = e_{20}+e_{02},$$

uniformly on $\mathcal{I}^2$. Applying Theorem (2.1) of [12] gives the result. □

Theorem 3.2 *For $\varphi \in C(\mathcal{I}^2)$, the operators $\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(\varphi)$ converges to φ as $j_1, j_2 \to \infty$, uniformly on $\mathcal{I}^2$.*

Proof: The proof is straightforward when Lemma (2.2) and Theorem 3.1 are applied. □

In the case of two variables and for $\varphi \in C(\mathcal{I}^2)$, the complete modulus of continuity is given as
$\omega(\varphi;\delta) = \sup\left\{|\varphi(\iota,\theta) - \varphi(\varkappa,\varrho)| : (\iota,\theta),(\varkappa,\varrho) \in \mathcal{I}^2 \text{ and } \sqrt{(\iota-\varkappa)^2+(\theta-\varrho)^2} \le \delta\right\}$.
For given $\varkappa$ and ϱ, the partial moduli of continuity are defined as
$\omega^{(1)}(\varphi;\delta) = \sup\left\{|\varphi(\varkappa_1,\varrho) - \varphi(\varkappa_2,\varrho)| : \varrho \in \mathcal{I} \text{ and } |\varkappa_1-\varkappa_2| \le \delta\right\}$,
$\omega^{(2)}(\varphi;\delta) = \sup\left\{|\varphi(\varkappa,\varrho_1) - \varphi(\varkappa,\varrho_2)| : \varkappa \in \mathcal{I} \text{ and } |\varrho_1-\varrho_2| \le \delta\right\}$.
The details of the modulus of continuity for the bivariate case can be found in [10].

Theorem 3.3 *Suppose $\varphi \in C(\mathcal{I}^2)$, then $\forall$ $(\varkappa,\varrho) \in \mathcal{I}^2$, we obtain*

$$||\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho) - \varphi(\varkappa,\varrho)|| \le 4\omega\left(\varphi; \sqrt{\lambda_{j_1,\zeta_1}(\varkappa)}, \sqrt{\lambda_{j_2,\zeta_2}(\varrho)}\right).$$

Proof: Using complete modulus of continuity of $\varphi(\varkappa,\varrho)$, we may express

$$\begin{aligned}
|\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho) - \varphi(\varkappa,\varrho)| &\le \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(|\varphi(\iota,\theta) - \varphi(\varkappa,\varrho)|;\varkappa,\varrho) \\
&\le \omega\Big(\varphi; \sqrt{\lambda_{j_1,\zeta_1}(\varkappa)}, \sqrt{\lambda_{j_1,\zeta_1}(\varkappa)}\Big)\Big(\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(1;\varkappa,\varrho) + \frac{1}{\sqrt{\lambda_{j_1,\zeta_1}(\varkappa)}}\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(|\iota-\varkappa|;\varkappa,\varrho) \\
&+ \frac{1}{\sqrt{\lambda_{j_2,\zeta_2}(\varrho)}}\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(|\theta-\varrho|;\varkappa,\varrho) + \frac{1}{\sqrt{\lambda_{j_1,\zeta_1}(\varkappa)}\sqrt{\lambda_{j_2,\zeta_2}(\varrho)}}\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(|\iota-\varkappa||\theta-\varrho|;\varkappa,\varrho)\Big).
\end{aligned}$$

Utilizing the Cauchy-Schwarz inequality and Lemma (2.2), the desired outcome is achieved. □

We now examine the approximation rate for the operators (2.2) utilizing the Lipschitz class.

For $0 < \mathfrak{r}_1, \mathfrak{r}_2 \le 1$, the Lipschitz class denoted by $\text{Lip}_M(\mathfrak{r}_1,\mathfrak{r}_2)$ [4] for two variables is defined as

$$\text{Lip}_M(\mathfrak{r}_1,\mathfrak{r}_2) = \{\varphi : C(\mathcal{I}^2) : |\varphi(\iota,\theta) - \varphi(\varkappa,\varrho)| \le M|\iota-\varkappa|^{\mathfrak{r}_1}|\theta-\varrho|^{\mathfrak{r}_2}\},$$

where $M > 0$.

Theorem 3.4 *If $\varphi \in \text{Lip}_M(\mathfrak{r}_1,\mathfrak{r}_2)$, then we have the following inequality*

$$\left|\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho) - \varphi(\varkappa,\varrho)\right| \le M\left[\lambda_{j_1,\zeta_1}(\varkappa)\right]^{\frac{\mathfrak{r}_1}{2}}\left[\lambda_{j_2,\zeta_2}(\varrho)\right]^{\frac{\mathfrak{r}_2}{2}}$$

hold for all $(\varkappa,\varrho) \in \mathcal{I}^2$, where $\lambda_{j_1,\zeta_1}(\varkappa) = ||\mathcal{Q}_{j_1}^{\zeta_1}((\iota-\cdot)^2;\cdot)||$ and $\lambda_{j_2,\zeta_2}(\varrho) = ||\mathcal{Q}_{j_2}^{\zeta_2}((\theta-\cdot)^2;\cdot)||$.

Proof: Let $\varphi \in \mathrm{Lip}_M(\mathfrak{r}_1, \mathfrak{r}_2)$, then we have

$$\begin{aligned}\left|\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right| &\leq \mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}\left(\left|\varphi(\iota,\theta)-\varphi(\varkappa,\varrho)\right|;\varkappa,\varrho\right)\\ &\leq M\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}\left(|\iota-\varkappa|^{\mathfrak{r}_1}|\theta-\varrho|^{\mathfrak{r}_2};\varkappa,\varrho\right)\\ &\leq M\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}\left(|\iota-\varkappa|^{\mathfrak{r}_1};\varkappa,\varrho\right)\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}\left(|\theta-\varrho|^{\mathfrak{r}_2};\varkappa,\varrho\right).\end{aligned}$$

Using Hölder's inequality with $p_1 = \frac{2}{\mathfrak{r}_1}, v_1 = \frac{2}{2-\mathfrak{r}_1}$ and $p_2 = \frac{2}{\mathfrak{r}_2}, v_2 = \frac{2}{2-\mathfrak{r}_2}$, we get

$$\begin{aligned}\left|\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right| &\leq M\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}\left((\iota-\varkappa)^2;\varkappa,\varrho\right)^{\frac{\mathfrak{r}_1}{2}}\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}\left((\theta-\varrho)^2;\varkappa,\varrho\right)^{\frac{\mathfrak{r}_2}{2}}\\ &= M\left[\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)\right]^{\frac{\mathfrak{r}_1}{2}}\left[\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)\right]^{\frac{\mathfrak{r}_2}{2}}.\end{aligned}$$

This completes the proof. □

Theorem 3.5 *Let $\varphi \in C^1(\mathcal{I}^2)$. Then, we have*

$$||\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varphi)-\varphi|| \leq ||\varphi_\varkappa||\sqrt{\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)}+||\varphi_\varrho||\sqrt{\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)},$$

where $\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)$ and $\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)$ are given in Theorem 3.4.

Proof: According to hypothesis, we can express

$$\varphi(\iota,\theta)-\varphi(\varkappa,\varrho)=\int_\varkappa^\iota \varphi_{\mathsf{w}}(\mathsf{w},\theta)d\mathsf{w}+\int_\varrho^\theta \varphi_{\mathsf{u}}(\varkappa,\mathsf{u})d\mathsf{u}.$$

Now, applying $\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\cdot;\varkappa,\varrho)$ on both sides, we have

$$\begin{aligned}\left|\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right| &\leq \mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}\left(\int_\varkappa^\iota \varphi_{\mathsf{w}}(\mathsf{w},\theta)d\mathsf{w};\varkappa,\varrho\right)\\ &\quad+\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}\left(\int_\varrho^\theta \varphi_{\mathsf{u}}(\varkappa,\mathsf{u})d\mathsf{u};\varkappa,\varrho\right).\end{aligned}$$

Since

$$\left|\int_\varkappa^\iota \varphi_{\mathsf{w}}(\mathsf{w},\theta)d\mathsf{w}\right| \leq ||\varphi_\varkappa||\;|\iota-\varkappa| \text{ and } \left|\int_\varrho^\theta \varphi_{\mathsf{u}}(\varkappa,\mathsf{u})d\mathsf{u}\right| \leq ||\varphi_\varrho||\;|\theta-\varrho|,$$

we have

$$\left|\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right| \leq ||\varphi_\varrho||\mathcal{Q}_{\mathrm{j}_2}^{\zeta_2}(|\theta-\varrho|;\varrho)+||\varphi_\varkappa||\mathcal{Q}_{\mathrm{j}_1}^{\zeta_1}(|\iota-\varkappa|;\varkappa).$$

Now, utilizing Lemma (2.3) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right| &\leq ||\varphi_\varkappa||\left(\mathcal{Q}_{\mathrm{j}_1}^{\zeta_1}((\iota-\varkappa)^2;\varkappa)\right)^{\frac{1}{2}}\left(\mathcal{Q}_{\mathrm{j}_1}^{\zeta_1}(e_0;\varkappa)\right)^{\frac{1}{2}}\\ &\quad+||\varphi_\varrho||\left(\mathcal{Q}_{\mathrm{j}_2}^{\zeta_2}((\theta-\varrho)^2;\varrho)\right)^{\frac{1}{2}}\left(\mathcal{Q}_{\mathrm{j}_2}^{\zeta_2}(e_0;\varrho)\right)^{\frac{1}{2}}\\ &\leq ||\varphi_\varkappa||\sqrt{\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)}+||\varphi_\varrho||\sqrt{\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)}, \quad \forall(\varkappa,\varrho)\in\mathcal{I}^2.\end{aligned}$$

This concludes the proof of the theorem. □

Theorem 3.6 *Suppose $\varphi \in C(\mathcal{I}^2)$. Then we have the inequalities*

$$|\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)| \le 2\Big[\omega^{(1)}(\varphi;\sqrt{\lambda_{j_1,\zeta_1}(\varkappa)})+\omega^{(2)}(\varphi;\sqrt{\lambda_{j_2,\zeta_2}(\varrho)})\Big].$$

Proof: Using the Cauchy-Schwarz inequality and partial moduli of continuity, we may write

$$\begin{aligned}
|\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)| &\le \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(|\varphi(\iota,\theta)-\varphi(\varkappa,\varrho)|;\varkappa,\varrho)\\
&\le \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(|\varphi(\iota,\theta)-\varphi(\iota,\varrho)|;\varkappa,\varrho)+\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}(|\varphi(\iota,\varrho)-\varphi(\varkappa,\varrho)|;\varkappa,\varrho)\\
&\le \mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}\left(\omega^{(2)}(\varphi;|\theta-\varrho|);\varkappa,\varrho\right)+\mathcal{Q}_{j_1,j_2}^{\zeta_1,\zeta_2}\left(\omega^{(1)}(\varphi;|\iota-\varkappa|);\varkappa,\varrho\right)\\
&\le \omega^{(2)}(\varphi;\delta_{j_2})\left[1+\frac{1}{\delta_{j_2}}\mathcal{Q}_{j_2}^{\zeta_2}(|\theta-\varrho|;\varrho)\right]\\
&\quad+\omega^{(1)}(\varphi;\delta_{j_1})\left[1+\frac{1}{\delta_{j_1}}\mathcal{Q}_{j_1}^{\zeta_1}(|\iota-\varkappa|;\varkappa)\right]\\
&\le \omega^{(2)}(\varphi;\delta_{j_2})\left[1+\frac{1}{\delta_{j_2}}\left(\mathcal{Q}_{j_2}^{\zeta_2}((\theta-\varrho)^2;\varrho)\right)^{1/2}\right]\\
&\quad+\omega^{(1)}(\varphi;\delta_{j_1})\left[1+\frac{1}{\delta_{j_1}}\left(\mathcal{Q}_{j_1}^{\zeta_1}((\iota-\varkappa)^2;\varkappa)\right)^{1/2}\right].
\end{aligned}$$

Taking $\delta_{j_2}=\sqrt{\lambda_{j_2,\zeta_2}(\varrho)}$ and $\delta_{j_1}=\sqrt{\lambda_{j_1,\zeta_1}(\varkappa)}$, we obtain the required result. □

4. Approximation properties and Voronovskaja type theorems

Theorem 4.1 (Voronovskaja type theorem). *Let $\varphi \in C^2(\mathcal{I}^2)$. Then, we have*

$$\begin{aligned}
\lim_{j\to\infty} j\left(\mathcal{Q}_{j,j}^{\zeta,\zeta}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right) &= \left(\frac{1}{\zeta_1+1}\right)\varphi_{\varkappa}(\varkappa,\varrho)+\left(\frac{1}{\zeta_2+1}\right)\varphi_{\varrho}(\varkappa,\varrho)\\
&+\left(\frac{-\varkappa^2+\varkappa(1+\frac{2}{\zeta_1+1})}{2}\right)\varphi_{\varkappa\varkappa}(\varkappa,\varrho)+\left(\frac{-\varrho^2+\varrho(1+\frac{2}{\zeta_2+1})}{2}\right)\varphi_{\varrho\varrho}(\varkappa,\varrho),
\end{aligned}$$

uniformly on $\mathcal{I}^2$.

Proof: Let $(\varkappa,\varrho)\in\mathcal{I}^2$ be arbitrary. According to Taylor's formula, we have

$$\begin{aligned}
\varphi(\iota,\theta) &= \varphi(\varkappa,\varrho)+\varphi_{\varkappa}(\varkappa,\varrho)(\iota-\varkappa)+\varphi_{\varrho}(\varkappa,\varrho)(\theta-\varrho)+\frac{1}{2}\{\varphi_{\varkappa\varkappa}(\varkappa,\varrho)(\iota-\varkappa)^2+2\varphi_{\varkappa\varrho}(\varkappa,\varrho)(\iota-\varkappa)(\theta-\varrho)\\
&\quad+\varphi_{\varrho\varrho}(\varkappa,\varrho)(\theta-\varrho)^2\}+\Omega(\iota,\theta;\varkappa,\varrho)\sqrt{(\iota-\varkappa)^4+(\theta-\varrho)^4},
\end{aligned} \tag{4.1}$$

for $(\iota,\theta)\in\mathcal{I}^2$, where $\Omega(\iota,\theta;\varkappa,\varrho)\in C(\mathcal{I}^2)$ and $\Omega(\iota,\theta;\varkappa,\varrho)\to 0$ as $(\iota,\theta)\to(\varkappa,\varrho)$.
Applying $\mathcal{Q}_{j,j}^{\zeta,\zeta}(\varphi;\varkappa,\varrho)$ on both sides of (4.1), we get

$$\begin{aligned}
\mathcal{Q}_{j,j}^{\zeta,\zeta}(\varphi;\varkappa,\varrho) &= \varphi(\varkappa,\varrho)+\varphi_{\varkappa}(\varkappa,\varrho)\mathcal{Q}_{j}^{\zeta}((\iota-\varkappa);\varkappa)+\varphi_{\varrho}(\varkappa,\varrho)\mathcal{Q}_{j}^{\zeta}((\theta-\varrho);\varkappa)\\
&\quad+\frac{1}{2}\{\varphi_{\varkappa\varkappa}(\varkappa,\varrho)\mathcal{Q}_{j}^{\zeta}((\iota-\varkappa)^2;\varkappa)+\varphi_{\varrho\varrho}(\varkappa,\varrho)\mathcal{Q}_{j}^{\zeta}((\theta-\varrho)^2;\varrho)\\
&\quad+2\varphi_{\varkappa\varrho}(\varkappa,\varrho)\mathcal{Q}_{j,j}^{\zeta,\zeta}((\iota-\varkappa)(\theta-\varrho);\varkappa,\varrho)\}\\
&\quad+\mathcal{Q}_{j,j}^{\zeta,\zeta}\left(\Omega(\iota,\theta;\varkappa,\varrho)\sqrt{(\iota-\varkappa)^4+(\theta-\varrho)^4};\varkappa,\varrho\right).
\end{aligned} \tag{4.2}$$

Using Holder's inequality, we obtain

$$\begin{aligned}
&\left|\mathcal{Q}_{j,j}^{\zeta,\zeta}\left(\Omega(\iota,\theta;\varkappa,\varrho)\sqrt{(\iota-\varkappa)^4+(\theta-\varrho)^4};\varkappa,\varrho\right)\right|\\
&\quad\le\left\{\mathcal{Q}_{j,j}^{\zeta,\zeta}(\Omega^2(\iota,\theta;\varkappa,\varrho);\varkappa,\varrho)\right\}^{1/2}\left\{\mathcal{Q}_{j,j}^{\zeta,\zeta}((\iota-\varkappa)^4+(\theta-\varrho)^4;\varkappa,\varrho)\right\}^{1/2}\\
&\quad\le\left\{\mathcal{Q}_{j,j}^{\zeta,\zeta}(\Omega^2(\iota,\theta;\varkappa,\varrho);\varkappa,\varrho)\right\}^{1/2}\left\{\mathcal{Q}_{j}^{\zeta}((\iota-\varkappa)^4;\varkappa)+\mathcal{Q}_{j}^{\zeta}((\theta-\varrho)^4;\varrho)\right\}^{1/2}.
\end{aligned}$$

In view of Theorem 3.1, $\mathcal{Q}_{\mathrm{j,j}}^{\zeta,\zeta}(\Omega^2(\iota,\theta;\varkappa,\varrho);\varkappa,\varrho)\to 0$ as $\mathrm{j}\to\infty$ uniformly in $\mathcal{I}^2$ and $\mathcal{Q}_{\mathrm{j}}^{\zeta}((\iota-\varkappa)^4;\varkappa)=O\left(\frac{1}{\mathrm{j}^2}\right)$, uniformly in $\mathcal{I}^2$, and $\mathcal{Q}_{\mathrm{j}}^{\zeta}((\theta-\varrho)^4;\varrho)=O\left(\frac{1}{\mathrm{j}^2}\right)$, uniformly in $\mathcal{I}^2$, we get

$$\lim_{\mathrm{j}\to\infty}\mathrm{j}\mathcal{Q}_{\mathrm{j,j}}^{\zeta,\zeta}\left(\Omega(\iota,\theta;\varkappa,\varrho)\sqrt{(\iota-\varkappa)^4+(\theta-\varrho)^4};\varkappa,\varrho\right)=0,$$

uniformly in $(\varkappa,\varrho)\in\mathcal{I}^2$. Using Lemma (2.3), we obtain

$$\lim_{\mathrm{j}\to\infty}\mathrm{j}\mathcal{Q}_{\mathrm{j}}^{\zeta}((\iota-\varkappa);\varkappa)=\frac{1}{\zeta_1+1}\ \text{ and }\ \lim_{\mathrm{j}\to\infty}\mathrm{j}\mathcal{Q}_{\mathrm{j}}^{\zeta}((\theta-\varrho);\varrho)=\frac{1}{\zeta_2+1},$$

$$\lim_{\mathrm{j}\to\infty}\mathrm{j}\mathcal{Q}_{\mathrm{j}}^{\zeta}((\iota-\varkappa)^2;\varkappa)=\left(-\varkappa^2+\varkappa(1+\frac{2}{\zeta_1+1})\right)\ \text{ and }\ \lim_{\mathrm{j}\to\infty}\mathrm{j}\mathcal{Q}_{\mathrm{j}}^{\zeta}((\theta-\varrho)^2;\varrho)=\left(-\varrho^2+\varrho(1+\frac{2}{\zeta_2+1})\right).$$

By applying Lemma (2.3), we obtain

$$\lim_{\mathrm{j}\to\infty}\mathrm{j}\mathcal{Q}_{\mathrm{j}}^{\zeta}((\iota-\varkappa);\varkappa)\mathcal{Q}_{\mathrm{j}}^{\zeta}((\theta-\varrho);\varrho)=0,$$

uniformly in $(\varkappa,\varrho)\in\mathcal{I}^2$. We obtain the intended outcome through combining the estimations with equation (4.2). □

Theorem 4.2 (Grüss-Voronovskaja type theorem). *Let $\varphi,\mathrm{g}\in C^2(\mathcal{I}^2)$, then the following equality holds true*

$$\lim_{\mathrm{j}\to\infty}\mathrm{j}\left\{\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi\mathrm{g};\varkappa,\varrho)-\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\mathrm{g};\varkappa,\varrho)\right\}=\left(-\varkappa^2+\varkappa(1+\frac{2}{\zeta_1+1})\right)\varphi'_{\varkappa}(\varkappa,\varrho)\mathrm{g}'_{\varkappa}(\varkappa,\varrho)$$
$$+\left(-\varrho^2+\varrho(1+\frac{2}{\zeta_2+1})\right)\varphi'_{\varrho}(\varkappa,\varrho)\mathrm{g}'_{\varrho}(\varkappa,\varrho),\quad (4.3)$$

uniformly in $(\varkappa,\varrho)\in\mathcal{I}^2$.

Proof: Using Taylor's expansion of φ, g and $\varphi\mathrm{g}$, we obtain

$$\begin{aligned}
&\mathrm{j}\left\{\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi\mathrm{g};\varkappa,\varrho)-\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\mathrm{g};\varkappa,\varrho)\right\}\\
&=\mathrm{j}\Big[\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi\mathrm{g};\varkappa,\varrho)-\varphi(\varkappa,\varrho)\mathrm{g}(\varkappa,\varrho)-\left(\varphi'_{\varkappa}(\varkappa,\varrho)\mathrm{g}(\varkappa,\varrho)+\varphi(\varkappa,\varrho)\mathrm{g}'_{\varkappa}(\varkappa,\varrho)\right)\Lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)\\
&-\left(\varphi'_{\varrho}(\varkappa,\varrho)\mathrm{g}(\varkappa,\varrho)+\varphi(\varkappa,\varrho)\mathrm{g}'_{\varrho}(\varkappa,\varrho)\right)\Lambda_{\mathrm{j}_2,\zeta_2}(\varrho)-\frac{1}{2}\left(\varphi''_{\varkappa\varkappa}(\varkappa,\varrho)\mathrm{g}(\varkappa,\varrho)+2\varphi'_{\varkappa}(\varkappa,\varrho)\mathrm{g}'_{\varkappa}(\varkappa,\varrho)\right.\\
&\left.+\varphi(\varkappa,\varrho)\mathrm{g}''_{\varkappa\varkappa}(\varkappa,\varrho)\right)\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)-\frac{1}{2}\left(\varphi''_{\varrho\varrho}(\varkappa,\varrho)\mathrm{g}(\varkappa,\varrho)+2\varphi'_{\varrho}(\varkappa,\varrho)\mathrm{g}'_{\varrho}(\varkappa,\varrho)+\varphi(\varkappa,\varrho)\mathrm{g}''_{\varrho\varrho}(\varkappa,\varrho)\right)\\
&\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)-\left(\varphi(\varkappa,\varrho)\mathrm{g}''_{\varkappa\varrho}(\varkappa,\varrho)+\varphi'_{\mathrm{y}}(\varkappa,\varrho)\mathrm{g}'_{\varkappa}(\varkappa,\varrho)+\varphi'_{\varkappa}(\varkappa,\varrho)\mathrm{g}'_{\mathrm{y}}(\varkappa,\varrho)+\varphi''_{\varkappa\varrho}(\varkappa,\varrho)\mathrm{g}(\varkappa,\varrho)\right)\\
&\Lambda_{\mathrm{j}_1,\mathrm{j}_2,\zeta_1,\zeta_2}(\varkappa,\varrho)-\mathrm{g}(\varkappa,\varrho)\Big\{\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)-\varphi'_{\varkappa}(\varkappa,\varrho)\Lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)-\varphi'_{\varrho}(\varkappa,\varrho)\\
&\Lambda_{\mathrm{j}_2,\zeta_2}(\varrho)-\frac{1}{2}\varphi''_{\varkappa\varkappa}(\varkappa,\varrho)\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)-\frac{1}{2}\varphi''_{\varrho\varrho}\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)-\varphi''_{\varkappa\varrho}(\varkappa,\varrho)\Lambda_{\mathrm{j}_1,\mathrm{j}_2,\zeta_1,\zeta_2}(\varkappa,\varrho)\Big\}\\
&-\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\Big\{\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\mathrm{g};\varkappa,\varrho)-\mathrm{g}(\varkappa,\varrho)-\mathrm{g}'_{\varkappa}(\varkappa,\varrho)\Lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)\\
&-\mathrm{g}'_{\varrho}(\varkappa,\varrho)\Lambda_{\mathrm{j}_2,\zeta_2}(\varrho)-\frac{1}{2}\mathrm{g}''_{\varkappa\varkappa}(\varkappa,\varrho)\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)-\frac{1}{2}\mathrm{g}''_{\varrho\varrho}(\varkappa,\varrho)\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)\\
&-\mathrm{g}''_{\varkappa\varrho}(\varkappa,\varrho)\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)(\varkappa,\varrho)\Big\}+\mathrm{g}'_{\varkappa}(\varkappa,\varrho)\Lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)\left\{\varphi(\varkappa,\varrho)-\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\right\}\\
&+\mathrm{g}'_{\varrho}(\varkappa,\varrho)\Lambda_{\mathrm{j}_2,\zeta_2}(\varrho)\left\{\varphi(\varkappa,\varrho)-\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\right\}\\
&+\frac{1}{2}\mathrm{g}''_{\varkappa\varkappa}(\varkappa,\varrho)\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)\left\{\varphi(\varkappa,\varrho)-\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\right\}+\frac{1}{2}\mathrm{g}''_{\varrho\varrho}(\varkappa,\varrho)\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)\{\varphi(\varkappa,\varrho)\\
&-\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\Big\}+\mathrm{g}''_{\varkappa\varrho}(\varkappa,\varrho)\Lambda_{\mathrm{j}_1,\mathrm{j}_2,\zeta_1,\zeta_2}(\varkappa,\varrho)\left\{\varphi(\varkappa,\varrho)-\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\right\}+\varphi'_{\varkappa}(\varkappa,\varrho)\mathrm{g}'_{\varkappa}(\varkappa,\varrho)\\
&\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)+\varphi'_{\varkappa}(\varkappa,\varrho)\mathrm{g}'_{\varrho}(\varkappa,\varrho)\Lambda_{\mathrm{j}_1,\mathrm{j}_2,\zeta_1,\zeta_2}(\varkappa,\varrho)+\varphi'_{\varrho}(\varkappa,\varrho)\mathrm{g}'_{\varkappa}(\varkappa,\varrho)\Lambda_{\mathrm{j}_1,\mathrm{j}_2,\zeta_1,\zeta_2}(\varkappa,\varrho)\\
&+\varphi'_{\varrho}(\varkappa,\varrho)\mathrm{g}'_{\varrho}(\varkappa,\varrho)\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)\Big].
\end{aligned}$$

Applying Theorem 3.1 and Lemma (2.4), we reach the assertion. □

Theorem 4.3 *For given function $\varphi\in C(\mathcal{I}^2)$, we obtain the inequality given as*

$$\begin{aligned}
\left|\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right| &\leq 4\mathcal{K}(\varphi;\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho))+\omega\left(\varphi;\sqrt{\left(\frac{1}{\mathrm{j}_1(\zeta_1+1)}\right)^2+\left(\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)^2}\right)\\
&\leq M\left\{\overline{\omega_2}\left(\varphi;\sqrt{\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)}\right)+\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)||\varphi||_{C(\mathcal{I}^2)}\right\}\\
&\quad+\omega\left(\varphi;\sqrt{\left(\frac{1}{\mathrm{j}_1(\zeta_1+1)}\right)^2+\left(\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)^2}\right).
\end{aligned}$$

The constant $M\geq 0$ is independent of φ and $\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)$,
where $\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)=\left(\lambda_{\mathrm{j}_1,\zeta_1}(\varkappa)+\left(\frac{1}{\mathrm{j}_1(\zeta_1+1)}\right)^2+\lambda_{\mathrm{j}_2,\zeta_2}(\varrho)+\left(\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)^2\right)$.

Proof: We describe the auxiliary operators as follows

$$\overline{\mathcal{Q}}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)=\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi\left(\varkappa+\frac{1}{\mathrm{j}_1(\zeta_1+1)},\varrho+\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)+\varphi(\varkappa,\varrho). \tag{4.4}$$

Then, using Lemma (2.3), we have $\overline{\mathcal{Q}}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}((\iota-\varkappa);\varkappa,\varrho)=0$ and $\overline{\mathcal{Q}}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}((\theta-\varrho);\varkappa,\varrho)=0$.

Suppose $\mathsf{g} \in C^2(\mathcal{I}^2)$ and $\iota, \theta \in \mathcal{I}$. Employing Taylor's formula, we may express

$$\begin{aligned}\mathsf{g}(\iota,\theta)-\mathsf{g}(\varkappa,\varrho) &= \mathsf{g}(\iota,\varrho)-\mathsf{g}(\varkappa,\varrho)+\mathsf{g}(\iota,\theta)-\mathsf{g}(\iota,\varrho)\\ &= \frac{\partial \mathsf{g}(\varkappa,\varrho)}{\partial\varkappa}(\iota-\varkappa)+\int_{\varkappa}^{\iota}(\iota-\mathsf{u})\frac{\partial^2\mathsf{g}(\mathsf{u},\varrho)}{\partial \mathsf{u}^2}d\mathsf{u}+\frac{\partial \mathsf{g}(\varkappa,\varrho)}{\partial\varrho}(\theta-\varrho)\\ &\quad+\int_{\varrho}^{\theta}(\theta-v)\frac{\partial^2\mathsf{g}(\varkappa,v)}{\partial v^2}dv. \end{aligned} \tag{4.5}$$

Applying the operators $\mathcal{Q}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}$ on (4.5), we get

$$\begin{aligned}\overline{\mathcal{Q}}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}(\mathsf{g};\varkappa,\varrho)-\mathsf{g}(\varkappa,\varrho) &= \overline{\mathcal{Q}}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}\left(\int_{\varkappa}^{\iota}(\iota-\mathsf{u})\frac{\partial^2\mathsf{g}(\mathsf{u},\varrho)}{\partial \mathsf{u}^2}d\mathsf{u};\varkappa,\varrho\right)\\ &\quad+\overline{\mathcal{Q}}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}\left(\int_{\varrho}^{\theta}(\theta-v)\frac{\partial^2\mathsf{g}(\varkappa,v)}{\partial v^2}dv;\varkappa,\varrho\right)\\ &= \mathcal{Q}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}\left(\int_{\varkappa}^{\iota}(\iota-\mathsf{u})\frac{\partial^2\mathsf{g}(\mathsf{u},\varrho)}{\partial \mathsf{u}^2}d\mathsf{u};\varkappa,\varrho\right)\\ &\quad-\int_{\varkappa}^{\varkappa+\frac{1}{\mathsf{j}_1(\zeta_1+1)}}\left(\varkappa+\frac{1}{\mathsf{j}_1(\zeta_1+1)}-\mathsf{u}\right)\frac{\partial^2\mathsf{g}(\mathsf{u},\varrho)}{\partial\mathsf{u}^2}d\mathsf{u}\\ &\quad+\mathcal{Q}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}\left(\int_{\varrho}^{\theta}(\theta-v)\frac{\partial^2\mathsf{g}(\varkappa,v)}{\partial v^2}dv;\varkappa,\varrho\right)\\ &\quad-\int_{\varrho}^{\varrho+\frac{1}{\mathsf{j}_2(\zeta_2+1)}}\left(\varrho+\frac{1}{\mathsf{j}_2(\zeta_2+1)}-v\right)\frac{\partial^2\mathsf{g}(\varkappa,v)}{\partial v^2}. \end{aligned}$$

Hence,

$$\begin{aligned}&\left|\overline{\mathcal{Q}}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}(\mathsf{g};\varkappa,\varrho)-\mathsf{g}(\varkappa,\varrho)\right|\\ &\leq \mathcal{Q}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}\left(\left|\int_{\varkappa}^{\iota}|\iota-\mathsf{u}|\left|\frac{\partial^2\mathsf{g}(\mathsf{u},\varrho)}{\partial\mathsf{u}^2}\right|d\mathsf{u}\right|;\varkappa,\varrho\right)+\left|\int_{\varkappa}^{\varkappa+\frac{1}{\mathsf{j}_1(\zeta_1+1)}}\left|\varkappa+\frac{1}{\mathsf{j}_1(\zeta_1+1)}-\mathsf{u}\right|\left|\frac{\partial^2\mathsf{g}(\mathsf{u},\varrho)}{\partial\mathsf{u}^2}\right|d\mathsf{u}\right|\\ &\quad+\mathcal{Q}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}\left(\left|\int_{\varrho}^{\theta}|\theta-v|\left|\frac{\partial^2\mathsf{g}(\varkappa,v)}{\partial v^2}\right|dv\right|;\varkappa,\varrho\right)+\left|\int_{\varrho}^{\varrho+\frac{1}{\mathsf{j}_2(\zeta_2+1)}}\left|\varrho+\frac{1}{\mathsf{j}_2(\zeta_2+1)}-v\right|\left|\frac{\partial^2\mathsf{g}(\varkappa,v)}{\partial v^2}\right|dv\right|\\ &\leq\left\{\mathcal{Q}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}((\iota-\varkappa)^2;\varkappa,\varrho)+\left(\varkappa+\frac{1}{\mathsf{j}_1(\zeta_1+1)}-\varkappa\right)^2\right\}||\mathsf{g}||_{C^2(\mathcal{I}^2)}\\ &\quad+\left\{\mathcal{Q}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}((\theta-\varrho)^2;\varkappa,\varrho)+\left(\varrho+\frac{1}{\mathsf{j}_2(\zeta_2+1)}-\varrho\right)^2\right\}||\mathsf{g}||_{C^2(\mathcal{I}^2)}\\ &\leq\left(\lambda_{\mathsf{j}_1,\zeta_1}(\varkappa)+\left(\frac{1}{\mathsf{j}_1(\zeta_1+1)}\right)^2+\lambda_{\mathsf{j}_2,\zeta_2}(\varrho)+\left(\frac{1}{\mathsf{j}_2(\zeta_2+1)}\right)^2\right)||\mathsf{g}||_{C^2(\mathcal{I}^2)}. \end{aligned}$$

Also,

$$\begin{aligned}\left|\overline{\mathcal{Q}}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\right| &\leq \left|\mathcal{Q}_{\mathsf{j},\mathsf{j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)\right|+\left|\varphi\left(\varkappa+\frac{1}{\mathsf{j}_1(\zeta_1+1)},\varrho+\frac{1}{\mathsf{j}_2(\zeta_2+1)}\right)\right|+|\varphi(\varkappa,\varrho)|\\ &\leq 3||\varphi||_{C(\mathcal{I}^2)}. \end{aligned} \tag{4.6}$$

Hence, in view of (4.6), we obtain

$$\begin{aligned}
\left|\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right| &= \left|\overline{\mathcal{Q}}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)+\varphi\left(\varkappa+\frac{1}{\mathrm{j}_1(\zeta_1+1)},\varrho+\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)-\varphi(\varkappa,\varrho)\right| \\
&\leq |\overline{\mathcal{Q}}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi-\mathrm{g};\varkappa,\varrho)|+|\overline{\mathcal{Q}}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\mathrm{g};\varkappa,\varrho)-\mathrm{g}(\varkappa,\varrho)| \\
&\quad +|\mathrm{g}(\varkappa,\varrho)-\varphi(\varkappa,\varrho)|+\left|\varphi\left(\varkappa+\frac{1}{\mathrm{j}_1(\zeta_1+1)},\varrho+\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)-\varphi(\varkappa,\varrho)\right| \\
&\leq 3||\varphi-\mathrm{g}||_{C(\mathcal{I}^2)}+||\varphi-\mathrm{g}||_{C(\mathcal{I}^2)}+|\overline{\mathcal{Q}}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\mathrm{g};\varkappa,\varrho)-\mathrm{g}(\varkappa,\varrho)| \\
&\quad +\left|\varphi\left(\varkappa+\frac{1}{\mathrm{j}_1(\zeta_1+1)},\varrho+\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)-\varphi(\varkappa,\varrho)\right| \\
&\leq 4||\varphi-\mathrm{g}||_{C(\mathcal{I}^2)}+\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)||\mathrm{g}||_{C^2(\mathcal{I}^2)} \\
&\quad +\left|\varphi\left(\varkappa+\frac{1}{\mathrm{j}_1(\zeta_1+1)},\varrho+\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)-\varphi(\varkappa,\varrho)\right| \\
&\leq \left(4||\varphi-\mathrm{g}||_{C(\mathcal{I}^2)}+\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)||\mathrm{g}||_{C^2(\mathcal{I}^2)}\right) \\
&\quad +\omega\left(\varphi;\sqrt{\left(\varkappa+\frac{1}{\mathrm{j}_1(\zeta_1+1)}-\varkappa\right)^2+\left(\varrho+\frac{1}{\mathrm{j}_2(\zeta_2+1)}-\varrho\right)^2}\right).
\end{aligned}$$

Using (2.1) and right-hand side infimum over $\mathrm{g}\in C^2(\mathcal{I}^2)$, we get

$$\begin{aligned}
\left|\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)-\varphi(\varkappa,\varrho)\right| &\leq 4\mathcal{K}(\varphi;\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho))+\omega\left(\varphi;\sqrt{\left(\frac{1}{\mathrm{j}_1(\zeta_1+1)}\right)^2+\left(\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)^2}\right) \\
&\leq M\left\{\overline{\omega_2}\left(\varphi;\sqrt{\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)}\right)+\min\{1,\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)\}||\varphi||_{C(\mathcal{I}^2)}\right\} \\
&\quad +\omega\left(\varphi;\sqrt{\left(\frac{1}{\mathrm{j}_1(\zeta_1+1)}\right)^2+\left(\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)^2}\right). \\
&\leq M\left\{\overline{\omega_2}\left(\varphi;\sqrt{\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)}\right)+\mathcal{S}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varkappa,\varrho)||\varphi||_{C(\mathcal{I}^2)}\right\} \\
&\quad +\omega\left(\varphi;\sqrt{\left(\frac{1}{\mathrm{j}_1(\zeta_1+1)}\right)^2+\left(\frac{1}{\mathrm{j}_2(\zeta_2+1)}\right)^2}\right).
\end{aligned}$$

Hence, the proof is completed. □

5. Numerical examples

Example 5.1 *Figures 1–4, represent the convergence for the operators* $\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}$ *for the functions* $\varphi(\varkappa,\varrho)(blue)=\varkappa\varrho e^{\varkappa\varrho},\varkappa^3-\varkappa^2\cos(2\pi\varrho),\varkappa\sin(2\pi\varrho),\varkappa\varrho+\varkappa^2\sin(3\pi\varrho)+\varrho^2\cos(6\pi\varkappa)$*, respectively. The outcomes indicate that the operators converge more effectively as the values of* j_1 *and* j_2 *increase. In Figure, 5 and 6 we take functions* $\varkappa^3e^{\varkappa}\varrho\sin(4\pi\varkappa)(blue),\varkappa^2\varrho^2(blue)$*, respectively and we observe that as we increase values of* ζ_1,ζ_2*, the convergence rate improves.*

Example 5.2 *In Figure 7–10, the error estimation of operators* $\mathcal{Q}_{\mathrm{j,j}}^{\zeta_1,\zeta_2}$*, improve for larger values of* $\mathrm{j}_1,\mathrm{j}_2$*. We take* $\varphi(\varkappa,\varrho)(blue)=\varkappa\sin(2\pi\varrho),\varrho\sin(2\pi\varkappa),\varkappa^3-\varkappa^2\cos(2\pi\varrho),\varkappa\varrho+\varkappa^2\sin(3\pi\varrho)+\varrho^2\cos(6\pi\varkappa)$*, respectively.*

In all these figures, we have taken $\mathrm{z}_1,\mathrm{z}_2,\mathrm{r}_1,\mathrm{r}_2=1$.

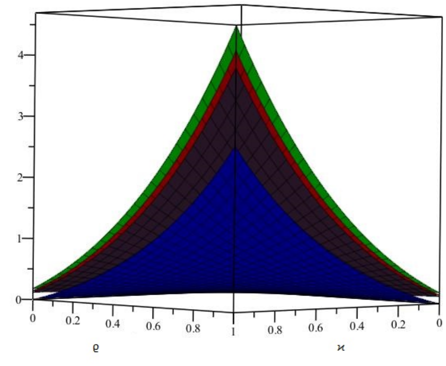

Figure 1: Convergence behaviour of $\mathcal{Q}_{5,5}^{0.5,0.5}$(green), $\mathcal{Q}_{6,6}^{0.5,0.5}$(red), $\mathcal{Q}_{7,7}^{0.5,0.5}$(violet).

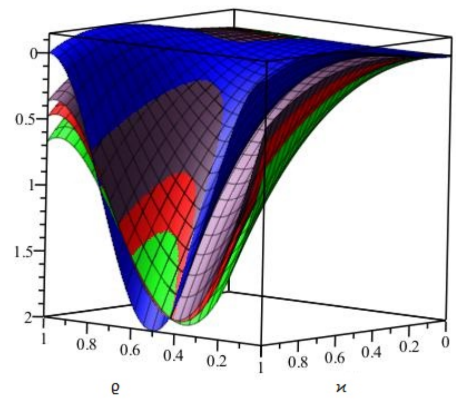

Figure 2: Convergence behaviour of $\mathcal{Q}_{5,5}^{0.5,0.5}$(green), $\mathcal{Q}_{6,6}^{0.5,0.5}$(red), $\mathcal{Q}_{7,7}^{0.5,0.5}$(violet).

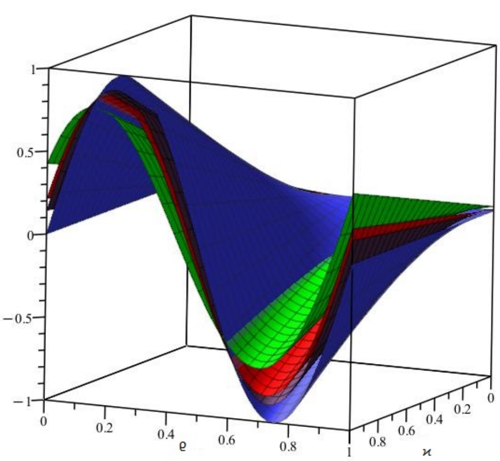

Figure 3: Convergence behaviour of $\mathcal{Q}_{10,10}^{0.5,0.5}$(green), $\mathcal{Q}_{20,20}^{0.5,0.5}$(red), $\mathcal{Q}_{30,30}^{0.5,0.5}$(violet).

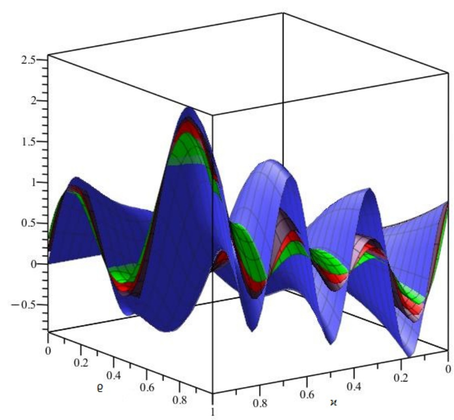

Figure 4: Convergence behaviour of $\mathcal{Q}_{20,20}^{0.7,0.7}$(green), $\mathcal{Q}_{30,30}^{0.7,0.7}$(red), $\mathcal{Q}_{40,40}^{0.7,0.7}$(violet).

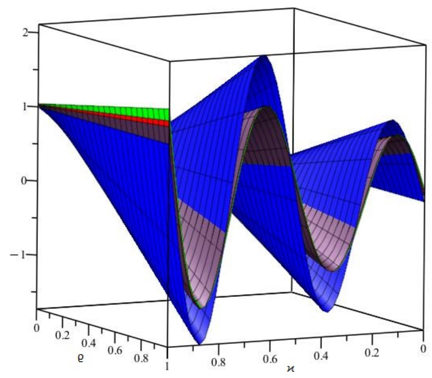

Figure 5: Convergence behaviour of $\mathcal{Q}_{40,40}^{1,1}$(green), $\mathcal{Q}_{40,40}^{2,2}$(red), $\mathcal{Q}_{40,40}^{3,3}$(violet).

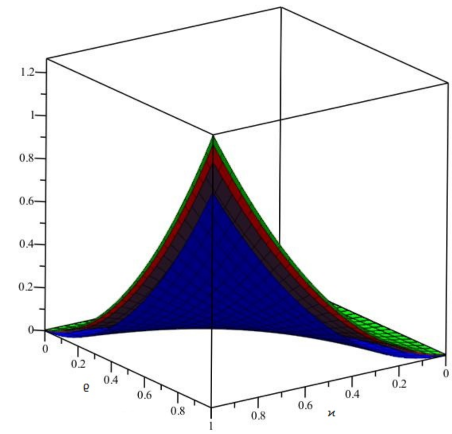

Figure 6: Convergence behaviour of $\mathcal{Q}_{15,15}^{0.1,0.1}$(green), $\mathcal{Q}_{15,15}^{0.3,0.3}$(red), $\mathcal{Q}_{15,15}^{1,1}$(violet).

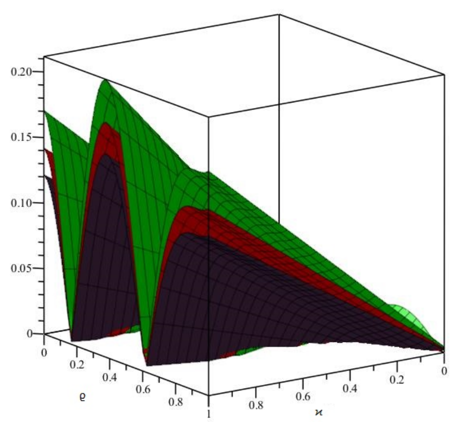

Figure 7: Error behaviour of $\mathcal{Q}_{25,25}^{0.5,0.5}$(green), $\mathcal{Q}_{30,30}^{0.5,0.5}$(red), $\mathcal{Q}_{35,35}^{0.5,0.5}$(violet).

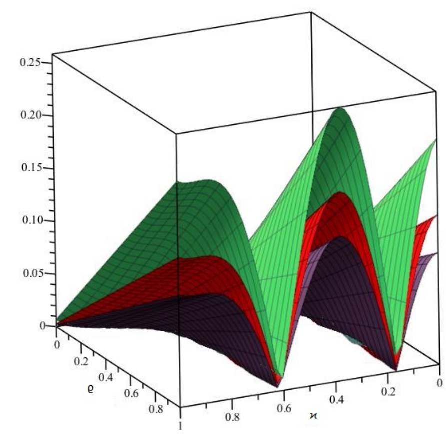

Figure 8: Error behaviour of $\mathcal{Q}_{20,20}^{0.5,0.5}$(green), $\mathcal{Q}_{30,30}^{0.5,0.5}$(red), $\mathcal{Q}_{40,40}^{0.5,0.5}$(violet).

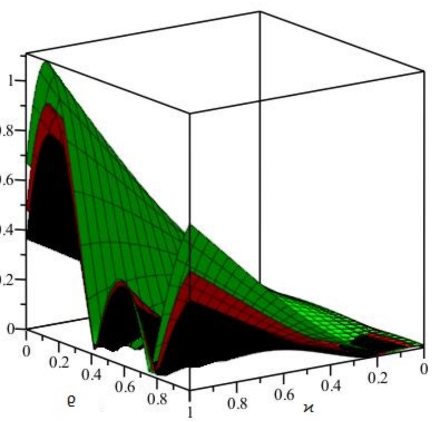

Figure 9: Error behaviour of $\mathcal{Q}_{5,5}^{0.5,0.5}$(green), $\mathcal{Q}_{6,6}^{0.5,0.5}$(red), $\mathcal{Q}_{7,7}^{0.5,0.5}$(violet).

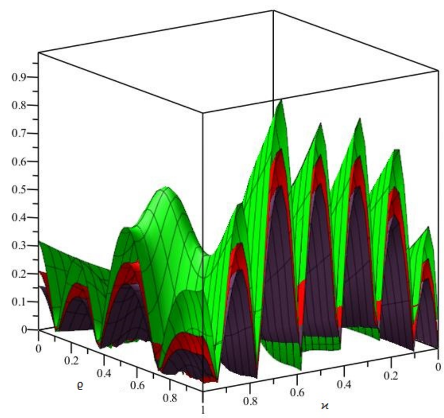

Figure 10: Error behaviour of $\mathcal{Q}_{20,20}^{0.5,0.5}$(green), $\mathcal{Q}_{30,30}^{0.5,0.5}$(red), $\mathcal{Q}_{40,40}^{0.5,0.5}$(violet).

6. Conclusion

In this paper, we introduced a new class of bivariate Riemann-Liouville type fractional Stancu-Kantorovich operators and studied their approximation properties. The uniform convergence of the operators was established, and the rate of convergence was obtained using the modulus of continuity and Lipschitz-type functions. Moreover, Voronovskaja-type and Grüss-Voronovskaja-type theorems were derived to describe the asymptotic behaviour of the operators. Numerical examples and graphical illustrations confirmed the theoretical results. Furthermore, for $\zeta_1 = \zeta_2 = 1$, our proposed operators $\mathcal{Q}_{\mathrm{j}_1,\mathrm{j}_2}^{\zeta_1,\zeta_2}(\varphi;\varkappa,\varrho)$ reduce to the bivariate Kantorovich Stancu operators. This reduction shows that our proposed operators are natural extension of the existing operators. Consequently, the results of this paper generalize the known approximation results and include several earlier results as particular cases.

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