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Existence of Solutions for Some Quasilinear Elliptic System with Weight and Measure-Valued Right Hand Side

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ABSTRACT: We prove the existence of a solution u for the nonlinear elliptic system:

$$\begin{array}{rcl} -\operatorname{div}\sigma(x,u,Du) & = & \mu+g(x,u,Du) \text{ in } \Omega \\ u & = & 0 \text{ on } \partial\Omega \end{array}$$

where μ is Radon measure on Ω with finite mass and g satisfies some standards continuity and growth conditions. In particular, we show that if the coercivity rate of σ lies in the range $]\frac{s+1}{s},(\frac{s+1}{s})(2-\frac{1}{n})]$ then u is approximately differentiable and the equation holds with Du replaced by $\operatorname{ap}Du$. The proof relies on an approximation of μ by smooth functions f_k and a compactness result for the corresponding solutions u_k. This follows from a detailed analysis of the Young measure $\{\delta_u(x)\otimes\vartheta(x)\}$ generated by the sequence (u_k, Du_k), and the div-curl type inequality $<\vartheta(x),\sigma(x,u,\cdot)>\leq\overline{\sigma}(x)<\vartheta(x),.>$ for the weak limit $\overline{\sigma}$ of the sequence.

Keywords: Nonlinear elliptic system, mesure-valued, young measure, the div-curl type inequality.

Contents

1. Introduction

We consider the existence and compactness questions for elliptic systems of the form

$$-div\sigma(x,u(x),Du(x))=\mu+g(x,u,Du) \text{ in } \Omega \tag{1.1}$$

$$u=0 \text{ on } \partial\Omega \tag{1.2}$$

with measure-valued right hand side $\mu\in M(\Omega,\mathbb{R}^m)$ on an open, bounded domain Ω in $\mathbb{R}^n$. And the weight functions ω^* defined by $\omega=\{\omega_{ij},0\leq i\leq n,1\leq j\leq m\}$ with $\omega^*_{ij}=\omega_{ij}^{1-p'}$ on this paper we are interested in the solution u in the Sobolev space $W_0^{1,p}(\Omega,w,\mathbb{R}^m)$, and estimation of the weak Lebesgues spaces. We assume that σ satisfies the following hypotheses $(H_0)-(H_3)$ and g satisfies G_0-G_1. Here $\mathbb{M}^{m\times n}$ denotes the space of real $m\times n$ matrices equipped with inner product $M:N=M_{ij}N_{ij}$ (we use the usual summation convention) and the tensor product $a\otimes b$ of two vectors $a,b\in\mathbb{R}^n$ is defined to be the matrix $(a_ib_j)_{i,j=1,\dots,m}$. Let $\omega=\{\omega_{ij}|\ \ 0\leq i\leq n;\ 1\leq j\leq m\}$ the system for weight functions on Ω satisfying:

$$\omega_{ij}\in L^1(\Omega),\quad \omega_{ij}^{\frac{-1}{p-1}}\in L^1_{loc}(\Omega) \tag{1.3}$$

$$\omega_{ij}^{-s}\in L^1(\Omega) \tag{1.4}$$

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for same s. The Sobolev space $W^{1,p}(\Omega,\omega,I\!R^m)$ with

$$W^{1,p}(\Omega,\omega,I\!R^m)=\{u\in L^p(\Omega,\overline{\omega}_0,I\!R^m)|\ \overline{\omega}_0=\omega_{0j};\ 1\le j\le m;\ \frac{\partial u_j}{\partial x_i}\in L^p(\Omega,\omega_{ij});\ 1\le j\le m;\ 1\le i\le n\}$$

The Sobolev space $W^{1,p}(\Omega,\omega,I\!R^m)$ equipped with the norm

$$\| u \|_{1,p,\omega}^p=\sum_{j=1}^m\int_\Omega \omega_{0j}|u_j|^p dx+\sum_{i,j}\int_\Omega \omega_{ij}|D_{ij}u|^p dx.$$

The condition (1.3) we give $(W^{1,p}(\Omega,\omega,I\!R^m),\|..\|_{1,p,\omega})$ Banach space and $C_0^\infty(\Omega,I\!R^m)$ subspace of $(W^{1,p}(\Omega,\omega,I\!R^m)$. The space $(W_0^{1,p}(\Omega,\omega,I\!R^m)$ is the fermenter of $C_0^\infty(\Omega,I\!R^m)$ in $(W^{1,p}(\Omega,\omega,I\!R^m)$ for the norm $\|..\|_{1,p,\omega}^p)$. The condition (1.4), we give

$$(W^{1,p}(\Omega,\omega,I\!R^m)\hookrightarrow(W^{1,p_s}(\Omega,I\!R^m)\hookrightarrow L^r(\Omega,I\!R^m),\tag{1.5}$$

for all $1\le r\prec p_s^*$ if $p.s\prec n(s+1)$, and
$\forall r\ge 1$ if $p.s\succ n(s+1)$ with $p_s=\frac{p.s}{s+1}$ and $p_s^*=\frac{n.p.s}{n(s+1)-p.s}$

2. Hypothesis

(H_0) (Continuity) $\sigma:\Omega\times I\!R^m\times I\!M^{m\times n}\longrightarrow I\!M^{m\times n}$ is a Carathéodory function, i.e:
$x\longmapsto\sigma(x,u,p)$ is measurable for every (u,p) and $(u,p)\longmapsto\sigma(x,u,p)$ is continuous for almost every $x\in\Omega$.

(H_1) (Coercivity and growth): There exist constants c_1, c_2, $\beta>0$ and $\lambda_1\in L^{p'}(\Omega)$, $\lambda_2\in L^1(\Omega)$, $\lambda_3\in L^{(\frac{p}{\theta})'}(\Omega)$, $0<\theta<p$, $1<q<\frac{p^2}{\theta}$, such that, for all $1\le r\le n$, and $1\le s\le m$:

$$|\sigma_{rs}(x,u,F)|\le\beta\omega_{rs}^{\frac{1}{p}}[\lambda_1+c_1\sum_{j=1}^m\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}+c_1\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{p-1}]$$

$$\sigma(x,u,F):F\ge-\lambda_2-\sum_{j=1}^m\lambda_3\gamma_j^{\frac{\theta}{p}}|u_j|^{\frac{q\theta}{p}}+c_2\sum_{i,j}\omega_{ij}|F_{ij}|^p.$$

(H_2) (Monotonicity) σ satisfies one following conditions:

a)- For all $x\in\Omega$, $u\in I\!R^m$ the function $F\longmapsto\sigma(x,u,F)$ is a C^1 and monotone function, which means $(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)\ge 0$, for all $x\in\Omega$, $u\in I\!R^m$, and $F,G\in I\!M^{m\times n}$.

b)- There exist a function $W:\Omega\times I\!R^m\times I\!M^{m\times n}\to R$ such that:
$\sigma(x,u,F)=\frac{\partial W}{\partial F}(x,u,F)$, and the function $F\longmapsto W(x,u,F)$ is a C^1 and convex function.

c)- σ is strictly monotone, i-e; σ is monotone and $(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)=0.$ implies $F=G$.

d)A function $F\longmapsto\sigma(x,u,F)$ is strictly $p-$quasi-monotone, i-e:
$\int_{I\!M^{m\times n}}(\sigma(x,u,\lambda)-\sigma(x,u,\overline{\lambda})):(\lambda-\overline{\lambda})d\nu(\lambda)>0$, for all homogeneous $W^{1,p}-gradient$ Young measures ν with center of mass $\overline{\lambda}=<\nu;Id>=\int_{I\!M^{m\times n}}\lambda d\nu(\lambda)$ which are not a single Dirac mass.

(H_3) (structure conditions)

i)- (Angle condition) for all $x\in\Omega$, $u\in I\!R^m$ and $F\in I\!M^{m\times n}$ there holds
$\sigma(x,u,F):MF\ge 0$, for all matrix $M\in I\!M^{m\times m}$ of the form $M=Id-a\otimes a$ with $|a|\le 1$.

ii)- (The sign condition) for all $x\in\Omega$, $u\in I\!R^m$ and $F\in I\!M^{m\times n}$, $\sigma_j(x,u,F):F_j\ge 0$, for all $1\le j\le m$ with F_j and σ_j are the cologne j of matrix F and σ.

(H_4) (The Hardy-Type Inequality) There exist $c > 0$, one weighted function $\gamma = (\gamma_j)_{1\leq j\leq m}$, and the parameter $1 < q < \frac{p^2}{\theta}$ (H_1), such that:

$$\left(\sum_{j=1}^{m}\int_{\Omega}\gamma_j \mid u_j \mid^q dx\right)^{\frac{1}{q}} \leq c\left(\sum_{i,j}\int_{\Omega}\omega_{ij} \mid D_{ij}u \mid^p dx\right)^{\frac{1}{p}}$$

and the expression $\left(\sum_{i,j}\int_{\Omega}\omega_{ij} \mid D_{ij}u \mid^p dx\right)^{\frac{1}{p}}$ is a norm equivalent to the norm $\|.\|_{1;p:\omega}$ in $W_0^{1,p}(\Omega,\omega,I\!R^m)$.

(G_0) (Continuity) $g : \Omega \times I\!R^n \longrightarrow I\!R^m$ is Caratheodory function.

(G_1) (Growth) There exist:$b_1 \in L^{p'}(\Omega)$, $C_1, C_1' > 0$ such that $\mid g_j(x,u,F) \mid \leq [b_1(x) + C_1\gamma_j^{\frac{1}{p'}} \mid u_j \mid^{\frac{q}{p'}} + C_1'\sum_{r,s}\omega_{rs}^{\frac{1}{p'}} \mid F_{rs} \mid^{p-1}]\omega_{0j}^{\frac{1}{p}}$

Remark 2.1

1. Assumption (H_0) ensures that $\sigma(x,u(x),U(x))$ is measurable on Ω for measurable function $u : \Omega \longrightarrow I\!R^m$ and $U : \Omega \longrightarrow I\!M^{m\times n}$. A typical example for a function σ satisfying (H_0) is $\sigma(x,u,p) = \zeta(x,u,p)p$ with a real valued non-negative function ζ.
2. A serious technical obstacle is that for $p_s \in (1, 2-1/n]$ solutions of the system (1.1) in general do not belongs to the Sobolev space $W^{1,1}(\Omega,\omega,I\!R^m)$.

This fact has led to the use of normalized solutions in [2] and generalized entropy solutions in [5] for elliptic equations of the above type .We will use a notion of solution where the weak derivative Du is replaced by the approximate derivative $apDu$. Recall that a measurable function u is said to be approximately differentiable at $x \in \Omega$ if there exists a matrix $F_x \in I\!M^{m\times n}$ such that:
for all $\epsilon \succ 0, \lim\limits_{r\to 0}\dfrac{1}{r^n} meas\left\{y \in B(x,r) : |u(y)-u(x)-F_x(y-x)| > \epsilon r\right\} = 0$. We write:
$apDu(x) = F_x$.

Definition 2.1 *A measurable function $u : \Omega \longrightarrow I\!R^m$ is called a solution of the system (1.1) if:*

(i) *u is almost everywhere approximately differentiable.*

(ii) *$\eta\circ u \in W^{1,1}(\Omega,\omega,I\!R^m)$, for all, $\eta \in C_0^1(I\!R^m,I\!R^m)$.*

(iii) *$\sigma(x,u,apDu) \in L^1(\Omega,I\!M^{m\times n})$;*

(iv) *The equation: $-div\sigma(x,u(x),Du(x)) = \mu + g(x,u,Du)$ holds in the sense of distributions. Moreover we say that u satisfies the boundary condition (1.2) if $\eta\circ u \in W_0^{1,1}(\Omega,\omega,I\!R^m)$, for all, $\eta \in C^1(I\!R^m,I\!R^m)\cap L^\infty(I\!R^m,I\!R^m)$ with $\eta = id$ on $B(0,r)$, for some $r \succ 0$, and $|D\eta(y)| \leq c.(1+|y|)^{-1}$, with $c < \infty$.*

Remark 2.2

1. The conditions in Definition (2.1) (except (ii)) are the weakest possible in order to define the system (1.1) in the sense of distributions. Note that if u is approximately differentiable, then $apDu$ is measurable, so $\sigma(,u,apDu)$ is measurable.
2. The assumption $\eta ou \in W^{1,1}(\Omega,I\!R^m)$ ensures minimal regularity of u. For example, if $\mu = 0$, and $\sigma(x,u,p) = \sigma(p)$ with $\sigma(0) = 0$, then piecewise constant functions u satisfy $apDu = 0$ a,e, but are not admissible solutions. The following theorem is the main result in this paper .

Theorem 2.1 *Let Ω be a bounded, open set. We suppose that the hypotheses $(H_0)-(H_2)-(H_3)$ and the coercivity condition in (H_1) are satisfied and g satisfies $(G_0)-(G_1)$. Let μ denote a $I\!R^m$- valued Radon measure on Ω with finite mass. Then the system (1.1)-(1.2) has a solution u in the sense of definition1, which satisfies the weak Lebesgue space estimate:*

$$\|u\|^*_{L^{t^*_{p_s},\infty}(\Omega,I\!R^m)} + \|apDu\|^*_{L^{t_{p_s},\infty}(\Omega,I\!M^{m\times n})} \leq C, \tag{2.1}$$

with the constant C depends of $|\Omega|, c, c_2,$ and $\parallel \lambda_3 \parallel_{L^{(\frac{p}{\theta})}(\Omega)}$, with $t_{p_s} = \frac{n(p_s-1)}{n-1}$ and
$t^*_{p_s} = \frac{n(p_s-1)}{n-p_s}$ is the Sobolev exposed of t_{p_s}. If $c_2 = 0$ the right hand side of (1.3) reduces to $C(c_1)\left\|\mu^{\frac{1}{p-1}}\right\|_M$.

Remark 2.3

1. If $p_s > 2 - \frac{1}{n}$, then $t_{p_s} > 1$ and $Du \in L^1(\Omega, M^{m\times n})$.
2. If $p \succ n$ one can replace the $L^{s,\infty}$-norm of u in (1.3) by the C^0, β-norm with $\beta = 1 - \frac{n}{p}$. For $p = q = n$ it is an open question whether $Du \in L^{n,\infty}$. See Section 7 [4] for the (weaker) inclusion $u \in BMO_{loc}$.
3. The exponent in (1.3) are optimal as can be seen from the nonlinear Green's function $G_p(x) = c\,|x|^{\frac{-n}{s^*}}$ for the p-Laplace equation: $-div(|Du|^{p-2}\,Du) = \delta_0$ in $I\!R^m$, $n \geq 3$. In particular, $L^{s,\infty}$ cannot be replaced by L^s. with $(L^{s,\infty})$, is a Laurent space.
4. The pointiest monotonicity condition can be replaced by a weaker integrated version, called quasi-monotonicity,
   The Key point in the proof of the theorem, is the div-curl inequality for the Young measure $\{\{\vartheta_x\}_{x\epsilon\Omega}\}$ generated by a sequence Du_k of gradients of approximate solutions. Together with the identity. (1.5):
   $apDu(x) =< \vartheta_x, Id >$. The div-curl inequality implies easily that $\sigma(., u_k, Du_k)$ converges weakly in L^1 to $\sigma(., u, apDu)$. (1.5) is a consequence of general properties of young measures if $p_s > 2 - \frac{1}{n}$ since in this case Du_k is bounded in L^s for some $s > 1$. If $1 < p_s \leq 2 - \frac{1}{n}$ one only has the weaker bounds.

3. Some Preliminary Lemmas

In this section, we will also use the Young measures, and Inequality div-curl for assume the convergence of subsequence $u_k \longrightarrow u$ in measure and for almost every subsequence, with u is approximately differentiable, and $apDu =< \nu_x, id >$, ν_x is the Young measures generated by a sequence Du_k.

Lemma 3.1 *Let $u_{k:} : \Omega \longrightarrow I\!R^m$ a sequence of measurable functions such that:*

$$\sup_{k\in I\!N} \int_\Omega |u_k|^s dx < +\infty \quad \textit{for some } s > 0. \tag{3.1}$$

We suppose that for each $\alpha > 0$ the sequence of truncated functions $\{T_\alpha(u_k)\}_{k\in I\!N}$ is precompact in $L^1(\Omega, I\!R^m)$. Then there exists a measurable function u on Ω such that for a subsequence $u_k \longrightarrow u$ in measure.

Proof
Choose a subsequence of $\{u_k\}$ (not relabeled) which generates a Young measure $\{\vartheta_x\}_{x\in\Omega}$. By 3.1 and Theorem (Young, Tartar, Ball) the measure ν_x are probability measure for almost every a $x \in \Omega$ and $T_\alpha(u_k) \longrightarrow v_\alpha =< \nu_x; T_\alpha >$, weakly in $L^1(\Omega, I\!R^m)$ and in fact strongly since $T_\alpha(u_k)$ is precompact in L^1. Consequently there exists a subsequence such that: $T_\alpha(u_{k_l}) \longrightarrow v_\alpha$ almost uniformly, i-e:

$$T_\alpha(u_{k_l}) \longrightarrow v_\alpha \quad \text{uniformly up to a set of arbitrary small measure.} \tag{3.2}$$

Let $M_\alpha = \{x \in \Omega : |v_\alpha(x)| < \alpha\}$. Then for each $\epsilon > 0$ and $\delta > 0$ there exists a set E_ϵ of measure $\text{meas}(E_\epsilon) < \epsilon$ and an index $l_0(\epsilon; \delta)$ such that: $|T_\alpha(u_{k_l})| < |v_\alpha(x)| + \delta$ for all $x \in M_\alpha \setminus E_\epsilon$ and all $l > l_0$.

It follows that $u_{k_l}(x) \longrightarrow v_\alpha(x)$ for almost every$x \in M_\alpha \setminus E_\epsilon$ consider first $x \in M_\beta$; $\beta < \alpha$ and then the union over $\beta < \alpha$). Since $\varepsilon > 0$ was arbitrary it follows that $\nu_x = \delta_{v_\alpha}(x)$ for almost every $x \in M_\alpha$ In view of the Ball's theorem it suffices to show that $\cup M_\alpha$ has full measure. Now clearly $M_\alpha \subset M_\beta$ for $\alpha < \beta$ since $T_\beta(u_{k_l}) \longrightarrow T_\beta(v_\alpha) = v_\alpha$ almost everywhere in M_α and therefore$v_\alpha = v_\beta$ on M_α. By (3.2) there exists for each $\epsilon > 0$ a set E_ϵ, and an index $l_0(\epsilon, \alpha)$ such that meas $(E_\epsilon) < \epsilon$ and $\mid u_{k_l} \mid \geq \mid T_\alpha(u_{k_l}) \mid \geq \frac{\alpha}{2}$ on $(\Omega \setminus E_\epsilon) \setminus M_\alpha$ for all $l \geq l_0$. In view of (3.2) this implies meas $((\Omega \setminus E_\epsilon) \setminus M_\alpha) \leq \frac{c}{\alpha^s}\epsilon \longrightarrow 0$ we deduce meas$(\Omega \setminus \cup M_\alpha)$=$\lim_{\alpha \longrightarrow \infty} meas(\Omega \setminus M_\alpha) = 0$ □

Lemma 3.2 *Let Ω be a domain in $I\!R^n$ with $|\Omega| < \infty$ and $u_k \in W^{1,1}(\Omega, I\!R^m)$. Suppose that there exist $p > 1$ and $s > 0$ such that:*

$$\sup_k \sum_{i,j} \int_{|u_k| \leq \alpha} \omega_{ij} |D_{ij} u_k|^p dx \leq c(\alpha) < \infty, \ \forall \alpha > 0, \tag{3.3}$$

and $\sup_{k \in I\!N} \int_\Omega |u_k|^s dx \leq c < \infty$. Then there exist a subsequence u_{k_j} and a measurable function $u : \Omega \longrightarrow I\!R^m$ such that $u_{k_j} \longrightarrow u$ in measure. Moreover u is for almost every $x \in \Omega$ approximately differentiable, for all $\eta \in C_0^1(\Omega, I\!R^m)$ there holds $\eta \circ u \in W^{1,p}(\Omega, \omega, I\!R^m)$. if $u_k \in W_0^{1,1}(\Omega, I\!R^m)$ then $\eta \circ u \in W_0^{1,1}(\Omega, I\!R^m) \cap W^{1,p}(\Omega, \omega, I\!R^m)$ provided that $\eta = id$ on $B(0, r)$ for some $r > 0$.

Proof
Choose

$$(u_k)_\alpha = \begin{cases} u_k & \text{if} \quad |u_k| \leq \alpha, \\ 0 & \text{if} \quad |u_k| > \alpha. \end{cases}$$

For the hypotheses:

$$\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}(u_k)_\alpha|^p dx = \sum_{i,j} \int_{|u_k| \leq \alpha} \omega_{ij} |D_{ij} u_k|^p dx \leq c(\alpha) < \infty.$$

Then, $(u_k)_\alpha \in W_0^{1,1}(\Omega, w, I\!R^m)$ and for (1.5), (H_4) and $|D|u|| \leq |Du|$ we have:

$$\int_\Omega |DT_\alpha(|u_k|)|^{p_s} dx = \int_{|u_k| \leq \alpha} |D|u_k|)|^{p_s} dx \leq$$

$$\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}(u_k)_\alpha|^p dx \leq c(\alpha) < +\infty$$

Hence by the compact Sobolev embedding $W_s^{1,p_s}(\Omega) \hookrightarrow\hookrightarrow L^{p_s}(\Omega)$, we have $\{T_\alpha(|u_k|)\}$ is precompact in $L^1(\Omega)$. And, if $\eta \in C_0^\infty(B(0, 3\alpha), I\!R^m)$ a symmetric radial such that $\eta = id$ on $B(0, 2\alpha)$, then by (1.3) and (3.3): $\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}(\eta(u_k))|^p dx = \sum_{ij} \int_{|u_k| \leq \alpha} \omega_{ij} |D_{ij}(u_k)|^p dx + \sum_{i,j} \int_{\alpha < |u_k| \leq 2\alpha} \omega_{ij} |D_{ij}(id)|^p dx + \sum_{i,j} \int_{2\alpha < |u_k| \leq 3\alpha} \omega_{ij} |D_{ij}(\eta(u_k))|^p dx \leq c(\alpha) + c \sum_{i,j} \parallel \omega_{ij} \parallel_{L^1_{loc}(\Omega)} + c < \infty$. Then, by (1.5), $\eta(u_k)$ is precompact in $L^{p_s}(\Omega, I\!R^m)$, and as in Lemma 8 [2], there exist a measurable function $u : \Omega \longrightarrow I\!R^m$ such that $u_k \longrightarrow u$ in measure, with $u(x) =< \vartheta_x, id >$ for almost every $x \in \Omega$ and u is approximately differentiable because $\eta(u_k) \rightharpoonup \eta(u)$ in $W^{1,P}(\Omega, \omega, I\!R^m)$ and $apDu = ap(\eta \circ u)$.□

Lemma 3.3 *Let u_k be as in Lemma (3.2) with $p \succ 1$. Then the Young measure ϑ_x generated by (a subsequence of) Du_k has the following properties:*

(a) *ϑ_x is a probability measure for almost every $x \in \Omega$.*

(b) *ϑ_x has finite p_s-th moment for almost every $x \in \Omega$, i-e $\int_{M^{m \times n}} \mid \lambda \mid^{p_s} d\vartheta_x(\lambda)$ is finite for almost every $x \in \Omega$.*

(c) *ϑ_x satisfies $< \vartheta_x, id >= apDu(x)$ almost everywhere in Ω.*

(d) *ϑ_x is a homogeneous $W^{1,P_s}-$ gradient young measure for almost every $x \in \Omega$.*

Proof Let $\widetilde{\vartheta_x}$ denote the Young measure generated by (a subsequence of) the sequence $\{(u_k, Du_k)\}$. By Lemma 3.2 we have :

$$\widetilde{\vartheta_x} = \delta_{u(x)} \otimes \vartheta_x.$$

Let $\eta \in C_0^\infty(B(0,2\alpha),\ I\!R^m)$, $\eta = Id$ on $B(0,\alpha)$, and let ϑ^η be the Young measure generated by

$$D(\eta \circ u_k) = (D\eta)(u_k)Du(x),$$

then ϑ^η is a probability measure, has finite $p-$th moment and

$$< \vartheta^\eta, Id >= (D(\eta \circ u))(x) = D\eta(u(x))apDu(x).$$

It follows for $\varphi \in C_0^\infty(M^{m\times n})$, that:

$$\varphi(D(\eta \circ u_k)) \rightharpoonup < \vartheta^\eta, \varphi >= \int_{M^{m\times n}} \varphi(\lambda) d\vartheta_x^\eta(\lambda).$$

Based on the proof (3.2), we have $\sum_{i,j} \int_\Omega |\omega_{ij} D_{ij}(\eta \circ u_k)|^p dx \prec \infty$, and by (1.5) $\sup_{k\in I\!N} \int_\Omega |D(\eta \circ u_k)|^{p_s} dx \prec \infty$, and the (Ball's Theorem, proof lemma 9 [2]) we conclude: (a)-(b)-(c)-and (d) □

4. Approximate Solutions and a Priori Bounds

We introduce the following approximating problems:

$$-div\sigma(x, u_k, Du_k) = f_k + g(x, u_k, Du_k)\ in\ \Omega. \tag{4.1}$$

$$u_k = 0\ \ on\ \partial\Omega. \tag{4.2}$$

With $f_k \in W^{-1,p'}(\Omega, \omega^*, I\!R^m) \cap L^1(\Omega, I\!R^m)$ and $f_k \rightharpoonup^* \mu\ in\ M(\Omega, I\!R^m)$ such that: $\| f_k \|_{L^1(\Omega,I\!R^m)} \leq \| \mu \|_{M(\Omega,I\!R^m)}$. By [7] and [6], and using the assumptions: (H_0), (H_1),(H_2), (H_4),G_0 and G_0, the problem (4.1)-(4.2) has a solution u_k with $u_k \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ u_k is the subsequence approximates solutions of (1.1)-(1.2). The results of theorems (2.1) is the consequence of the following proposition:

Proposition 4.1 *Let, $f \in L^1(\Omega, I\!R^m)$ and σ satisfies (H_0), the coercivity of $(H_1) - (H_3)$ and g satisfies $(G_0) - (G_1)$. If $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ is a solution of:*

$$-div\sigma(x, u, Du) = f + g \quad\ in\ \Omega. \tag{4.3}$$

in the sense of distributions. Then

$$u \in L^{t^*_{p_s},\infty}(\Omega, I\!R^m), \qquad Du \in L^{t_{p_s},\infty}(\Omega, I\!R^m)$$

and

$$\| u \|^*_{L^{t^*_{p_s},\infty}(\Omega,I\!R^m)} + \| Du \|^*_{L^{t_{p_s},\infty}(\Omega,M^{m\times n})}$$

$$\leq C\Big(|\Omega|, \| \lambda_1 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\theta})'}(\Omega)}, \| f \|_{L^1(\Omega;I\!R^m)}, \| b_1 \|_{L^{p'}}\Big) \sum_{i,j} \int_{|u|\leq\alpha} \omega_{ij} |D_{ij}u|^p dx \leq M\alpha + L, \quad \forall \alpha \succ 0,$$

M and L are the constants dependents on:

$$\| \lambda_1 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\theta})'}(\Omega)}; \| f \|_{L^1(\Omega,I\!R^m)}, c_2, \| b_1 \|_{L^{p'}}$$

Proof i)- We suppose the condition of l'angle in (H_3). Let $\alpha \succ 0$. Testing $T_\alpha(u)$ in (4.3) and we use the coercivity condition in (H_1), the growth condition in (G_1), the Hardy type inequality in (H_4) and Hölder inequality, we have:

$$c_2.\sum_{i,j}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u|^p dx \le \alpha \parallel f \parallel_{L^1(\Omega,I\!R^m)} + \alpha\sum_j(\int_{|u|\le\alpha}\omega_{0j})^{\frac{1}{p}}\left(\parallel b_1 \parallel_{L^{p'}} + \left(\sum_j\int_{|u|\le\alpha}\gamma_j \mid u_j\mid^q dx\right)^{\frac{1}{p'}}\right)$$
$$+ \parallel \lambda_2 \parallel_{L^1(\Omega)} + c \parallel \lambda_3 \parallel_{L^{(\frac{p}{\theta})'}}\left(\sum_{j=1}^m\int_{|u|\le\alpha}\gamma_j \mid u_j\mid^q dx\right)^{\frac{\theta}{p}} \tag{4.4}$$

Choose :

$$(u)_\alpha = \begin{cases} u & \text{if} \quad |u|\le\alpha, \\ 0 & \text{if} \quad |u|>\alpha. \end{cases}$$

Then $u \in W_0^{1,p}(\Omega,\omega,I\!R^m)$ because $u \in W_0^{1,p}(\Omega,\omega,I\!R^m)$ and by Hardy-Type inequality:

$$\sum_j^m\int_{|u|\le\alpha}\gamma_j|u_j|^q dx = \sum_j^m\int_{|u|\le\alpha}\gamma_j|(u_\alpha)_j|^q dx$$
$$\le c.\left(\sum_{i,j}\int_\Omega\omega_{ij}|D_{ij}u_\alpha|^p dx\right)^{\frac{q}{p}}$$
$$\le c.\left(\sum_{i;j}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u_\alpha|^p dx\right)^{\frac{q}{p}}$$

By (4.4)

$$c_2.\left(\sum_{ij}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u|^p dx\right) \le \alpha \parallel f \parallel_{L^1(\Omega,I\!R^m)} + \alpha\sum_j(\int_{|u|\le\alpha}\omega_{0j})^{\frac{1}{p}}(\parallel b_1 \parallel_{L^{p'}} + c')$$
$$+ \parallel \lambda_2 \parallel_{L^1(\Omega)} + c \parallel \lambda_3 \parallel_{L^{(\frac{p}{\theta})'}} . \left(\sum_{i,j}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u|^p dx\right)^{\frac{\theta q}{p^2}}$$

and $\frac{\theta q}{p^2} \prec 1$. Then

$$\left(\sum_{ij}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u|^p dx\right) \le c.\left(\alpha \parallel f \parallel_{L^1(\Omega,I\!R^m)} + \alpha\sum_j(\int_{|u|\le\alpha}\omega_{0j})^{\frac{1}{p}}(\parallel b_1 \parallel_{L^{p'}} + c') + \parallel \lambda_2 \parallel_{L^1(\Omega)}\right) \le M\alpha + L, \tag{4.5}$$

with $L = L(c, \parallel \lambda_2 \parallel_{L^1(\Omega)}, c \parallel \lambda_3 \parallel_{L^{(\frac{p}{\theta})'}})$ and $M = M(c_1, c_2 \parallel \lambda_3 \parallel, \parallel f \parallel_{L^1(\Omega,I\!R^m)})$, we choose:
$u^\alpha = min(|u|,\alpha)$, then by $|D|u|| \le |Du|$

$$\int_\Omega|Du^\alpha|^{p_s}dx = \int_{|u|\le\alpha}|D|u||^{p_s}dx + 0 \le \int_{|u|\le\alpha}|Du|^{p_s}dx = \int_\Omega|Du_\alpha|^{p_s}dx$$

$\le \left(\sum_{i,j}\int_\Omega\omega_{ij}|D_{ij}u_\alpha|^p dx\right)^{\frac{p_s}{p}} = \left(\sum_{i,j}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u|^p dx\right)^{\frac{p_s}{p}}$ And by (4.5), and $p_s \le p$, we have:

$$\int_\Omega|Du^\alpha|^{p_s}dx \le c.(\alpha \parallel f \parallel_{L^1(\Omega,I\!R^m)} + \alpha\sum_j(\int_{|u|\le\alpha}\omega_{0j})^{\frac{1}{p}}(\parallel b_1 \parallel_{L^{p'}} + c') + \parallel \lambda_2 \parallel_{L^1(\Omega)}). \tag{4.6}$$

By (1.5) and (4.6), we have:

$$\int_\Omega |u^\alpha|^{p_s^*}dx \le c.\left(\int_\Omega |Du^\alpha|^{p_s}dx\right)^{\frac{p_s^*}{p}} \le c.\left(\alpha \parallel f \parallel_{L^1(\Omega;\mathbb{R}^m)} +\alpha\sum_j(\int_{|u|\le\alpha}\omega_{0j})^{\frac{1}{p}}\,(\parallel b_1\parallel_{L^{p'}}+c')+\parallel \lambda_2\parallel_{L^1(\Omega)}\right)^{\frac{p_s^*}{p}} \tag{4.7}$$

Then:

$$\begin{aligned}\lambda_{|u|}(\alpha) = \alpha^{-p_s^*}\textstyle\int_{|u|\succ\alpha}\alpha^{p_s^*}dx \le \alpha^{-p_s^*}\int_{|u|\succ\alpha}|u^\alpha|^{p_s^*}dx\\ \le c.\alpha^{-p_s^*}\left(\alpha \parallel f \parallel_{L^1(\Omega,\mathbb{R}^m)} +\alpha\textstyle\sum_j(\int_{|u|\le\alpha}\omega_{0j})^{\frac{1}{p}}\,(\parallel b_1\parallel_{L^{p'}}+c')+\parallel \lambda_2\parallel_{L^1(\Omega)}\right)^{\frac{p_s^*}{p}}\end{aligned} \tag{4.8}$$

and we continue in the same way as in a case that is non-degenerated [2] by replacing p by p_s as well as

$$\begin{aligned}\|u\|^*_{L^{t^*_{p_s}},\infty}(\Omega,\mathbb{R}^m) &= \sup_{\alpha\succ 0}\alpha|\lambda_{|u|}(\alpha)|^{\frac{1}{t^*_{p_s}}}\\ &\le |\Omega| + \sup_{\alpha\succ 1}\alpha|\lambda_{|u|}(\alpha)|^{\frac{1}{t^*_{p_s}}}\\ &\le |\Omega| + c.(\parallel f\parallel^{\frac{1}{p_s-1}}_{L^1(\Omega,\mathbb{R}^m)},\parallel b_1\parallel_{L^1},\parallel\lambda_2\parallel^{\frac{1}{p_s-1}}_{L^1(\Omega)})\end{aligned}$$

i-e:

$$\|u\|^*_{L^{t^*_{p_s}},\infty}(\Omega,\mathbb{R}^m) \le c.\left(|\Omega|,\parallel b_1\parallel_{L^1},\parallel\lambda_2\parallel_{L^1(\Omega)},\parallel\lambda_3\parallel_{L^{(\frac{p}{\theta})'}(\Omega)};\, c_2,\parallel f\parallel_{L^1(\Omega,\mathbb{R}^m)}\right), \tag{4.9}$$

on the other hen, by using ($p_s \le p$) and thinks to 1.5, we obtain:

$$\begin{aligned}\lambda_{|Du|}(s) &\le s^{-p_s}\int_{|u|\le\alpha}|Du|^{p_s}dx+\lambda_{|u|}(\alpha)\\ &= s^{-p_s}\int_{|u|\le\alpha}|Du_\alpha|^{p_s}dx+\lambda_{|u|}(\alpha)\\ &\le s^{-p_s}\left(\sum_{i,j}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u_\alpha|^{p}dx\right)+\lambda_{|u|}(\alpha)\\ &\le s^{-p_s}\left(\sum_{i,j}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u|^{p}dx\right)+\lambda_{|u|}(\alpha)\end{aligned}$$

By4.5 and 4.8:

$$\lambda_{|Du|}(s) \le c.\left(\max(\frac{\alpha}{s^{p_s}},\frac{1}{s^{p_s}})+\max(\alpha^{-p_s^*},\alpha^{\frac{p_s^*}{p_s}-p_s^*})\right)$$

or $-t^*_{p_s}=\frac{p_s^*}{p_s}-p_s^*$, so as in [3]

$$\parallel Du\parallel^*_{L^{t_{p_s}},\infty(\Omega,M^{m\times n})}\le c.\left(|\Omega|,\parallel\lambda_2\parallel_{L^1(\Omega)},\parallel\lambda_3\parallel_{L^{(\frac{p}{\theta})'}(\Omega)},c_2,\parallel b_1\parallel_{L^1},\parallel f\parallel_{L^1(\Omega;\mathbb{R}^m)}\right). \tag{4.10}$$

From (4.5)-(4.9) and (4.10), we obtain the result of the proposition (4.1) in case i). ii)-Suppose the condition (of l'angle in H_3): Let $S_\alpha(y)=(T_\alpha(y_1);T_\alpha(y_2);.......T_\alpha(y_m))$, $y\in\mathbb{R}^m$,the cubic truncation, we have $Ds_\alpha(y)=Id$ if $|y|_{max}=\max_{1\le i\le m}|y_i|\le\alpha$, in the same way as in i)- by testing $S_\alpha(u)$ in (4.3). Then $\int_\Omega\sigma(x,u,Du):D(S_\alpha(u))dx=\int_\Omega f.S_\alpha(u)dx+\int_\Omega g.S_\alpha(u)dx$
or

$$\int_\Omega\sigma(x,u,Du):D(S_\alpha(u))dx=\sum_{i=1}^m{}_{|u_i|\le\alpha}\sigma_i(x,u,Du):Du_idx$$

$$\ge\int_{|u|=\max_{1\le j\le m}(|u_j|)}\sum_{i=1}^m\sigma_i(x,u,Du):Du_idx$$

and like $\sum_{i=1}^{m}\sigma_i(x,u,Du):Du_i dx=\sigma(x,u,Du):Du$. By the coercivity condition in (H_1) and the Hölder Inequality we obtain:

$$c_2.\sum_{i,j}\int_{|u|\le\alpha}\omega_{ij}|D_{ij}u|^p dx \le \sqrt{m}.\alpha(\parallel f \parallel_{L^1(\Omega;I\!R^m)})+\alpha\sum_j(\int_{|u|\le\alpha}\omega_{0j})^{\frac{1}{p}}(\parallel b_1\parallel_{L^{p'}}+c')+\parallel \lambda_2 \parallel_{L^1(\Omega)}$$

$$+c.\parallel\lambda_3\parallel_{L^{(\frac{p}{\theta})'}(\Omega)}\left(\sum_{j=1}^{m}\int_{|u|\le\alpha}\gamma_j|u_j|^q dx\right)^{\frac{q}{p}}$$

and we continue in the same way as in i), this completes the proof of the proposition (6) $\square$

5. A Div-Curl Inequality

The result of this section is the key ingredient for the proof that one can pass to the limit in the equation (4.1) for the solution $\{u_k\}_{k\in I\!N}$ of approximating problems. Since it is independent of the differential equation we state it a more general form using only the hypotheses (5.1)-(5.8) below:

$$\sigma;\tau:\Omega\times I\!R^m\times M^{m\times n}\longrightarrow M^{m\times n}, \tag{5.1}$$

is a Carathéodory function.

$$\sigma \text{ and } \tau \text{ satisfing one of the fellowing conditions:} \tag{5.2}$$

(i) $\sigma(x,u,F):MF\ge 0$, $\tau(x,u,F):MF\ge 0$; $M=Id-b\otimes b\in M^{m\times n}$, with $|b|\le 1$.

(ii) $\sigma_j(x;u;F):F_j\ge 0$, and $\tau_j(x,u,F):F_j\ge 0$; $1\le j\le m$, σ_j, τ_j and F_j is the j^{eme} colonnes of $\sigma,\ \tau,\ F$.

$$u_k\in W^{1;1}(\Omega,I\!R^m) \text{ and there exists an } s\ge 0 \text{ such that } \int_\Omega |\ Du_k\ |^s\,dx\le c \text{ uniformly in } k \tag{5.3}$$

$$\text{The sequence } \sigma_k(x)=\sigma(x,u_k,Du_k) \text{ is equiintegrale.} \tag{5.4}$$

$$\begin{gathered}\text{The sequence } u_k \text{ converges in measure to some function } u,\\ \text{and } u \text{ is almost everywhere approximately differentiable.}\end{gathered} \tag{5.5}$$

$$\text{The sequence } f_k=-div(\sigma_k+\tau_k)-\mu \text{ is bounded in } L^1(\Omega,I\!R^m). \tag{5.6}$$

$$\begin{aligned}D_{ij}u_k\in L^r_{loc}(\Omega,\omega_{ij},M^{m\times n}) \text{ and } (\sigma_k+\tau_k)\in L^{r'}_{loc}(\Omega,\omega^*,M^{m\times n}),\ \text{for some}\\ 1\le r\prec\infty \text{ and } 1\le i\le n,\ 1\le j\le m.\end{aligned} \tag{5.7}$$

$$\text{The sequence } \tau_k(x)=\tau[x](x,u,Du_k) \text{ converges to weakly to 0 in } L^1(\Omega,M^{m\times n}). \tag{5.8}$$

Lemma 5.1 *Suppose (5.1)-(5.8). Then (after passage to a subsequence) the sequence σ_k converges weakly i $L^1(\Omega,M^{m\times n})$ and the weak limit $\overline{\sigma}$ is given by $\overline{\sigma}(x)=<\nu_x;\sigma(x,u(x),\ .\ >$. Moreover the following inequality holds:*

$$\int_{M^{m\times n}},\sigma(x,u(x),\lambda):\lambda d\nu_x(\lambda)\le\overline{\sigma}(x):apDu(x) \text{ for a.e. } x\in\Omega. \tag{5.9}$$

Proof See [3] $\square$

6. Passage to the Limit

Proposition 6.1 *Suppose that the sequence $(u_k)_{k\in I\!N}$ satisfies the hypotheses (5.1)-(5.7), (H_2) and that the Young measure ν generated by the sequence $(Du_k)_{k\in I\!N}$ satisfies: a)-c) and d)- in lemma (3.3). Then the sequence (σ_k) is weakly converge in $L^1(\Omega,M^{m\times n})$, with $\overline{\sigma}$ is the limit and $\overline{\sigma(x)}=<\nu_x,u(x),apDu(x)>$. If in H_2 b)- c)-or d)-holds, $\sigma(x,u_k,Du_k)\to\sigma(x,u,apDu)$ strongly in $L^1(\Omega,M^{m\times n})$.*
In the cases (c) and (d) it follows addition that $Du_k\to apDu$ in measure.

Proof See [3].

Proof of the theoreme2.1

For using the results of proposition (6.1): we assume that (5.1)-(5.7) and the Young measure ν_x generated by the sequence Du_k satisfies :(i), (ii) and (iii) in Lemma(3.3), for the approximate systems (4.1)-(4.2). By the proposition 6.1, with $u_k \in W_0^{1,p}(\Omega, w, I\!R^m)$, we have: $\| u_k \|_{L^{t^*_{p_s},\infty}(\Omega,I\!R^m)} \leq c.\left(|\Omega|, \| \lambda_2 \|_{L^1(\Omega)}, \| \lambda_1 \|_{L^{p'}}, \| \lambda_3 \|_{L^{\left(\frac{p}{\theta}\right)'}(\Omega)}, c_2, \| \mu \|_{M(\Omega,\omega^*,I\!R^m)}\right)$, and

$$\sum_{i,j} \int_{|u_k|\leq \alpha} \omega_{ij} |D_{ij} u_k|^p dx \leq M\alpha + L < \infty. \tag{6.1}$$

By $L^{t^*_{p_s},\imath\infty}(\Omega, I\!R^m) \hookrightarrow L^p(\Omega, I\!R^m)$ for all $1 \prec p \prec t^*_{p_s}$, then

$$\| u_k \|_{L^p(\Omega,I\!R^m)} \leq c < \infty. \tag{6.2}$$

Now

- (5.1) is (H_0)
- (5.2) is (H_3)
- (5.3): $u_k \in W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow W_0^{1,p_s}(\Omega, I\!R^m)$ with $p_s > 1$, then $u_k \in W^{1,1}(\Omega, I\!R^m)$. Moreover, by the proposition

$$\| Du_k \|_{L^{t_{p_s},\infty}(\Omega,I\!R^m)} \leq c.\left(|\Omega|, \| \lambda_2 \|_{L^1(\Omega)}, \lambda_1 \|_{L^{p'}}, \| \lambda_3 \|_{L^{\left(\frac{p}{\theta}\right)'}(\Omega)}, c_2, \| \mu \|_{M(\Omega,\omega^*,I\!R^m)}\right)$$

  hence

$$\| Du_k \|_{L^s(\Omega,M^{m\times n})} \leq c > \infty, \ \forall\ 1 < s < t_{p_s}$$

  with $\sup_{k\in I\!N} \int_\Omega |Du_k|^s dx < \infty$.

- (5.4): Let A a measurable in Ω, by (H_1) and Hölder we have

$$\begin{aligned} \int_A |\sigma(x, u_k, Du_k)| dx &\leq c\left(\sum_{r,s} \int_\Omega \omega_{rs} dx\right)^{\frac{1}{p}} . \Bigg\{ \| \lambda_1 \|_{L^{p'}(\Omega)} + \\ &\left(\sum_{j=1}^m \int_\Omega \gamma_j |(u_k)_j|^q dx\right)^{\frac{1}{p'}} + \left(\sum_{i;j} \int_\Omega \omega_{ij} |D_{ij} u_k|^p dx\right)^{\frac{1}{p'}} \Bigg\}, \end{aligned}$$

  and with (1.5) and (1.3):

$$\begin{aligned} \int_A |\sigma(x, u_k, Du_k)| dx &\leq c\left(\sum_{r,s} \| \omega_{rs} \|_{L^1_{loc}(\Omega)}^{\frac{1}{p}}\right) \\ &\times \left\{ \| \lambda_1 \|_{L^{p'}(\Omega)} + \| u_k \|_{1,p,\omega}^{\frac{q}{pp'}} + \| u_k \|_{1,p,\omega}^{\frac{p}{p'}} \right\} < \infty. \end{aligned}$$

- (5.5): By (6.1) and (6.2) and Lemma (3.2).
- (5.6): $\| f_k \|_{L^1(\Omega,I\!R^m)} \leq \| \mu \|_{M(\Omega,\omega^*,I\!R^m)}$. And By (G_1) we have

$$\int_\Omega | g(x, u_k, Du_k) | dx \leq < \infty (by Holder and G_1)$$

- (5.7): $\forall \varepsilon \succ 0$ and $x_0 \in \Omega$ $\int_{B(x_0,\varepsilon)} |D_{ij}u_k|^p \omega_{ij} dx \leq \| u_k \|^p_{1,p,\omega} < \infty$. And by (H_3)

$$\begin{aligned}\int_{B(x_0,\varepsilon)} |\sigma_{rs}(x,u_k,Du_k)|^{p'} \omega^*_{rs} dx &= \int_{B(x_0,\varepsilon)} |\sigma_{rs}(x,u_k,Du_k)|^{p'} \omega_{rs}^{1-p'} dx \\ &\leq c. \int_{B(x_0,\varepsilon)} w^{1-p'+\frac{p'}{p}} \left[|\lambda_1|^{p'} + \sum_{j=1}^{m} \gamma_j |(u_k)_j|^q + \sum_{i,j} \omega_{ij} |D_{ij}u_k|^p \right] dx \\ &\leq c \left(\| \lambda_1 \|^{p'}_{L^{p'}(\Omega)} + \| u_k \|^{\frac{q}{p}}_{1,p,w} + \| u_k \|^p_{1,p,\omega} \right) < \infty.\end{aligned}$$

Then, by the proposition (6.1) $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,apDu)$ in $L^1(\Omega, I\!M^{m\times n})$ and $\forall \varphi \in C_0^\infty(\Omega, I\!R^m)$; $D\varphi \in L^\infty(\Omega, I\!M^{m\times n})$ hence:

$$\int_\Omega \sigma(x,u_k,Du_k) : D\varphi dx \longrightarrow \int_\Omega \sigma(x,u,apDu) : D\varphi dx$$

i.e:

$$-div\sigma(x,u_k,apDu_k) \longrightarrow -div\sigma(x,u,apDu)$$

In the sense of distributions. Moreover, since $u_k \to u$ in measure, it follows that (at least for a subsequence) $u_k \to u$ almost everywhere and hence that $g(x,u_k,Du_k) \to g(x,u,Du)$ almost everywhere from the continuity condition (G_0). Since $g(x,u_k,Du_k)$ is equi-integrable by the growth condition in G_1 and the uniform bounded 6.1- 6.2, we may infer that $g(x,u_k,Du_k) \to g(x,u,Du)$ in $L^1(\Omega, I\!R^m)$ by the Vitali's converge theorem. On the other hand $f_k \xrightarrow{*} \mu$ in $L^1(\Omega, I\!R^m)$. Then $\int_\Omega f_k.\varphi dx \longrightarrow \int_\Omega \mu.\varphi dx \quad \forall \varphi \in C_0^\infty(\Omega, I\!R^m)$ so μ is the solution in $W_0^{1,p}(\Omega,\omega,I\!R^m)$ of the system:
$-div\sigma(x,u,apDu) = \mu$ in Ω
$u = 0$ on $\partial\Omega$
to show the estimation (2.1), we take the function η in $C_0^1(B(0,2\alpha), I\!R^m)$; $\eta = Id$ in $B(0,\alpha)$ and $|D\eta| \leq c$, then:

$$\begin{aligned}\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}\eta(u_k)|^p dx &= \sum_{i,j} \int_\Omega \omega_{ij} |(D_{ij}\eta)(u_k)|^p |Du_k\|^p dx \\ &\leq c^p. \sum_{ij} \int_{|u_k|\leq\alpha} \omega_{ij} |D_{ij}u_k)| dx + c. \sum_{i,j} \int_{|u_k|\leq 2\alpha} \omega_{ij} D_{ij}u_k|^p dx \\ &\leq c.c(\alpha) + c.c(2\alpha) < \infty,\end{aligned}$$

thanks to (6.1).
Now, we have $\eta(u_k) \longrightarrow \eta(u)$, for every $x \in \Omega$ because η is C^∞. Then $\eta(u_k) \rightharpoonup \eta(u)$, in $W_0^{1,p}(\Omega,\omega,I\!R^m)$, and $apDu = apD(\eta \circ u)$ on $\{|u| \prec \alpha\}$. Hence,

$$\begin{aligned}\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}(\eta\circ u)|^p dx &\leq \liminf_{k\longrightarrow\infty} \int_\Omega w |D(\eta\circ u_k)|^p dx \\ &\leq \liminf_{k\longrightarrow\infty} \sum_{ij} \int_{|u_k|\leq 2\alpha} |D_{ij}\eta(u_k)|^p |D_{ij}u_k| \omega_{ij} dx \\ &\leq \leq c. \liminf_{k\longrightarrow\infty} \int_{|u_k|\leq 2\alpha} \omega_{ij} |D_{ij}u_k)|^p dx \\ &\leq c.c(2\alpha) < \infty.\end{aligned}$$

Then:

$$\sum_{i,j} \int_{|u|\leq\alpha} \omega_{ij} |apDu|^p dx = \sum_{i,j} \int_{|u_k|\leq 2\alpha} \omega_{ij} |D(\eta\circ u)|^p dx < \infty,$$

in the same as in the proof of the proposition (6) by replacing u_k by u and f_k by μ, we obtain the estimation (2.1) and this completes the proof of the theorem 2.1 □

Case: $0<\theta<\frac{n(p_s-1)}{n-1}$ (the general case) The idea is to consider the regularized problems:

$$-div\phi_\varepsilon(x,u_\varepsilon,Du_\varepsilon)=\mu \text{ in } \Omega, \tag{6.3}$$

$$u_\varepsilon=0 \text{ on } \partial\Omega \tag{6.4}$$

With

$$\phi_{\varepsilon,r,s}(x,u,F)=\sigma_{rs}(x,u,F)+\varepsilon\beta\left(\sum_{ij}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{s-2}\right)\omega_{rs}^{\frac{1}{p}}F_{rs},$$

$\forall 1\le r\le n$, $\forall 1\le s\le m$ with $s>n+1$, and $\varepsilon<\frac{1}{2}$, we have $p<s$, then $s'<p'$, and $(\frac{s}{\theta})'<(\frac{p}{\theta})'$. Moreover $\exists c>0$ which doesn't depend on p, s, such that $\omega_{rs}^{\frac{1}{p}}\le c.\omega_{rs}^{\frac{1}{s}}$ $\forall\ 1\le r\le n$ and $1\le s\le m$.
By (H_1) for σ, we obtain

$$\begin{aligned}|\phi_{\varepsilon,r,s}(x,u,F)| &\le \beta'.|\omega_{rs}|^{\frac{1}{p}}\left[\lambda_1+\textstyle\sum_{j=1}^m\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}+\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{\theta}\right]\\ &\quad+\varepsilon\beta\omega_{rs}^{\frac{1}{p}}\left(\textstyle\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{s-1}\right)\left(\theta<\frac{n(p_s-1)}{n-1}<n(s-1)\right)\\ &\le\ \le\beta'\omega_{rs}^{\frac{1}{p}}\left[\lambda_1+\textstyle\sum_{j=1}^m\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}+\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{s-1}\right].\end{aligned}$$

And $p<s$, then $\frac{1}{p'}<\frac{1}{s'}$ and like $\omega_{rs}^{\frac{1}{p}}\le c.\omega_{rs}^{\frac{1}{s}}$, then:
$|\phi_{\varepsilon,r,s}(x,u,F)|\le\beta'.|\omega_{rs}|^{\frac{1}{s}}\left[\lambda_1+\sum_{j=1}^m\gamma_j^{\frac{1}{s'}}|u_j|^{\frac{q}{s'}}+\sum_{ij}\omega_{ij}^{\frac{1}{s'}}|F_{ij}|^{s-1}\right]$, and by (H_3), we conclude that

$$\phi_\varepsilon(x,u,F):F=\sigma(x,u,F):F+\varepsilon\sum_{i,j,r,s}\omega_{ij}^{\frac{1}{p'}}\omega_{rs}^{\frac{1}{p}}|F_{ij}|^{s-2}F_{ij}.F_{rs}$$

$\ge-\lambda_2-\sum_{j=1}^m\lambda_3\gamma_j^{\frac{q}{s}}.|u_j|^{\frac{q\alpha}{s}}+\varepsilon\sum_{ij}\omega_{ij}|F_{ij}|^s$.
On the other hand, $0<\alpha<p-1<s-1$, $1<q<\frac{p^2}{\alpha}<\frac{s^2}{\alpha'}$, $\lambda_1\in L^{p'}(\Omega)\hookrightarrow L^{s'}(\Omega)$, and $\lambda_3\in L^{(\frac{p}{\alpha})'}(\Omega)\hookrightarrow L^{(\frac{s}{\alpha})'}(\Omega)$ and as σ_ε verifies the conditions of the structures (of l'angle and sign), the strict monotony, the s-quasi monotonous with regard to F is a C^1 monotony in relation with F or accepting a convex potential because: $F\longrightarrow\varepsilon\beta\left(\sum_{ij}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{s-2}\right)\omega_{rs}^{\frac{1}{p}}F_{rs}$ verify them as well, hence σ_ε verifies the hypotheses $(H_0)-(H_5)$, for the regularized problems (6.3)-(6.4), thus for the previous case, $\theta=s-1$, of theorem 2.1, there exists a solution, $u_\varepsilon\in W_0^{1,s}(\Omega,\omega,I\!R^m)$, of the system (6.3)-(6.4). Now showing that the conditions: i), ii) and iii), of lemma (3.3), and the hypotheses (5.1)-(5.8) of the div-curl inequality are verified for u_ε with order s in the place of p.
We suppose the condition of l'angle verifying that ϕ_ε by testing, $T_\alpha(u_\varepsilon)$ $\alpha\succ 0$ in (5.3)-(5.4), we get: $\int_\Omega\phi_\varepsilon(x,u_\varepsilon,Du_\varepsilon):DT_\alpha(u_\varepsilon)dx=\int_\Omega f.T_\alpha(u_\varepsilon)dx$, so

$$\int_{|u_\varepsilon|\le\alpha}\sigma(x,u_\varepsilon,Du_\varepsilon):Du_\varepsilon dx+\int_{|u_\varepsilon|>\alpha}\frac{\alpha}{|u_\varepsilon|}\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon):\left(Id-\frac{u_\varepsilon}{|u_\varepsilon|}\otimes\frac{u_\varepsilon}{|u_\varepsilon|}\right)Du_\varepsilon dx$$

$$+\varepsilon\beta\int_{|u_\varepsilon|\le\alpha}\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\sum_{r,s}\omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2dx$$

$$+\varepsilon\beta\int_{|u_\varepsilon|>\alpha}\sum_{ij}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\sum_{r,s}\omega_{rs}D_{rs}u_\varepsilon\left(Id-\frac{u_\varepsilon}{|u_\varepsilon|}\otimes\frac{u_\varepsilon}{|u_\varepsilon|}\right)$$

$$\leq \alpha. \parallel f \parallel_{L^1(\Omega,\mathbb{R}^m)} .$$

since

$$\sum_{rs} |D_{rs}u_\varepsilon|^{s-2} D_{rs}u_\varepsilon \left(Id - \frac{\alpha}{|u_\varepsilon|} (\frac{u_\varepsilon}{|u_\varepsilon|} \otimes \frac{u_\varepsilon}{|u_\varepsilon|}) \right) \geq 0$$

so

$$\int_{|u_\varepsilon|\leq\alpha} \sigma(x,u_\varepsilon,Du_\varepsilon) : Du_\varepsilon dx \leq \alpha \parallel f \parallel_{L^1(\Omega,\mathbb{R}^m)} .$$

And by the coercivity condition of σ in (H_1) and Hölder inequality, we get as in the proof of the proposition

$$\sum_{ij} \int_{|u_\varepsilon|\leq\alpha} \omega_{ij}|D_{ij}u_\varepsilon|^p dx \leq M'\alpha + L', \tag{6.5}$$

And the following a priori estimation:

$$\parallel u_\varepsilon \parallel^*_{L^{t^*_{p_s};\infty}(\Omega,\mathbb{R}^m)} + \parallel Du_\varepsilon \parallel^*_{L^{t_{p_s},\infty}(\Omega,\mathbb{R}^{m\times n})} < c < \infty, \tag{6.6}$$

and by the injection $L^{\beta',\infty} \hookrightarrow L^{\alpha'}$, $\forall 0<\alpha'<\beta'$, then $\forall$, $0<r<t^*_{p_s}$, $\forall 0<p<t_{p_s}$

$$\parallel u_\varepsilon \parallel_{L^r(\Omega,\mathbb{R}^m)} + \parallel Du_\varepsilon \parallel_{L^p(\Omega,\mathbb{M}^{m\times n})} + \parallel Du_\varepsilon \parallel^*_{L^{t_{p_s},\infty}(\Omega,\mathbb{M}^{m\times n})} < \infty. \tag{6.7}$$

We suppose that the condition of the sign is verify.
As in the same way in the proof of the proposition (6), we test $S_\alpha(u_\varepsilon)$ in (6.3)-(6.4), we obtain (6.5) and (6.7).
Starting with verifying that i), ii) et iii) of lemma (3.3) and the hypotheses (5.1) and(5.7) for σ_ε. By (6.5)and(6.7), the points i), ii) et iii) are a direct consequence of lemma(3.2) and lemma (3.3). On the other hand:
-(5.1): for σ is (H_0) and $\tau_{rs}(x,u,F) = \varepsilon\beta \left(\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{s-2}\right) .\omega_{rs}^{\frac{1}{p}} F_{rs}$ is a Carathéodory function, because $x \longmapsto \omega_{ij}(x)$, is measurable, so σ_ε is a Cathéodory function.
-(5.2)

(i) $\phi_\varepsilon(x,u,F) : MF = \sigma(x,u,F) : MF + \left(\sum_{rs} (\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{s-2}) \omega_{rs}^{\frac{1}{p}} F_{rs}\right) (MF)_{rs} \geq 0,$

with $M = Id - a\otimes a$ and $|a| \leq 1$.

(ii)

$$\begin{aligned} \phi_{rs}(x,u,F).F_j &= \sigma_j(x,u,F) : F_j + \tau_j(x,u,F).F_j \\ &= \sigma_j(x,u,F) : F_j + \sum_{l=1}^{m} \varepsilon\beta \left(\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{s-1}\right) .w_{lj}^{\frac{1}{p}} |F_{lj}|^2 \geq 0, \end{aligned}$$

$\forall 1 \leq j \leq m$.

-(5,3): $u_\varepsilon \in W_0^{1,s}(\Omega,\omega,\mathbb{R}^m) \hookrightarrow W_0^{1,s_s}(\Omega,\mathbb{R}^m)$, $s_s > 1$, so $u_\varepsilon \in W^{1,1}(\Omega,\mathbb{R}^m)$, and by (6.7) $\sup_{\varepsilon>0} \int_\Omega |Du_\varepsilon|^p dx < \infty$, $\forall$, $0<p<t_{p_s}$.
(4.5): $\sigma(x,u_\varepsilon,Du_\varepsilon)$ is equi-integrable as previously $\forall\ \Omega' \subset \Omega$, measurable, we have:

$$\int_{\Omega'} |\sum_{i,j} (\omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-2}) \omega_{rs}^{\frac{1}{p}} D_{rs}u_\varepsilon| dx$$

$$\leq \left(\sum_{i,j} \int_{\Omega'} \omega_{ij}|D_{ij}u_\varepsilon|^{s-1} dx\right)$$

$$\leq c. \sum_{ij} \int_{\Omega'} \omega_{ij}|D_{ij}u_\varepsilon|^s dx \leq c. \parallel u_\varepsilon \parallel^s_{1,s,w}.$$

-(5.5): by (6.7) and the lemma (3.2).
-(5.6): by (6.3), $-div(\sigma_l+\tau_k)-\mu=0$, with $\mu\in M(\Omega,I\!R^m)$ is bounded in $L^1(\Omega,I\!R^m)$.
-(5.7): $\forall\ \varepsilon>0$ and $x_0\in\Omega$, by the growth condition of σ_ε and previously with s in the place of p, $\int_{B(x_{\tilde{A}},\varepsilon)}|\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon)|^s\omega^*_{rs}dx<\infty$
and
-(5.8): $\int_{B(x_{\tilde{A}},\varepsilon)}|D_{ij}u_\varepsilon|^s\omega_{rs}dx<\parallel u_\varepsilon\parallel^\varepsilon_{1,s,w}<\infty$.
Testing that u_ε in (6.3)-(6.4)

$$\varepsilon\beta\int_\Omega\left(\sum_{ij}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\right)\left(\sum_{rs}\omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2\right)dx \\ \leq\parallel u_\varepsilon\parallel_{L^\infty(\Omega,I\!R^m)}\parallel\mu\parallel_{M(\Omega,\omega^*,I\!R^m)} \tag{6.8}$$

We have $W_0^{1,s}(\Omega,w,I\!R^m)\hookrightarrow W_0^{1,s_s}(\Omega,I\!R^m)\hookrightarrow L^\infty(\Omega,I\!R^m)$. Then

$$\begin{aligned}\parallel u_\varepsilon\parallel_{L^\infty(\Omega,I\!R^m)} &\leq c.\left(\sum_{i,j}\int_\Omega\omega_{ij}|D_{ij}u_\varepsilon|^sdx\right)^{\frac{1}{s}}\\ &\leq c.\left(\sum_{i,j}\int_\Omega\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\omega_{ij}^{\frac{1}{p}}|D_{ij}u_\varepsilon|^2dx\right)^{\frac{1}{s}}\\ &\leq c(\int_\Omega(\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}).(\sum_{r,s}\omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2dx)^{\frac{1}{s}}.\end{aligned} \tag{6.9}$$

Thanks to (6.8) and (6.9), we have

$$\int_\Omega\sum_{ij}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\sum_{rs}\omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2dx$$

$\leq\frac{c.\|\mu\|_{M(\Omega,\omega^*,I\!R^m)}}{\varepsilon}\left(\int_\Omega(\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}).(\sum_{rs}\omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2dx\right)$ So:

$$\left(\int_\Omega(\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}).(\sum_{r,s}\omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2dx\right)^{\frac{s-1}{s}}\leq\frac{c\parallel\mu\parallel_M}{\varepsilon},$$

which mean that

$$\left(\int_\Omega(\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}).(\sum_{r,s}\omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2dx\right)^{\frac{1}{s}}\leq\frac{c\parallel\mu\parallel_M}{\varepsilon}, \tag{6.10}$$

and

$$\parallel u_\varepsilon\parallel_{L^\infty(\Omega,I\!R^m)}\leq c.\left(c.\frac{\parallel\mu\parallel_M}{\varepsilon}\right)^{\frac{1}{s-1}}. \tag{6.11}$$

On the other hand and $\forall\ 1 < p < \frac{s}{s-1}$, can write

$$
\begin{aligned}
&\| \epsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-2} \omega_{rs}^{\frac{1}{p}} |F_{rs}| \, \|_{L^{\frac{s}{s-1}}(\Omega, I\!M^{m\times n})} \\
&\leq \varepsilon^{\frac{s}{s-1}} \left(\int_\Omega | \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-1} \omega_{rs}^{\frac{1}{p}} |^{\frac{s}{s-1}} dx \right)^{\frac{s-1}{s}} \\
&\leq c\varepsilon^{\frac{s}{s-1}} \left(| \sum_{i,j} \int_\Omega \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-1} \omega_{rs}^{\frac{1}{p}} |^{\frac{s}{s-1}} dx \right)^{\frac{s-1}{s}} \\
&\leq c.\varepsilon^{\frac{s}{s-1}} \left(\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}u_\varepsilon|^{s-2} \sum_{r,s} \omega_{rs}^{\frac{s}{(s-1)p}} |D_{rs}u_\varepsilon|^2 dx \right) < \infty.
\end{aligned}
$$

thanks to (6.10). Now, since $u_\varepsilon \in W_0^{1,s}(\Omega, \omega, I\!R^m) \hookrightarrow W_0^{1,s_s}(\Omega, I\!R^m) \hookrightarrow W_0^{1,p_s}(\Omega, I\!R^m)$, so by testing $T_\alpha(u_\varepsilon)$ in (6.3)-(6.4), we obtain as in the proof of the proposition (4.1)

$$
\| Du_\varepsilon \|^*_{L^{\frac{n(p_s-1)}{n-1},\infty}(\Omega, I\!M^{m\times n})} \leq c. \tag{6.12}
$$

By the Hölder inequality for the exponent a with a and ξ are the solutions of systems:

$$
\left\{ \begin{array}{rcl} a'\xi = \tau & > & \frac{n(p_s-1)}{n-1} \\ a\left((s-1)\rho - \xi\right) & = & s \end{array} \right.
$$

a given system accepting the solution when $\rho < \frac{s}{s-1}$. So

$$
\int_\Omega |\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-1} \omega_{ij}^{\frac{1}{p}} |^\rho dx
$$

$$
\leq c. \int_\Omega \varepsilon^\rho \left(\sum_{i,j} \omega_{ij}^{\frac{\rho}{p'}} |D_{ij}u_\varepsilon|^{(s-1)\rho-\xi} \omega_{ij}^{\frac{\rho}{p}} |D_{ij}u_\varepsilon|^\xi \right)^\rho dx
$$

$$
\leq c.\varepsilon^\rho \left(\sum_{i,j} \int_\Omega \omega_{ij}^{a\rho} |D_{ij}u_\varepsilon|^{a((s-1)\rho-\xi)} dx \right)^{\frac{1}{a}} . \left(\int_\Omega |Du_\varepsilon|^{a'\xi} dx \right)^{\frac{1}{a'}}
$$

$$
\leq c.\varepsilon^\rho \left(\sum_{i,j} \int_\Omega \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-2} \sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 \right)^{\frac{1}{a}} . \| Du_\varepsilon \|^{\frac{\tau}{a}}_{L^\tau(\Omega, I\!M^{m\times n})} .
$$

And by the injection: $L^{\frac{n(p_s-1)}{n-1}} \hookrightarrow L^\tau \qquad \forall \tau > \frac{n(p_s-1)}{n-1}$ and thanks to (6.10)-(6.12), we get:

$$
\begin{aligned}
\int_\Omega |\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-1} \omega_{ij}^{\frac{1}{p}} |^\rho dx &\leq c.\varepsilon^\rho \left(\frac{c\|\mu\|_M}{\varepsilon} \right)^{\frac{s}{(s-1)a}} .c^{\frac{\tau}{a}} \\
&\leq c.c^{\frac{\tau}{a}} \varepsilon^{\frac{a((s-1)\rho-s))}{a(s-1)}} \\
&\leq c.c^{\frac{\tau}{a}} \varepsilon^{\frac{a\xi}{a(s-1)}} \\
&\leq c.c^{\frac{\tau}{a}} .\varepsilon^{\frac{\xi}{s-1}}
\end{aligned}
$$

with $\frac{\xi}{s-1} > 0$. Hence

$$
\lim_{\varepsilon \longrightarrow} \| \varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-1} \omega_{rs}^{\frac{1}{p}} D_{rs}u_\varepsilon \|_{L^\rho(\Omega, I\!M^{m\times n})} = 0, \quad \forall \rho < \frac{s}{s-1}.
$$

In particular for $\rho = 1$
$$\lim_{\varepsilon \longrightarrow} \int_\Omega |\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-1} \omega_{rs}^{\frac{1}{p}} D_{rs} u_\varepsilon | dx = 0,$$
which mean that
$$\tau[\varepsilon]\,(x, u_\varepsilon, Du_\varepsilon) = \varepsilon \sum_{i,j}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2} \omega_{rs}^{\frac{1}{p}} D_{rs} u_\varepsilon \rightharpoonup 0$$
in $L^1(\Omega, I\!M^{m\times n})$.
As well as by the proposition 6.1, $div\sigma(x, u_\varepsilon, Du_\varepsilon)$ converges to $div\sigma(x, u, apDu)$, in the sense of the distributions, and as
$$\tau[\varepsilon]\,(x, u_\varepsilon, Du_\varepsilon) = \varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2} \omega_{rs}^{\frac{1}{p}} D_{rs} u_\varepsilon \rightharpoonup 0,$$
in $L^1(\Omega, I\!M^{m\times n})$. Then $div\sigma_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$ converge to $div\sigma(x, u, apDu)$ in the sense of distributions, i-e: u is the solution of the system
$$\begin{cases} -div\sigma(x, u, apDu) &= \mu \qquad \in \Omega \\ u &= 0, \text{ on } \partial\Omega. \end{cases}$$
In the same way as in the case of $\theta = p - 1$, we have
$$\int_{|u|\le\alpha} |apDu|^s dx < c(\alpha) < \infty \text{ and } p < s.$$
So we conclude as in the proof of the proposition 6.1, in order to get the estimation of theorem (2.1). This completes the proof of the theorem.

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