



On an Open Problem of Gui, Xu and Wang

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ABSTRACT: In this paper, we exhaustively study the uniqueness of meromorphic function under the aegis of truncated sharing five small functions in k -punctured complex plane. Most importantly, we have answered an open question asked by Gui et al. [AIMS Math., 5(6)(2020), 7438-7457].

Keywords: Small functions, uniqueness, k -punctured complex plane.

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1. Introduction

Let k be a positive integer, for k distinct points $c_j \in \mathbb{C}$, $j = 1, 2, \dots, k$, we say that $\Omega = \mathbb{C} \setminus \cup_{j=1}^k \{c_j\}$ is a k -punctured complex plane. Now we introduce the basic Nevanlinna theory in k -punctured complex plane. Let $d = \frac{1}{2} \min\{|c_s - c_j| : j \neq s\}$ and $r_0 = \frac{1}{d} + \max\{|c_j| : j = 1, 2, \dots, k\}$, thus $\frac{1}{r_0} < d$ and $\overline{D}_{1/r_0}(c_j) \cap \overline{D}_{1/r_0}(c_s) = \emptyset$, for $j \neq s$, and $\overline{D}_{1/r_0}(c_j) \subset D_{r_0}(0)$, for $j = 1, 2, \dots, k$, where $D_\delta(c) = \{z : |z - c| < \delta\}$ and $\overline{D}_\delta(c) = \{z : |z - c| \leq \delta\}$. We define $\Omega_r = D_r(0) \setminus \cup_{j=1}^k \overline{D}_{1/r}(c_j)$, for any $r \geq r_0$. Thus $\Omega_r \supset \Omega_{r_0}$ for $r_0 < r \leq +\infty$.

In 2007, Hanyak and Kondratyuk [6] gave some extension of the Nevanlinna value distribution theory for meromorphic functions in a k -punctured complex plane. Consider f be a meromorphic function in a k -punctured complex plane Ω . Let $n_0(r, a; f)$ counts the number of a -points with counting multiplicities in $\overline{\Omega}_r$, for $r_0 \leq r < +\infty$ and the counting function is defined as

$$N_0(r, a; f) = \int_{r_0}^r \frac{n_0(t, a; f)}{t} dt.$$

Similarly, the reduced counting function $\tilde{N}_0(r, a; f)$ is defined as

$$\tilde{N}_0(r, a; f) = \int_{r_0}^r \frac{\tilde{n}_0(t, a; f)}{t} dt,$$

where $\tilde{n}_0(t, a; f)$ counts the number of distinct a -points (i.e., ignoring multiplicities) in $\overline{\Omega}_r$. Moreover we denote

$$\tilde{N}_0^{(l)}(r, a; f) = \int_{r_0}^r \frac{\tilde{n}_0(t, a; f \leq l)}{t} dt,$$

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where $\tilde{n}_0(t, a; f | \leq l)$ counts the number of distinct a -points of multiplicity $\leq l$ in $\overline{\Omega}_r$. In particular, $N_0(r, f) := N_0(r, \infty; f)$. Also,

$$\begin{aligned} m_0(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta + \frac{1}{2\pi} \sum_{j=1}^k \int_0^{2\pi} \log^+ \left| f \left(c_j + \frac{1}{r} e^{i\theta} \right) \right| d\theta \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r_0 e^{i\theta})| d\theta - \frac{1}{2\pi} \sum_{j=1}^k \int_0^{2\pi} \log^+ \left| f \left(c_j + \frac{1}{r_0} e^{i\theta} \right) \right| d\theta, \end{aligned}$$

where $\log^+ x = \max\{\log x, 0\}$ and $r_0 \leq r < +\infty$. The Nevanlinna's characteristic function is defined as $T_0(r, f) = m_0(r, f) + N_0(r, f)$. By $S(r, f)$ we denote any quantity satisfying $S(r, f) = o(T_0(r, f))$ for all r outside a possible exceptional set of finite linear measure. Let $a(z)$ be a meromorphic function in Ω , we call $a(z)$ a small function with respect to f if $T_0(r, a) = S(r, f)$ and the collection of all small functions with respect to f is denoted by $S(f)$.

Let f be a non-constant meromorphic function in the k -punctured complex plane Ω , a small function $\alpha(z) \in S(f) \cup \{\infty\}$ and l be a positive integer or $+\infty$. We use $E_\Omega(\alpha, l; f)$ ($\tilde{E}_\Omega(\alpha, l; f)$) to denote the set of zeros of $f(z) - \alpha$ in Ω of multiplicity $\leq l$ with counting (ignoring) multiplicities.

Moreover, for any two non-constant meromorphic functions f and g in Ω and $\alpha \in (S(f) \cap S(g)) \cup \{\infty\}$, if $E_\Omega(\alpha, l; f) = E_\Omega(\alpha, l; g)$ ($\tilde{E}_\Omega(\alpha, l; f) = \tilde{E}_\Omega(\alpha, l; g)$), then we call f and g share α with counting (ignoring) multiplicity, truncated up to multiplicity l . In particular, for $l = +\infty$, we usually denote $E_\Omega(\alpha, l; f) := E_\Omega(\alpha; f)$ and $\tilde{E}_\Omega(\alpha, l; f) := \tilde{E}_\Omega(\alpha; f)$. Also, we call f and g share α CM (IM) in Ω , whenever $E_\Omega(\alpha; f) = E_\Omega(\alpha; g)$ ($\tilde{E}_\Omega(\alpha; f) = \tilde{E}_\Omega(\alpha; g)$). The following definition is required for our paper.

Definition 1.1 [17] *Let f be a non-constant meromorphic function in k -punctured complex plane Ω . The function f is called transcendental or admissible in Ω if*

$$\limsup_{r \rightarrow +\infty} \frac{T_0(r, f)}{\log r} = +\infty, \quad r_0 \leq r < +\infty.$$

2. Backgrounds and Main Result

Over complex field, in 1926, R. Nevanlinna proved the following famous five values theorem.

Theorem 2.1 [18] *If f and g are two non-constant meromorphic functions, and a_j ($j = 1, 2, \dots, 5$) be five distinct values in extended complex plane. If f and g share a_j ($j = 1, 2, \dots, 5$) IM, then $f \equiv g$.*

Latter on, in [19], Yi extend *Theorem 2.1* by considering small function sharing instead of value sharing and proved the next result.

Theorem 2.2 [19] *Let f and g be two non-constant meromorphic functions in complex plane, and a_j ($j = 1, 2, 3, 4, 5$) be five distinct small functions with respect to f and g . If f and g share a_j ($j = 1, 2, 3, 4, 5$) IM in \mathbb{C} , then $f \equiv g$.*

In last few decades, many researchers are showing their interest to investigate the uniqueness of meromorphic functions on the unit disc, the annuli, the angular domain, the whole complex plane etc., and a ample numbers of important results were obtained in [1,2,4,7,8,10,11,12,13,16,18]. In 2003 and 2004, Zheng [20,21] studied the uniqueness problem and obtained five values theorem and four values theorem in some angular domain of \mathbb{C} . In 2015, Liu and Mao [9], extended *Theorem 2.2* to an angular domain and proved the following theorem.

Theorem 2.3 [9] *Let f and g be two non-constant meromorphic functions in an angular domain $\mathbb{X}(\alpha, \beta) := \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leq 2\pi$ such that*

$$\lim_{r \rightarrow +\infty} \frac{\mathfrak{T}_{\alpha, \beta}(r, f)}{\log r} = +\infty,$$

and let a_j ($j = 1, \dots, 5$) be five distinct small functions with respect to f and g in $\mathbb{X}(\alpha, \beta)$. If f and g share a_j ($j = 1, \dots, 5$) IM in $\mathbb{X}(\alpha, \beta)$, then $f \equiv g$.

We all know that, the whole complex plane, unit disc and angular domain can all be considered as a simply connected region. For doubly connected region such as for annulus, in 2009, Cao [3] investigated the uniqueness of meromorphic functions sharing some values, and obtained an analog of Nevanlinna's famous five values theorem.

Theorem 2.4 [3] *Let f and g be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_j (j = 1, 1, \dots, 5)$ be five distinct numbers in extended complex plane. If f and g share a_j IM for $j = 1, 2, \dots, 5$, then $f(z) \equiv g(z)$.*

Later on, Wu and Ge [14] extended the last theorem for small functions sharing. Recently, Xu and Liu [17] studied the Nevanlinna's five value theorem in a multiply connected region namely k -punctured complex plane Ω and obtained the following result.

Theorem 2.5 [17] *Let f and g be two admissible meromorphic functions in Ω , if f, g share five distinct values a_1, a_2, \dots, a_5 IM in Ω , then $f(z) \equiv g(z)$.*

In 2020, Gui et al. [5] improved *Theorem 2.5* by considering truncated IM sharing of small functions instead of IM sharing of values. They proved the following results.

Theorem 2.6 [5] *Let f and g be two transcendental or admissible meromorphic functions in Ω , $\alpha_j \in S(f) \cap S(g)$, $j = 1, 2, \dots, 5$ be five distinct small functions and l be a positive integer. If $\tilde{E}_\Omega(\alpha_j, l; f) = \tilde{E}_\Omega(\alpha_j, l; g)$ for $j = 1, 2, \dots, 5$, and $l \geq 22$, then $f(z) \equiv g(z)$.*

Very recently, in [10], the first author of the present paper improved *Theorem 2.6* by reducing the lower bound of l from 22 to 14.

In [5], Gui et al. posted the following question, which is still open.

Question 2.1 *What condition on l_j ($j = 1, 2, \dots, 5$) can guarantee that the conclusion of *Theorem 2.6* still holds if $\tilde{E}_\Omega(\alpha_j, l; f) = \tilde{E}_\Omega(\alpha_j, l; g)$ for $j = 1, 2, \dots, 5$, are replaced by $\tilde{E}_\Omega(\alpha_j, l_j; f) = \tilde{E}_\Omega(\alpha_j, l_j; g)$ for $j = 1, 2, \dots, 5$, and l_j are five positive integers?*

In our main result, we give the answer to the *Question 2.1*.

Theorem 2.7 *Let f and g be two transcendental or admissible meromorphic functions in Ω , $\alpha_j \in S(f) \cap S(g)$, $j = 1, 2, \dots, 5$ be five distinct small functions and l_1, l_2, \dots, l_5 are positive integers or $+\infty$ and $\sum_{j=1}^5 \frac{1}{l_j + 1} < \frac{2}{9}$. If $\tilde{E}_\Omega(\alpha_j, l_j; f) = \tilde{E}_\Omega(\alpha_j, l_j; g)$ for $j = 1, 2, \dots, 5$, then $f(z) \equiv g(z)$.*

Note that *Theorem 2.7* answers the *Question 2.1* as well as extends and improves *Theorem 2.6*.

3. Lemmas

Lemma 3.1 [6] *Let f be a non-constant meromorphic function in a k -punctured plane Ω and k a positive integer, then*

$$m_0 \left(r, \frac{f^{(k)}}{f} \right) = O(\log T_0(r, f)) + O(\log^+ r) = S(r, f), \quad r \rightarrow +\infty,$$

outside a set of finite linear measure.

Remark 3.1 Obviously, if f is transcendental or admissible in a k -punctured plane, then

$$m_0 \left(r, \frac{f^{(k)}}{f} \right) = S(r, f) = o(T_0(r, f)),$$

outside a set of finite linear measure.

Lemma 3.2 [15] *Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in a k -punctured plane Ω , $\alpha_t(z) (\neq 0, 1) \in S(f_1) \cap S(f_2)$, $t = 1, 2$, be not equal to constants simultaneously, and for $s = 1, 2$ let*

$$F_s(z) = \begin{vmatrix} f_s f'_s & f'_s & f_s^2 - f_s \\ \alpha_1 \alpha'_1 & \alpha'_1 & \alpha_1^2 - \alpha_1 \\ \alpha_2 \alpha'_2 & \alpha'_2 & \alpha_2^2 - \alpha_2 \end{vmatrix}.$$

Then:

(i) For $s = 1, 2$, $F_s(z) \neq 0$.

(ii) For $s = 1, 2$,

$$\begin{aligned} 2T_0(r, f_s) &< \tilde{N}_0\left(r, \frac{1}{f_s - 1}\right) + \tilde{N}_0\left(r, \frac{1}{f_s}\right) + \tilde{N}_0(r, f_s) + \tilde{N}_0\left(r, \frac{1}{f_s - \alpha_1}\right) \\ &+ \tilde{N}_0\left(r, \frac{1}{f_s - \alpha_2}\right) + S(r, f_1) + S(r, f_2). \end{aligned}$$

The next lemma extended (ii) of Lemma 3.2, by considering α_j ($j = 1, 2, \dots, 5$) are any five distinct small functions of f_1 and f_2 .

Lemma 3.3 [10] *Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in a k -punctured plane Ω , $\alpha_j(z) \in S(f_1) \cap S(f_2)$, $j = 1, 2, \dots, 5$, be five distinct meromorphic functions in a k -punctured plane Ω , then*

$$2T_0(r, f_s) \leq \sum_{j=1}^5 \tilde{N}_0\left(r, \frac{1}{f_s - \alpha_j}\right) + S(r, f_1) + S(r, f_2), \quad \text{for } s = 1, 2. \quad (3.1)$$

Lemma 3.4 [5] *Let f be a transcendental or admissible meromorphic function in Ω , and $\alpha(z), \beta(z) \in S(f)$ be two distinct small functions with respect to f . Set*

$$L(f, \alpha, \beta) := \begin{vmatrix} f & f' & 1 \\ \alpha & \alpha' & 1 \\ \beta & \beta' & 1 \end{vmatrix}.$$

Then we have $L(f, \alpha, \beta) \neq 0$ and for $i = 0, 1$,

$$m_0\left(r, \frac{L(f, \alpha, \beta)f^i}{(f - \alpha)(f - \beta)}\right) = S(r, f).$$

Lemma 3.5 *Let f and g be two transcendental or admissible meromorphic functions in Ω , $\alpha_j \in S(f) \cap S(g)$, $j = 1, 2, \dots, 5$ be five distinct small functions and l_1, l_2, \dots, l_5 are positive integers or $+\infty$. Suppose $\tilde{E}_\Omega(\alpha_j, l_j; f) = \tilde{E}_\Omega(\alpha_j, l_j; g)$ for $j = 1, 2, \dots, 5$. If $f \neq g$, then for every subset $\{i_1, i_2, i_3, i_4\}$ of $\{1, 2, \dots, 5\}$ we have*

$$\begin{aligned} \tilde{N}_0^{l_j}\left(r, \frac{1}{f - \alpha_j}\right) &\leq \sum_{s=1}^4 \left(\tilde{N}_0^{(l_{i_s}+1)}\left(r, \frac{1}{f - \alpha_{i_s}}\right) + \tilde{N}_0^{(l_{i_s}+1)}\left(r, \frac{1}{g - \alpha_{i_s}}\right) \right) \\ &+ S(r, f) + S(r, g), \end{aligned}$$

for $j \in \{1, 2, \dots, 5\} \setminus \{i_1, i_2, i_3, i_4\}$.

Proof: Without loss of generality we may assume $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$, then we need to prove

$$\begin{aligned} \tilde{N}_0^{l_5}\left(r, \frac{1}{f - \alpha_5}\right) &\leq \sum_{t=1}^4 \left(\tilde{N}_0^{(l_t+1)}\left(r, \frac{1}{f - \alpha_t}\right) + \tilde{N}_0^{(l_t+1)}\left(r, \frac{1}{g - \alpha_t}\right) \right) \\ &+ S(r, f) + S(r, g). \end{aligned} \quad (3.2)$$

If $\tilde{N}_0^{l_5} \left(r, \frac{1}{f-\alpha_5} \right) = S(r, f) + S(r, g)$, then (3.2) holds trivially. Thus we may assume

$$\tilde{N}_0^{l_5} \left(r, \frac{1}{f-\alpha_5} \right) \neq S(r, f) + S(r, g). \quad (3.3)$$

By applying the quasi-Möbius transformation

$$\frac{w - \alpha_1}{w - \alpha_2} \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1},$$

we may assume $\alpha_1 = 0$, $\alpha_2 = \infty$, $\alpha_3 = 1$ and α_4, α_5 are five distinct small functions of f and g such that $\alpha_4, \alpha_5 \neq 0, 1, \infty$.

Consider L as defined in Lemma 3.4, and set

$$\Phi := \frac{L(f, 0, 1)(f-g)L(g, \alpha_4, 0)}{f(f-1)g(g-\alpha_4)} - \frac{L(g, 0, 1)(f-g)L(f, \alpha_4, 0)}{g(g-1)f(f-\alpha_4)}, \quad (3.4)$$

which implies

$$\Phi = \frac{f'(\alpha'_4 g - \alpha_4 g')(f-g)}{f(f-1)g(g-\alpha_4)} - \frac{g'(\alpha'_4 f - \alpha_4 f')(f-g)}{g(g-1)f(f-\alpha_4)}. \quad (3.5)$$

First assume $\Phi \equiv 0$, then from (3.5) we get

$$\frac{f'(\alpha'_4 g - \alpha_4 g')(f-g)}{f(f-1)g(g-\alpha_4)} \equiv \frac{g'(\alpha'_4 f - \alpha_4 f')(f-g)}{g(g-1)f(f-\alpha_4)}. \quad (3.6)$$

If α_4 is constant then from (3.6) we obtain $(1-\alpha_4)f = (1-\alpha_4)g$. Since $\alpha_4 \neq 1$ thus $f \equiv g$, which contradicts our assumption. Hence α_4 is not a constant. Now from (3.6)

$$\begin{aligned} & \frac{(f-1)(g-\alpha_4)}{(g-1)(f-\alpha_4)} - 1 \equiv \frac{f'(\alpha'_4 g - \alpha_4 g')}{g'(\alpha'_4 f - \alpha_4 f')} - 1 \\ \implies & \frac{(f-g)(1-\alpha_4)}{(g-1)(f-\alpha_4)} \equiv \frac{\alpha'_4 [(f'-g')g - (f-g)g']}{g'(\alpha'_4 f - \alpha_4 f')} \\ \implies & \frac{f'-g'}{f-g} \equiv \frac{(1-\alpha_4)g'(\alpha'_4 f - \alpha_4 f')}{\alpha'_4 g(g-1)(f-\alpha_4)} + \frac{g'}{g}. \end{aligned} \quad (3.7)$$

Now from $\tilde{E}_\Omega(\alpha_5, l_5; f) = \tilde{E}_\Omega(\alpha_5, l_5; g)$ and (3.3) we get $\tilde{N}_0^{l_5}(r, \alpha_5; f, g) \neq S(r, f) + S(r, g)$, where $\tilde{N}_0^{l_5}(r, \alpha_5; f, g)$ denotes the reduced counting function of common α_5 points of f, g up to multiplicity l_5 . Since $\alpha_4, \alpha'_4, \alpha_5, \alpha_5 - 1, \alpha_5 - \alpha_4$ all are small functions with respect to f and g then

$$\begin{aligned} & \tilde{N}_0^{l_5}(r, \alpha_5; f, g) - N_0(r, \alpha_4) - N_0(r, \alpha'_4) - N_0(r, \alpha_5) - N_0(r, \alpha_5 - 1) - N_0(r, \alpha_5 - \alpha_4) \\ & - N_0 \left(r, \frac{1}{\alpha_4} \right) - N_0 \left(r, \frac{1}{\alpha'_4} \right) - N_0 \left(r, \frac{1}{\alpha_5} \right) - N_0 \left(r, \frac{1}{\alpha_5 - 1} \right) - N_0 \left(r, \frac{1}{\alpha_5 - \alpha_4} \right) \\ & \neq S(r, f) + S(r, g). \end{aligned}$$

Thus there exists a point z such that z is a common zero of $f - \alpha_5$ and $g - \alpha_5$ that is not a zero or a pole of $\alpha_4, \alpha'_4, \alpha_5, \alpha_5 - 1, \alpha_5 - \alpha_4$. From (3.7) note that z must be a pole of the left hand side and not be pole of the right hand side, which is a contradiction. Hence $\Phi \neq 0$.

Note that (3.5) can also be written as

$$\Phi = \frac{(f-g)H}{f(f-1)(f-\alpha_4)g(g-1)(g-\alpha_4)}, \quad (3.8)$$

where

$$\begin{aligned}
H &= f'(\alpha_4'g - \alpha_4g')(f - \alpha_4)(g - 1) - g'(\alpha_4'f - \alpha_4f')(g - \alpha_4)(f - 1) \\
&= \alpha_4'f f' g^2 - \alpha_4'f f' g - \alpha_4(\alpha_4 - 1)f f' g' - \alpha_4\alpha_4'f' g^2 + \alpha_4\alpha_4'f' g \\
&\quad - \alpha_4'f^2 g g' + \alpha_4'f g g' + \alpha_4(\alpha_4 - 1)f' g g' + \alpha_4\alpha_4'f^2 g' - \alpha_4\alpha_4'f g'.
\end{aligned} \tag{3.9}$$

Let z_* be a common zero of $f - \alpha_5$ and $g - \alpha_5$ that is not a zero or a pole of $\alpha_4, \alpha_4', \alpha_5, \alpha_5 - 1, \alpha_5 - \alpha_4$. Then, it is a zero of $f - g$ and is not a pole of $\frac{H}{f(f-1)(f-\alpha_4)g(g-1)(g-\alpha_4)}$. Hence z_* is a zero of Φ . From $\tilde{E}_\Omega(\alpha_5, l_5; f) = \tilde{E}_\Omega(\alpha_5, l_5; g)$ we have

$$\begin{aligned}
&\tilde{N}_0^{l_5}\left(r, \frac{1}{f - \alpha_5}\right) = \tilde{N}_0^{l_5}\left(r, \frac{1}{g - \alpha_5}\right) \\
&\leq N_0\left(r, \frac{1}{\Phi}\right) + S(r, f) + S(r, g) \\
&\leq T_0(r, \Phi) + S(r, f) + S(r, g) \\
&\leq m_0(r, \Phi) + N_0(r, \Phi) + S(r, f) + S(r, g).
\end{aligned} \tag{3.10}$$

First we calculate $m_0(r, \Phi)$. Using *Lemma 3.4*, from (3.4),

$$\begin{aligned}
&m_0(r, \Phi) \\
&= m_0\left(r, \left\{ \frac{L(f, 0, 1)(f - g)L(g, \alpha_4, 0)}{f(f - 1)g(g - \alpha_4)} - \frac{L(g, 0, 1)(f - g)L(f, \alpha_4, 0)}{g(g - 1)f(f - \alpha_4)} \right\}\right) \\
&\leq m_0\left(r, \left\{ \frac{L(f, 0, 1)f L(g, \alpha_4, 0)}{f(f - 1)g(g - \alpha_4)} - \frac{L(f, 0, 1)L(g, \alpha_4, 0)g}{f(f - 1)g(g - \alpha_4)} \right\}\right) \\
&\quad + m_0\left(r, \left\{ \frac{L(g, 0, 1)g L(f, \alpha_4, 0)}{g(g - 1)f(f - \alpha_4)} - \frac{L(g, 0, 1)L(f, \alpha_4, 0)f}{g(g - 1)f(f - \alpha_4)} \right\}\right) \\
&= S(r, f) + S(r, g).
\end{aligned} \tag{3.11}$$

Next we estimate $N_0(r, \Phi)$. In (3.5), note that the poles of Φ can only occur at the zeros of $f, g, f - 1, g - 1, f - \alpha_4, g - \alpha_4$ and the poles of f, g, α_4 . Let A be the set of all zeros, 1-points and poles of α_4 . Now we discuss the following cases.

If $z_0 \notin A$, is a common pole of f and g of order p_1 and q_1 , respectively. From (3.9) it is clear that z_0 is a pole of H of order at most $2p_1 + 2q_1 + 1$. Hence, from (3.8) we see that z_0 is a pole of the numerator of Φ of order at most $2p_1 + 2q_1 + 1 + \max\{p_1, q_1\}$ and it is a pole of the denominator of Φ of order $3p_1 + 3q_1$. Now as,

$$2p_1 + 2q_1 + 1 + \max\{p_1, q_1\} - (3p_1 + 3q_1) = 1 + \max\{p_1, q_1\} - (p_1 + q_1) \leq 0,$$

so z_0 is not a pole of Φ .

If $z_1 \notin A$ is a common zero of f and g of order p_2 and q_2 , respectively. Then z_1 is a zero of $f - g$ of order $\min\{p_2, q_2\}$. From (3.9) we have z_1 is a zero of H of order at least $p_2 + q_2 - 1$. From (3.8), z_1 is a zero of order

$$\min\{p_2, q_2\} + p_2 + q_2 - 1 - (p_2 + q_2) = \min\{p_2, q_2\} - 1 \geq 0.$$

Thus z_1 is not a pole of Φ .

If $z_2 \notin A$ is a common zero of $f - \alpha_j$ and $g - \alpha_j$ with $j = 3, 4$, note that $\alpha_3 = 1$. Then obviously z_2 is a zero of $f - g$ and z_2 pole of $\frac{f'}{f - \alpha_j}$ and $\frac{g'}{g - \alpha_j}$ (for $j = 3, 4$) with multiplicity 1. Then from (3.8), z_2 is not a pole of Φ .

If $z_3 \notin A$ is a pole of f but is not a pole of g and is not a zero of $g, g - 1$ and $g - \alpha_4$ or $z_3 \notin A$ is a pole of g but is not a pole of f and is not a zero of $f, f - 1$ and $f - \alpha_4$. Then from (3.8) that z_3 is a pole of Φ of order at most 1.

If $z_4 \notin A$ is a zero of f but is not a pole of g and is not a zero of g , $g - 1$ and $g - \alpha_4$ or $z_4 \notin A$ is a zero of g but is not a pole of f and is not a zero of f , $f - 1$ and $f - \alpha_4$. Then, from (3.8) it is easy to see that, z_4 is a pole of Φ of order at most 1.

If $z_5 \notin A$ is a zero of $f - \alpha_j$ with $j = 3, 4$ but is not a pole of g and is not a zero of g , $g - 1$ and $g - \alpha_4$ or $z_5 \notin A$ is a zero of $g - \alpha_j$ with $j = 3, 4$ but is not a pole of f and is not a zero of f , $f - 1$ and $f - \alpha_4$. Then, from (3.8) we get z_5 is a pole of Φ of order at most 1.

If $z_6 \notin A$ is a common zero of $f - \alpha_i$ and $g - \alpha_j$, where $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$. Then, from (3.8) it follows that, z_6 is a pole of Φ of order at most 2.

Thus from the above discussion we get

$$N_0(r, \Phi) \leq \sum_{t=1}^4 \left(\tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{g - \alpha_t} \right) \right) + S(r, f) + S(r, g). \quad (3.12)$$

Now combining (3.10), (3.11), (3.12) we get the conclusion (3.2). This completes the proof of Lemma 3.5. \square

4. Proof of the Theorem

Proof: [Proof of Theorem 2.7] Assume $f \neq g$. From Lemma 3.5 we get, for every subset $\{i_1, i_2, i_3, i_4\}$ of $\{1, 2, \dots, 5\}$,

$$\begin{aligned} \tilde{N}_0^{(l_j)} \left(r, \frac{1}{f - \alpha_j} \right) &\leq \sum_{s=1}^4 \left(\tilde{N}_0^{(l_{i_s+1})} \left(r, \frac{1}{f - \alpha_{i_s}} \right) + \tilde{N}_0^{(l_{i_s+1})} \left(r, \frac{1}{g - \alpha_{i_s}} \right) \right) \\ &\quad + S(r, f) + S(r, g), \end{aligned} \quad (4.1)$$

where $j \in \{1, 2, \dots, 5\} \setminus \{i_1, i_2, i_3, i_4\}$. Now summing up of (4.1) over all possible subsets $\{i_1, i_2, i_3, i_4\}$ of $\{1, 2, \dots, 5\}$. Note that, after summing up each index $i_s \in \{1, 2, \dots, 5\}$ will appear 4 times. Thus we obtain

$$\begin{aligned} \sum_{t=1}^5 \tilde{N}_0^{(l_t)} \left(r, \frac{1}{f - \alpha_t} \right) &\leq 4 \sum_{t=1}^5 \left(\tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{g - \alpha_t} \right) \right) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (4.2)$$

Similarly for the function g ,

$$\begin{aligned} \sum_{t=1}^5 \tilde{N}_0^{(l_t)} \left(r, \frac{1}{g - \alpha_t} \right) &\leq 4 \sum_{t=1}^5 \left(\tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{g - \alpha_t} \right) \right) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (4.3)$$

Adding (4.2) and (4.3)

$$\begin{aligned} &\sum_{t=1}^5 \left(\tilde{N}_0^{(l_t)} \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0^{(l_t)} \left(r, \frac{1}{g - \alpha_t} \right) \right) \\ &\leq 8 \sum_{t=1}^5 \left(\tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{g - \alpha_t} \right) \right) + S(r, f) + S(r, g) \\ \Rightarrow &\sum_{t=1}^5 \left(\tilde{N}_0 \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0 \left(r, \frac{1}{g - \alpha_t} \right) \right) \\ &\leq 9 \sum_{t=1}^5 \left(\tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0^{(l_{t+1})} \left(r, \frac{1}{g - \alpha_t} \right) \right) + S(r, f) + S(r, g). \end{aligned} \quad (4.4)$$

From *Lemma 3.3* we obtain

$$2(T_0(r, f) + T_0(r, g)) \leq \sum_{t=1}^5 \left(\tilde{N}_0 \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0 \left(r, \frac{1}{g - \alpha_t} \right) \right) + S(r, f) + S(r, g). \quad (4.5)$$

Combining (4.4) and (4.5)

$$\frac{2}{9}(T_0(r, f) + T_0(r, g)) \leq \sum_{t=1}^5 \left(\tilde{N}_0^{(l_t+1)} \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0^{(l_t+1)} \left(r, \frac{1}{g - \alpha_t} \right) \right) + S(r, f) + S(r, g). \quad (4.6)$$

Now we will estimate the right hand side of (4.6).

$$\begin{aligned} & \sum_{t=1}^5 \left(\tilde{N}_0^{(l_t+1)} \left(r, \frac{1}{f - \alpha_t} \right) + \tilde{N}_0^{(l_t+1)} \left(r, \frac{1}{g - \alpha_t} \right) \right) \\ & \leq \sum_{t=1}^5 \frac{1}{l_t + 1} \left(N_0^{(l_t+1)} \left(r, \frac{1}{f - \alpha_t} \right) + N_0^{(l_t+1)} \left(r, \frac{1}{g - \alpha_t} \right) \right) \\ & \leq \sum_{t=1}^5 \frac{1}{l_t + 1} (T_0(r, f) + T_0(r, g)). \end{aligned} \quad (4.7)$$

Combining (4.6) and (4.7)

$$\left(\frac{2}{9} - \sum_{t=1}^5 \frac{1}{l_t + 1} \right) (T_0(r, f) + T_0(r, g)) \leq S(r, f) + S(r, g). \quad (4.8)$$

Since we have $\sum_{t=1}^5 \frac{1}{l_t + 1} < \frac{2}{9}$, so $\left(\frac{2}{9} - \sum_{t=1}^5 \frac{1}{l_t + 1} \right) > 0$, hence from (4.8) a contradiction arises.

Therefore $f \equiv g$. \square

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