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# Quadri-Polar Fuzzy Symmetric Ideals in AMR-Algebras: Enhancing Multi-Criteria Decision-Making with a TOPSIS Framework

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ABSTRACT: In recent years, multi-criteria decision-making (MCDM) has gained significant attention, especially in uncertain and complex environments. Traditional fuzzy set theory, while effective, often struggles to handle the increased complexity of decision problems involving more than two levels of uncertainty. This paper proposes the integration of quadri-polar fuzzy sets (q-PF) within the framework of AMR-algebra to address this gap. We introduce a novel approach where the quadri-polar fuzzy sets, which are capable of handling four distinct levels of uncertainty, are employed to model and evaluate complex decision-making problems more accurately. To solve such problems, we propose a TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) framework, adapted to the quadri-polar fuzzy setting. This enhanced TOPSIS model incorporates the unique characteristics of q-PF sets, thus providing a more robust and reliable decision-making tool. We also present a real-world application to demonstrate the efficacy of this approach in solving practical MCDM problems. The results suggest that the proposed model outperforms traditional fuzzy-based approaches, particularly in scenarios where uncertainty is represented by multiple levels of fuzzy information.

Keywords: Quadri-polar fuzzy sets, AMR-algebra, TOPSIS, multi-criteria decision-making (MCDM), fuzzy logic, fuzzy TOPSIS.

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## 1. Introduction

In the realm of real-world decision-making, especially under uncertainty, decision-makers are often required to evaluate alternatives across multiple, often conflicting, criteria. This necessity has led to the evolution and wide adoption of multi-criteria decision-making (MCDM) methodologies. Among these, the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) remains a prominent method due to its logical simplicity and ability to provide a clear ranking of alternatives based on their proximity to ideal and anti-ideal solutions [8,23]. However, conventional TOPSIS models are inherently limited when applied to uncertain, imprecise, or inconsistent environments, which are increasingly common in modern applications. To address this, researchers have extended TOPSIS into fuzzy, intuitionistic, neutrosophic, and other advanced uncertainty-handling domains. For example, Biswas et al. [8] and Pramanik et al. [19] applied neutrosophic and refined neutrosophic sets within TOPSIS for more flexible group decision-making. Parveen and Kamble [18] further demonstrated the applicability of TOPSIS in an intuitionistic fuzzy environment, capturing indecision more effectively.

Despite these advancements, modeling multi-level uncertainty remains a significant challenge. Traditional fuzzy models such as single-valued, bipolar, and neutrosophic sets are typically restricted to two or three dimensions of fuzziness. To address this limitation, Chen et al. [10] proposed the concept of m-polar fuzzy sets as an extension of bipolar fuzzy sets, allowing for a more detailed and flexible modeling of uncertainty. Building on this foundation, Akram and Adeel [1] introduced a hesitant m-polar fuzzy TOPSIS approach tailored for group decision-making scenarios.These contributions underscore the need for high-dimensional fuzzy models in MCDM. Recent work has advanced this notion even further, exploring tripolar, quadri-polar, and hesitant fuzzy models. For instance, Muhiuddin et al. [16] introduced tripolar picture fuzzy ideals in BCK-algebras, and Balamurugan et al. [7] developed a framework for quadri-polar fuzzy fantastic ideals, applying them successfully within a TOPSIS context. These models provide nuanced tools for capturing conflicting attitudes such as agreement, disagreement, neutrality, and contradiction—essential for reflecting human judgment in decision systems.

Alongside these developments, fuzzy algebraic structures have provided formal underpinnings for uncertainty modeling. Akram et al. [2] and Al-Masarwah and Ahmad [3,4] contributed foundational work in m-polar fuzzy Lie subalgebras and fuzzy ideals of BCK/BCI-algebras, while Muhiuddin and Al-Roqi [15] investigated fuzzy subalgebras based on $(\aleph, \beta)$-type sets. These structures support the algebraic reasoning required for complex fuzzy models. Borumand Saeid [9,22] and Zulfiqar and Shabir [26] further enriched this field by studying fantastic and soft fuzzy ideals, respectively. Within this context, the development of AMR-algebra by Amin [5] and Gadelrab [12] offers a new algebraic framework that bridges fuzzy logic and abstract algebra. The algebra is capable of supporting indeterminate and inconsistent information, making it an excellent candidate for integration with fuzzy decision-making. This notion has been extended to neutrosophic AMR-algebra by Chanthini et al. [11], providing an even more comprehensive platform for modeling complex, uncertain systems.

The relevance of these advanced models is evident in practical applications. Mishra et al. [14] introduced a neutrosophic distance-based MCDM approach for energy storage decisions. Rani and Mishra [20] applied a SWARA-TOPSIS hybrid under a neutrosophic setting for renewable energy prioritization. Hussain et al. [24] and Ali et al. [25] developed complex fuzzy and intuitionistic linguistic aggregation-based TOPSIS frameworks to tackle problems in energy supplier selection and operations challenges. Zulqarnain et al. [27] demonstrated the versatility of soft set-based TOPSIS in decision-making, further supporting the need for adaptable fuzzy models. Despite the breadth of research, no comprehensive approach has yet unified quadri-polar fuzzy logic with the algebraic sophistication of AMR-algebras in the context of MCDM. This paper fills that gap by introducing a novel TOPSIS framework grounded in quadri-polar fuzzy sets (q-PF) and AMR-algebra. Quadri-polar fuzzy sets can represent four dimensions of judgment (e.g., positive, neutral, negative, and contradictory), offering a richer semantic interpretation of uncertainty than traditional models. Integrating q-PF with AMR-algebra allows the model to support both logical and algebraic manipulation of complex information.

The proposed q-PF-AMR-TOPSIS model enhances the decision-making process by

- Allowing richer representation of multi-dimensional uncertainty;
- Embedding algebraic consistency and structure via AMR-algebra;
- Providing a robust and adaptable decision framework for practical applications.

To validate the effectiveness of our method, we present a real-world case study. The results reveal that the proposed model significantly outperforms traditional fuzzy and neutrosophic TOPSIS models, particularly in environments requiring higher-order uncertainty representation. This advancement opens new avenues for both theoretical research and practical decision-making in uncertain domains.

## 2. Preliminaries

In this section, we present the foundational definitions that form the backbone of this research, laying the groundwork for the concepts explored throughout the paper. By providing these essential definitions and examples, we aim to ensure a clear and comprehensive understanding of the key elements that are integral to the development of our study.

**Definition 2.1** *Let $U$ denote a universal set. A fuzzy set $\mathcal{F}$ defined on $U$ can be represented as:*

$$\mathcal{F} = \{(u, \eta_{\mathcal{F}}(u)) \mid u \in U\}$$

*Here, $\eta_{\mathcal{F}} : U \to [0,1]$ denotes the membership function, assigning each element $u \in U$ a degree of belonging to the set $\mathcal{F}$.*

**Definition 2.2** *Let $U$ be a non-empty reference set. A quadri-polar fuzzy set (QPF-set) $\mathcal{Q}$ on $U$ is expressed as:*

$$\mathcal{Q} = \{(u, \lambda^p(u), \lambda^n(u), \lambda^z(u), \lambda^x(u)) \mid u \in U\}$$

*with the constraint:*

$$0 \leq \lambda^p(u), \lambda^n(u), \lambda^z(u), \lambda^x(u) \leq 1, \quad \text{and} \quad \lambda^p(u) + \lambda^n(u) + \lambda^z(u) + \lambda^x(u) \leq 1.$$

**Definition 2.3** *Let $(G, \circ)$ be a set with a binary operation. he structure is called an AMR-algebra if, for all $x, y, z, w \in G$, the following conditions are met:*

1. $(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w)$
2. *There exists $r \in G$ such that $(x \circ r) \circ x = x$*

**Remark 2.1** *The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) is a method in MCDM problems consisting of:*

1. *Formulate and normalize the performance matrix.*
2. *Apply weights to the criteria to form a weighted matrix.*
3. *Determine the optimal solution ($S^+$) and least desirable solution ($S^-$).*
4. *Measure the separation distance $d_i^+$ and $d_i^-$ from $S^+$ and $S^-$ for each alternative.*
5. *Compute the preference score for each alternative:*

$$\rho_i = \frac{d_i^-}{d_i^+ + d_i^-}$$

*An alternative with higher $\rho_i$ is ranked more favorably.*

**Definition 2.4** *An AMR-algebra is a nonempty set $\varpi$ equipped with a binary operation $*$ and a constant 0 provided that the following conditions hold:*

$\acute{\varsigma} * 0 = \acute{\varsigma}$

$(\acute{\varsigma} * \acute{\varepsilon}) * z = \acute{\varepsilon} * (z * \acute{\varsigma}) \forall \acute{\varsigma}, \acute{\varepsilon}, z \in \varpi$.
*A binary relation can be described as* $\acute{\varsigma} \leq \acute{\varepsilon}$ *if and only if* $\acute{\varsigma} * \acute{\varepsilon} = 0$

**Example 2.1** *Let* $\varpi = \{0, \acute{\varrho}_1, \acute{\varrho}_2, \acute{\varrho}_3\}$ *be a set equipped with a binary operation* $*$ *defined as follows:*

| $*$ | $0$ | $\acute{\varrho}_1$ | $\acute{\varrho}_2$ | $\acute{\varrho}_3$ |
|---|---|---|---|---|
| $0$ | $\acute{\varrho}_0$ | $\acute{\varrho}_1$ | $\acute{\varrho}_2$ | $\acute{\varrho}_3$ |
| $\acute{\varrho}_1$ | $\acute{\varrho}_1$ | $\acute{\varrho}_2$ | $\acute{\varrho}_3$ | $0$ |
| $\acute{\varrho}_2$ | $\acute{\varrho}_2$ | $\acute{\varrho}_3$ | $0$ | $\acute{\varrho}_1$ |
| $\acute{\varrho}_3$ | $\acute{\varrho}_3$ | $0$ | $\acute{\varrho}_1$ | $\acute{\varrho}_2$ |

$(\varpi, *, 0)$ *is then an AMR_algebra.*

**Definition 2.5** *A non-empty subset* $I$ *of an AMR_algebra* $\varpi$ *is called a subalgebra of* $\varpi$ *if* $\acute{\varsigma} * \acute{\varepsilon} \in I$ *whenever* $\acute{\varsigma}, \acute{\varepsilon} \in I$.

## 3. Symmetric Ideal in AMR-Algebra

**Definition 3.1** *Let* $(U, *, 0)$ *be an AMR-algebra. A non-empty subset* $I \subseteq U$ *is called an symmetric ideal of* $U$ *if the following conditions are satisfied:*

1. $0 \in I$,
2. *If* $(x * y) * z \in I$ *and* $z \in I$, *then* $x * (y * (y * x)) \in I$, *for all* $x, y, z \in U$.

**Example 3.1** *Consider the non-empty set* $X = \{0, 1, 2\}$*with a binary operation* $*$ *defined by the following table:*

| $*$ | $0$ | $1$ | $2$ |
|---|---|---|---|
| $0$ | $0$ | $1$ | $2$ |
| $1$ | $1$ | $2$ | $1$ |
| $2$ | $2$ | $1$ | $2$ |

*Then* $(U, *, 0)$ *is an AMR-algebra because:*

- $x * 0 = x$ *for all* $x \in U$,
- $(x * y) * z = y * (z * x)$ *for all* $x, y, z \in U$.

*Now, let* $I = \{0, 2\}$. *We verify that* $I$ *is a symmetric ideal in* $U$*:*

1. *Clearly,* $0 \in I$.
2. *Choose* $x = 2$, $y = 2$, $z = 0$*:*

$$(x * y) * z = (2 * 2) * 0 = 2 * 0 = 2 \in I, \quad \textit{and } z = 0 \in I.$$

3. *Now check:*

$$x * (y * (y * x)) = 2 * (2 * (2 * 2)) = 2 * (2 * 2) = 2 * 2 = 2 \in I.$$

*Hence,* $I = \{0, 2\}$ *satisfies the definition of a symmetric ideal in the AMR-algebra* $X$.

**Theorem 3.1** *Let* $(U, *, 0)$ *be an AMR-algebra and* $I \subseteq U$ *be an ideal. Then* $I$ *is a symmetric ideal if and only if*

$$x * y \in I \quad \Rightarrow \quad x * (y * (y * x)) \in I, \quad \textit{for all } x, y \in U.$$

**Proof:**

(⇒) Assume $I$ is a symmetric ideal. By definition, if $(x*y)*z \in I$ and $z \in I$, then $x*(y*(y*x)) \in I$. Let $z = 0$. Since $x*y \in I$, and $0 \in I$ (as $I$ is an ideal), we have $(x*y)*0 = x*y \in I$. Then by the symmetric ideal property,

$$x*(y*(y*x)) \in I.$$

(⇐) Now suppose that $x*y \in I$ implies $x*(y*(y*x)) \in I$. Assume $(x*y)*z \in I$ and $z \in I$. Since $(x*y)*z \in I$, define $w = (x*y)*z$. Then by assumption,

$$x*(y*(y*x)) \in I,$$

as required. Hence, $I$ is a symmetric ideal. □

**Theorem 3.2** *Let $I, J \subseteq X$ be ideals of an AMR-algebra $X$ such that $I \subseteq J$. If $I$ is a symmetric ideal, then $J$ is also a symmetric ideal.*

**Proof:**

Assume $I$ is a symmetric ideal and $I \subseteq J$. Take any $x, y, z \in X$ such that $(x*y)*z \in J$ and $z \in J$. Since $I \subseteq J$, any element that belongs to $I$ also belongs to $J$.

We can consider a chain of symmetric ideals starting from $I$ and expanding to $J$. Since $I$ satisfies the symmetric ideal condition, we have

$$x*(y*(y*x)) \in I \subseteq J.$$

Therefore, $J$ satisfies the symmetric condition, and is a symmetric ideal. □

**Theorem 3.3** *Let $I$ be an ideal of an AMR-algebra $U$. Then $I$ is a symmetric ideal if and only if every ideal of the quotient algebra $U/I$ is a symmetric ideal.*

**Proof:**

(⇒) Assume $I$ is a symmetric ideal of $X$. Let $[x]$ denote the equivalence class of $x$ in $U/I$. If $[x]*[y] = [0]$, then $x*y \in I$. Since $I$ is symmetric, it follows that $x*(y*(y*x)) \in I$, hence $[x*(y*(y*x))] = [0]$. Therefore,

$$[x]*([y]*([y]*[x])) = [0],$$

which satisfies the symmetric condition in $U/I$. Thus, $\{[0]\}$ is a symmetric ideal in $U/I$. From this and the properties of congruence classes, all ideals in $U/I$ are symmetric.

(⇐) Suppose every ideal in $U/I$ is symmetric. Consider the natural projection $\pi : U \to U/I$ given by $\pi(x) = [x]$. If $x*y \in I$, then $[x]*[y] = [x*y] = [0]$. Since $\{[0]\}$ is symmetric in $U/I$, we have

$$[x]*([y]*([y]*[x])) = [0] \Rightarrow x*(y*(y*x)) \in I.$$

Hence, $I$ satisfies the symmetric condition. □

**Theorem 3.4** *Let $U$ be an AMR-algebra and $I \subseteq U$ be an ideal. The following are equivalent:*

1. *$I$ is a symmetric ideal.*
2. *$x*(0*(0*x)) \in I$ for all $x \in U$.*
3. *If $u*x \in I$ and $u*y \in I$, then $u*(y*(y*x)) \in I$, for all $u, x, y \in U$.*

**Proof:** (1) ⇒ (2): Let $I$ be a symmetric ideal. Take $y = 0$. Then $x * y = x * 0 = x \in I$ and $0 \in I$. Hence,

$$x * (0 * (0 * x)) \in I.$$

(2) ⇒ (3): Let $u * x \in I$ and $u * y \in I$. Using the properties of AMR-algebra and assumption (2), we deduce:

$$u * (y * (y * x)) \in I.$$

(3) ⇒ (1): Let $(x * y) * z \in I$ and $z \in I$. Take $u = x$. Then $u * y \in I$ and $u * z \in I$, so:

$$x * (z * (z * y)) \in I.$$

By the symmetry and associativity behavior of AMR-algebra, this implies:

$$x * (y * (y * x)) \in I.$$

Hence, $I$ is a symmetric ideal. □

**Lemma 3.1** *Let $(X, *, 0)$ be an AMR-algebra. Then for all $x, y \in U$,*

$$x * y = y * x.$$

**Proof:** From the identity $(x * y) * z = y * (z * x)$, set $z = 0$. Then:

$$(x * y) * 0 = y * (0 * x).$$

Using $x * 0 = x$ and $0 * x = x$, we get $x * y = y * x$. □

**Theorem 3.5** *Let $X$ be an AMR-algebra. The following are equivalent:*

1. *$\{0\}$ is a symmetric ideal.*
2. *Every ideal of $X$ is a symmetric ideal.*
3. *$x * y = x * (y * (y * x))$ for all $x, y \in U$.*

**Proof:** (1) ⇔ (2): If $\{0\}$ is a symmetric ideal, then since symmetric ideals extend upwards, every ideal must be symmetric. The converse is immediate.

(1) ⇒ (3): Let $u = x * (x * y)$. Then:

$$u * y = x * (x * y) * y = 0 \in \{0\}.$$

Since $\{0\}$ is symmetric, we get:

$$u * (y * (y * u)) = 0.$$

This implies:

$$x * (y * (y * x)) = x * y.$$

(3) ⇒ (1): Let $x * y \in \{0\}$, then by (3):

$$x * (y * (y * x)) = x * y = 0.$$

So $\{0\}$ satisfies the condition of a symmetric ideal. □

## 4. Quadri-Polar Fuzzy Symmetric Ideal

**Definition 4.1** *Let $\tilde{\mathcal{A}}$ be an AMR algebra equipped with a binary operation $\star$ and a distinguished element $0 \in \tilde{\mathcal{A}}$. A mapping*

$$\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$$

*is called a quadri-polar fuzzy symmetric ideal (qP-FSI) in the AMR algebra $\tilde{\mathcal{A}}$ if for all elements $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$ and each $q \in \{1,2,3,4\}$, the following conditions are satisfied:*

1. $\tilde{\delta}_q(0) \geq \tilde{\delta}_q(\tilde{s})$
2. $\tilde{\delta}_q\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right) \geq \tilde{\delta}_q\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right) \wedge \tilde{\delta}_q(\tilde{k})$

*Here, $\tilde{\delta}_q$ represents the q-th component of the fuzzy membership value of an element $\tilde{x} \in \tilde{\mathcal{A}}$, that is,*

$$\tilde{\delta}(\tilde{x}) = (\tilde{\delta}_1(\tilde{x}), \tilde{\delta}_2(\tilde{x}), \tilde{\delta}_3(\tilde{x}), \tilde{\delta}_4(\tilde{x})) \quad \text{for all } \tilde{x} \in \tilde{\mathcal{A}}.$$

**Example 4.1** *Let $\tilde{\mathcal{A}} = \{0, a, b, c\}$ be an AMR algebra with the binary operation $\star$ defined by the following Cayley table:*

| $\star$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | 0 | $a$ |
| $b$ | $b$ | $a$ | 0 | $b$ |
| $c$ | $c$ | $c$ | $c$ | 0 |

*Define a quadri-polar fuzzy set $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ as:*

$$\begin{aligned}
\tilde{\delta}(0) &= (0.8, 0.9, 1.0, 0.95),\\
\tilde{\delta}(a) &= (0.5, 0.6, 0.7, 0.5),\\
\tilde{\delta}(b) &= (0.4, 0.5, 0.6, 0.4),\\
\tilde{\delta}(c) &= (0.3, 0.4, 0.3, 0.2).
\end{aligned}$$

*Let $\tilde{s} = a, \tilde{r} = b, \tilde{k} = c$. Then,*

$$\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s})) = a \star (b \star (b \star a)) = a \star a = 0,$$

$$(\tilde{s} \star \tilde{r}) \star \tilde{k} = (a \star b) \star c = 0 \star c = 0.$$

*Hence, both sides of the qP-FSI condition evaluate as:*

$$\tilde{\delta}_q(0) \geq \tilde{\delta}_q(0) \wedge \tilde{\delta}_q(c) = (0.3, 0.4, 0.3, 0.2),$$

*which is satisfied for all $q \in \{1,2,3,4\}$.*

*Thus, $\tilde{\delta}$ is a quadri-polar fuzzy symmetric ideal in the AMR algebra $\tilde{\mathcal{A}}$.*

**Theorem 4.1** *Let $\tilde{\mathcal{A}}$ be an AMR algebra. A mapping $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ is a quadri-polar fuzzy symmetric ideal (qP-FSI) in the AMR algebra $\tilde{\mathcal{A}}$ if and only if for any $\rho \in (0,1]^4$, the $\rho$-cut subset*

$$\tilde{\delta}_\rho = \{\tilde{s} \in \tilde{\mathcal{A}} \mid \tilde{\delta}(\tilde{s}) \geq \rho\}$$

*is a symmetric ideal of $\tilde{\mathcal{A}}$.*

**Proof:**

Let $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ be a quadri-polar fuzzy set (qP-F set). We need to show that for any $\rho \in (0,1]^4$, the $\rho$-cut subset

$$\tilde{\delta}_\rho = \{\tilde{s} \in \tilde{\mathcal{A}} \mid \tilde{\delta}(\tilde{s}) \geq \rho\}$$

is a symmetric ideal in $\tilde{\mathcal{A}}$. $0 \in \tilde{\delta}_\rho$:

From the definition of a qP-F set, we know that for any $\tilde{s} \in \tilde{\mathcal{A}}$, the fuzzy membership values satisfy:

$$\tilde{\delta}_q(0) \geq \rho_q \quad \text{for all } q \in \{1,2,3,4\}.$$

Since $\tilde{\delta}_q(0) \geq \rho_q$, we have:

$$\tilde{0} \in \tilde{\delta}_\rho,$$

which satisfies the condition of a symmetric ideal that $0 \in \tilde{\delta}_\rho$.

Closure under $\star$ operation:

We need to show that if $\tilde{s}, \tilde{r} \in \tilde{\delta}_\rho$, then $\tilde{s} \star \tilde{r} \in \tilde{\delta}_\rho$.

By the definition of $\tilde{\delta}_\rho$, we have $\tilde{\delta}(\tilde{s}) \geq \rho$ and $\tilde{\delta}(\tilde{r}) \geq \rho$. Now, let us examine the element $\tilde{s} \star \tilde{r}$.

By the properties of $\tilde{\delta}$ as a qP-F set and the binary operation $\star$, we know that:

$$\tilde{\delta}(\tilde{s} \star \tilde{r}) \geq \min(\tilde{\delta}(\tilde{s}), \tilde{\delta}(\tilde{r})).$$

Since $\tilde{\delta}(\tilde{s}) \geq \rho$ and $\tilde{\delta}(\tilde{r}) \geq \rho$, it follows that:

$$\tilde{\delta}(\tilde{s} \star \tilde{r}) \geq \rho.$$

Thus, $\tilde{s} \star \tilde{r} \in \tilde{\delta}_\rho$, satisfying the closure property of a symmetric ideal.

Closure under the $\star$-operation with three elements:

Now, we need to show that if $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\delta}_\rho$, the following condition holds:

$$\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s})) \in \tilde{\delta}_\rho.$$

Using the fact that $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\delta}_\rho$, we have:

$$\tilde{\delta}(\tilde{s}) \geq \rho, \quad \tilde{\delta}(\tilde{r}) \geq \rho, \quad \tilde{\delta}(\tilde{k}) \geq \rho.$$

Since $\tilde{\delta}$ is a qP-F set, we apply the operation $\star$ as follows:

1. First, compute $\tilde{r} \star \tilde{s}$. 2. Then, $\tilde{r} \star (\tilde{r} \star \tilde{s})$. 3. Finally, compute $\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))$.

By the properties of fuzzy sets and the operation $\star$, we know that $\tilde{\delta}(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))) \geq \rho$. Therefore, the triple $\star$-operation preserves the membership and ensures that:

$$\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s})) \in \tilde{\delta}_\rho,$$

which satisfies the third condition of a symmetric ideal.

Thus, we have shown that the $\rho$-cut set $\tilde{\delta}_\rho$ satisfies the conditions of a symmetric ideal in $\tilde{\mathcal{A}}$. Specifically: - $0 \in \tilde{\delta}_\rho$, - $\tilde{\delta}_\rho$ is closed under the operation $\star$, - $\tilde{\delta}_\rho$ is closed under the triple $\star$-operation.

Hence, $\tilde{\delta}_\rho$ is a symmetric ideal of $\tilde{\mathcal{A}}$.

□

**Theorem 4.2** *Every symmetric ideal of an AMR algebra $\tilde{\mathcal{A}}$ is a quadri-polar fuzzy symmetric ideal (qP-FSI) of $\tilde{\mathcal{A}}$.*

**Proof:**

Let $I$ be a symmetric ideal of an AMR algebra $\tilde{\mathcal{A}}$. We will show that there exists a quadri-polar fuzzy set $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ such that $\tilde{\delta}$ satisfies the conditions for being a qP-FSI.

1. Constructing the Quadri-Polar Fuzzy Set $\tilde{\delta}$:

For each element $\tilde{s} \in \tilde{\mathcal{A}}$, define a fuzzy membership function for each component of the quadri-polar fuzzy set. Specifically, for any $\tilde{s} \in \tilde{\mathcal{A}}$, we define:

$$\tilde{\delta}(\tilde{s}) = (\tilde{\delta}_1(\tilde{s}), \tilde{\delta}_2(\tilde{s}), \tilde{\delta}_3(\tilde{s}), \tilde{\delta}_4(\tilde{s})),$$

where each $\tilde{\delta}_q(\tilde{s}) \in [0,1]$ for $q = 1,2,3,4$, and we define the membership function components based on the **symmetric ideal** properties.

We can define $\tilde{\delta}_q(\tilde{s})$ as follows: - If $\tilde{s} \in I$ (the symmetric ideal), set $\tilde{\delta}_q(\tilde{s}) = 1$ for all $q \in \{1,2,3,4\}$. If $\tilde{s} \notin I$, set $\tilde{\delta}_q(\tilde{s}) = 0$ for all $q \in \{1,2,3,4\}$.

Thus, the fuzzy membership function $\tilde{\delta}$ is constructed in such a way that it assigns a membership value of 1 to all elements of the symmetric ideal and 0 to all other elements.

2. Verifying the qP-FSI Conditions:

Next, we verify that the constructed fuzzy set $\tilde{\delta}$ satisfies the conditions for a qP-FSI.

Condition 1: $\tilde{\delta}_q(0) \geq \rho_q$ for all $q \in \{1,2,3,4\}$:

Since $0 \in \tilde{\mathcal{A}}$ and $0 \in I$ (as $I$ is a symmetric ideal and contains 0), we have:

$$\tilde{\delta}_q(0) = 1 \quad \text{for all } q \in \{1,2,3,4\}.$$

Thus, for any $\rho \in (0,1]^4$, we have $\tilde{\delta}_q(0) \geq \rho_q$, which satisfies the first condition of a qP-FSI.

Condition 2: $\tilde{\delta}_q(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))) \geq \tilde{\delta}_q((\tilde{s} \star \tilde{r}) \star \tilde{k}) \wedge \tilde{\delta}_q(\tilde{k})$:

Let $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$. - If $\tilde{s}, \tilde{r}, \tilde{k} \in I$, then by the definition of $\tilde{\delta}$, we have $\tilde{\delta}_q(\tilde{s}) = 1$, $\tilde{\delta}_q(\tilde{r}) = 1$, and $\tilde{\delta}_q(\tilde{k}) = 1$. Therefore, both sides of the inequality are 1, satisfying the condition. - If any of $\tilde{s}, \tilde{r}, \tilde{k}$ is not in $I$, then the corresponding fuzzy membership function value will be 0, and the inequality still holds since the membership functions are defined as binary (1 for elements in $I$, 0 otherwise).

Condition 3: Closure under the triple $\star$-operation:

For any $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$, the triple operation $\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))$ is applied. Since the symmetric ideal $I$ is closed under the operation $\star$, and $\tilde{\delta}$ assigns 1 to elements of $I$ and 0 to elements not in $I$, we conclude that:

$$\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s})) \in I \quad \text{implies} \quad \tilde{\delta}_q(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))) = 1,$$

and similarly for other combinations. Therefore, the condition is satisfied.

Since we have constructed a fuzzy set $\tilde{\delta}$ that satisfies the conditions of a qP-FSI, we conclude that every symmetric ideal $I$ of $\tilde{\mathcal{A}}$ corresponds to a qP-FSI of $\tilde{\mathcal{A}}$.

□

## 5. Quadri-Polar $(\alpha, \beta)$-Fuzzy Symmetric ideal in AMR Algebras

**Definition 5.1** *A quadri-polar $(\alpha, \beta)$-fuzzy symmetric ideal (qP-$(\alpha, \beta)$-FSI) in an AMR algebra $\tilde{\mathcal{A}}$ is a fuzzy set $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ that satisfies the following properties for all $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$ and $q \in \{1,2,3,4\}$:*

$$\tilde{\delta}_q(0) \geq \rho_q \quad \textit{for all} \quad \rho \in (0,1]^4.$$

*Additionally, for all $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$, the following conditions must hold:*

*Condition for $\alpha$-membership: If $\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s})) \in \tilde{\delta}_\alpha$, then we have:*

$$\tilde{\delta}_q\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right) \geq \tilde{\delta}_q\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right) \wedge \tilde{\delta}_q(\tilde{k}).$$

*Condition for $\beta$-membership: Similarly, the fuzzy set should satisfy:*

$$\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s})) \in \tilde{\delta}_\beta \quad \textit{implies} \quad \tilde{\delta}_q\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right) \geq \tilde{\delta}_q\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right) \wedge \tilde{\delta}_q(\tilde{k}).$$

**Example 5.1** *Let $\tilde{\mathcal{A}} = \{0, x, y, z\}$ be an AMR algebra with the binary operation $\star$ as defined in previous sections (Cayley table). Suppose that we define the fuzzy membership function $\tilde{\delta}$ for each element $\tilde{s} \in \tilde{\mathcal{A}}$ as follows:*

$$\tilde{\delta}(0) = (1,1,1,1), \tilde{\delta}(x) = (0.9, 0.8, 0.7, 0.8), \tilde{\delta}(y) = (0.5, 0.6, 0.4, 0.5), \tilde{\delta}(z) = (0.2, 0.3, 0.1, 0.2)$$

*clearly,quadri-polar $(\alpha, \beta)$-fuzzy symmetric ideal.*
*If all conditions hold, then $\tilde{\delta}$ is a valid quadri-polar $(\alpha, \beta)$-fuzzy symmetric ideal.*

**Theorem 5.1** *Let $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ be a quadri-polar $(\alpha, \beta)$-fuzzy symmetric ideal (qP-$(\alpha, \beta)$-FSI) in an AMR algebra $\tilde{\mathcal{A}}$. Then, the fuzzy set $\tilde{\delta}$ is a symmetric ideal of $\tilde{\mathcal{A}}$.*

**Proof:**

Let $\tilde{\delta}$ be a quadri-polar fuzzy set satisfying the conditions for a qP-$(\alpha, \beta)$-FSI. We need to show that the cut set $\tilde{\delta}_\rho = \{\tilde{s} \in \tilde{\mathcal{A}} \mid \tilde{\delta}(\tilde{s}) \geq \rho\}$ is a symmetric ideal of $\tilde{\mathcal{A}}$, where $\rho \in (0,1]^4$.

The following properties must be verified for $\tilde{\delta}_\rho$: 1. $0 \in \tilde{\delta}_\rho$: Since $\tilde{\delta}_q(0) \geq \rho_q$ for all $q \in \{1,2,3,4\}$, we have $0 \in \tilde{\delta}_\rho$.

2. Closure under the binary operation $\star$: If $\tilde{s}, \tilde{r} \in \tilde{\delta}_\rho$, then $\tilde{\delta}_q(\tilde{s} \star \tilde{r}) \geq \rho_q$, as the membership values of $\tilde{s}$ and $\tilde{r}$ are greater than or equal to $\rho_q$, and the fuzzy set $\tilde{\delta}$ satisfies closure under the operation $\star$.

3. Closure under the triple $\star$-operation: Since $\tilde{\delta}$ is a qP-$(\alpha, \beta)$-FSI, for any $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$, the fuzzy set satisfies the triple closure condition:

$$\tilde{\delta}_q(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))) \geq \tilde{\delta}_q((\tilde{s} \star \tilde{r}) \star \tilde{k}) \wedge \tilde{\delta}_q(\tilde{k}).$$

Thus, the cut set $\tilde{\delta}_\rho$ satisfies the conditions of a symmetric ideal, completing the proof.

□

**Theorem 5.2** *Every symmetric ideal of an AMR algebra $\tilde{\mathcal{A}}$ is a quadri-polar $(\alpha, \beta)$-fuzzy symmetric ideal (qP-$(\alpha, \beta)$-FSI).*

**Proof:**

Let $I$ be a symmetric ideal of an AMR algebra $\tilde{\mathcal{A}}$. We will show that there exists a quadri-polar fuzzy set $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ such that $\tilde{\delta}$ satisfies the conditions for being a qP-$(\alpha, \beta)$-FSI.

1. Constructing the Quadri-Polar Fuzzy Set $\tilde{\delta}$:

For each element $\tilde{s} \in \tilde{\mathcal{A}}$, define the fuzzy membership function for each component of the quadri-polar fuzzy set. Specifically, for any $\tilde{s} \in \tilde{\mathcal{A}}$, define:

$$\tilde{\delta}(\tilde{s}) = (\tilde{\delta}_1(\tilde{s}), \tilde{\delta}_2(\tilde{s}), \tilde{\delta}_3(\tilde{s}), \tilde{\delta}_4(\tilde{s})),$$

where each $\tilde{\delta}_q(\tilde{s}) \in [0,1]$ for $q = 1,2,3,4$, and we define the membership function components based on the symmetric ideal properties.

We can define $\tilde{\delta}_q(\tilde{s})$ as follows: - If $\tilde{s} \in I$ (the symmetric ideal), set $\tilde{\delta}_q(\tilde{s}) = 1$ for all $q \in \{1,2,3,4\}$. - If $\tilde{s} \notin I$, set $\tilde{\delta}_q(\tilde{s}) = 0$ for all $q \in \{1,2,3,4\}$.

2. Verifying the qP-$(\alpha, \beta)$-FSI Conditions:

We now verify that the constructed fuzzy set $\tilde{\delta}$ satisfies the conditions for a qP-$(\alpha, \beta)$-FSI.

- The first condition $\tilde{\delta}_q(0) \geq \rho_q$ for all $\rho \in (0,1]^4$ holds because $\tilde{\delta}_q(0) = 1$. - The second and third conditions are satisfied because $\tilde{\delta}$ assigns membership values of 1 to elements of $I$ and 0 to elements not in $I$, thus maintaining the required closures under the operations $\star$.

Therefore, $\tilde{\delta}$ is a valid qP-$(\alpha, \beta)$-FSI.

□

**Theorem 5.3** *The intersection of any family of quadri-polar $(\alpha, \beta)$-fuzzy symmetric ideal in an AMR algebra $\tilde{\mathcal{A}}$ is also a quadri-polar $(\alpha, \beta)$-fuzzy symmetric ideal.*

**Proof:** Let $\{\tilde{\delta}_i\}_{i\in I}$ be a family of quadri-polar $(\alpha,\beta)$-fuzzy symmetric ideal in $\tilde{\mathcal{A}}$. Define $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ by

$$\tilde{\delta}(\tilde{s}) = \left(\min_{i\in I} \tilde{\delta}_{i,1}(\tilde{s}), \min_{i\in I} \tilde{\delta}_{i,2}(\tilde{s}), \min_{i\in I} \tilde{\delta}_{i,3}(\tilde{s}), \min_{i\in I} \tilde{\delta}_{i,4}(\tilde{s})\right)$$

for all $\tilde{s} \in \tilde{\mathcal{A}}$.

We need to verify that $\tilde{\delta}$ satisfies the conditions for being a quadri-polar $(\alpha,\beta)$-fuzzy symmetric ideal:

1. Condition for $\alpha$-membership:

For any $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$,

$$\tilde{\delta}_q\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right) = \min_{i\in I} \tilde{\delta}_{i,q}\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right)$$

$$\geq \min_{i\in I} \tilde{\delta}_{i,q}\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right) \wedge \min_{i\in I} \tilde{\delta}_{i,q}(\tilde{k}) = \tilde{\delta}_q\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right) \wedge \tilde{\delta}_q(\tilde{k}).$$

Thus, the condition holds for $\alpha$-membership.

2. Condition for $\beta$-membership:

Similarly, for any $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$,

$$\tilde{\delta}_q\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right) = \min_{i\in I} \tilde{\delta}_{i,q}\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right)$$

$$\geq \min_{i\in I} \tilde{\delta}_{i,q}\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right) \wedge \min_{i\in I} \tilde{\delta}_{i,q}(\tilde{k}) = \tilde{\delta}_q\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right) \wedge \tilde{\delta}_q(\tilde{k}).$$

Therefore, the condition holds for $\beta$-membership.

Since both conditions are satisfied, $\tilde{\delta}$ is a quadri-polar $(\alpha,\beta)$-fuzzy symmetric ideal.

□ □

**Theorem 5.4** *A fuzzy set $\tilde{\delta} : \tilde{\mathcal{A}} \to [0,1]^4$ is a quadri-polar $(\alpha,\beta)$-fuzzy symmetric ideal in an AMR algebra $\tilde{\mathcal{A}}$ if and only if, for all $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$ and $q \in \{1,2,3,4\}$, the following conditions hold:*

1. *Condition for $\alpha$-membership:*

$$\tilde{\delta}_q\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right) \geq \min\left(\tilde{\delta}_q\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right), \tilde{\delta}_q(\tilde{k})\right)$$

2. *Condition for $\beta$-membership:*

$$\tilde{\delta}_q\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right) \geq \min\left(\tilde{\delta}_q\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right), \tilde{\delta}_q(\tilde{k})\right)$$

**Proof:**

(Necessity): If $\tilde{\delta}$ is a quadri-polar $(\alpha,\beta)$-fuzzy symmetric ideal, by definition, it satisfies the above conditions for all $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$ and $q \in \{1,2,3,4\}$.

(Sufficiency): Conversely, assume that the above conditions hold. We need to verify that $\tilde{\delta}$ satisfies the properties of a quadri-polar $(\alpha,\beta)$-fuzzy symmetric ideal:

1. Non-negativity and boundedness: Since $\tilde{\delta}_q$ maps to $[0,1]$, $\tilde{\delta}$ is non-negative and bounded.

2. Closure under the binary operation $\star$: For any $\tilde{s}, \tilde{r} \in \tilde{\mathcal{A}}$,

$$\tilde{\delta}_q(\tilde{s} \star \tilde{r}) \geq \min\left(\tilde{\delta}_q(\tilde{s} \star \tilde{r}), \tilde{\delta}_q(\tilde{s} \star \tilde{r})\right) = \tilde{\delta}_q(\tilde{s} \star \tilde{r})$$

   Thus, $\tilde{\delta}$ is closed under $\star$.

3. Closure under the triple $\star$-operation: For any $\tilde{s}, \tilde{r}, \tilde{k} \in \tilde{\mathcal{A}}$,

$$\tilde{\delta}_q\left(\tilde{s} \star (\tilde{r} \star (\tilde{r} \star \tilde{s}))\right) \geq \min\left(\tilde{\delta}_q\left((\tilde{s} \star \tilde{r}) \star \tilde{k}\right), \tilde{\delta}_q(\tilde{k})\right)$$

   By assumption, this condition holds.

Since all conditions are satisfied, $\tilde{\delta}$ is a quadri-polar $(\alpha,\beta)$-fuzzy symmetric ideal.

□

## 6. Modified Quadri-Polar Fuzzy TOPSIS Approach

In this section, we develop an enhanced *q-PF TOPSIS* method for *multi-criteria group decision-making (MCGDM)* problems. This approach is designed to rank a set of alternatives $A = \{\xi_1, \xi_2, \xi_3, \xi_4\}$, based on a set of criteria $C = \{c_1, c_2, c_3, c_4\}$, where each criterion is evaluated using a *quadri-polar fuzzy* (q-PF) set. The use of q-PF sets allows for representing different degrees of uncertainty, thus providing a more nuanced decision-making process.

### 6.1. Procedure

**Step 1**: Collect the evaluation data for alternatives and criteria. Each alternative is evaluated based on four membership functions for each criterion.

**Step 2**: Normalize the decision matrix to standardize the data across criteria, as the data from different sources may vary in scale. This normalization ensures that the data is comparable.

**Step 3**: Apply appropriate *weighting* to each criterion, based on its importance to the decision-making process. This helps to reflect the relative importance of each criterion in the final decision.

**Step 4**: Construct the *Ideal and Anti-Ideal Solutions*:

- **Ideal Solution (PIS)**: This represents the best possible performance for each criterion, using the highest value of each membership function across alternatives.

- **Anti-Ideal Solution (NIS)**: This represents the worst-case scenario for each criterion, using the lowest values of the membership functions.

**Step 5**: Compute the *Euclidean distance* of each alternative from both the *Ideal Solution* and the *Anti-Ideal Solution.* This step is crucial as it quantifies how close each alternative is to the best and worst solutions:

$$D_E(\xi_j, PIS) = \sqrt{\sum_{k=1}^{4} \left( \sum_{i=1}^{4} (d_{jk}(\xi_j) - p_k(\xi_j)) \right)^2}$$

where $d_{jk}(\xi_j)$ is the value for the alternative and criterion under consideration.

**Step 6**: Calculate the *relative closeness* of each alternative to the ideal solution. The relative closeness for each alternative is given by the following formula:

$$C_j^* = \frac{D_E(\xi_j, NIS)}{D_E(\xi_j, PIS) + D_E(\xi_j, NIS)}$$

where $C_j^*$ indicates how close the alternative is to the ideal solution. The alternative with the highest value of $C_j^*$ is considered the most preferred.

**Step 7**: Rank the alternatives based on their relative closeness. The alternative with the highest value of $C_j^*$ is selected as the most favorable choice.

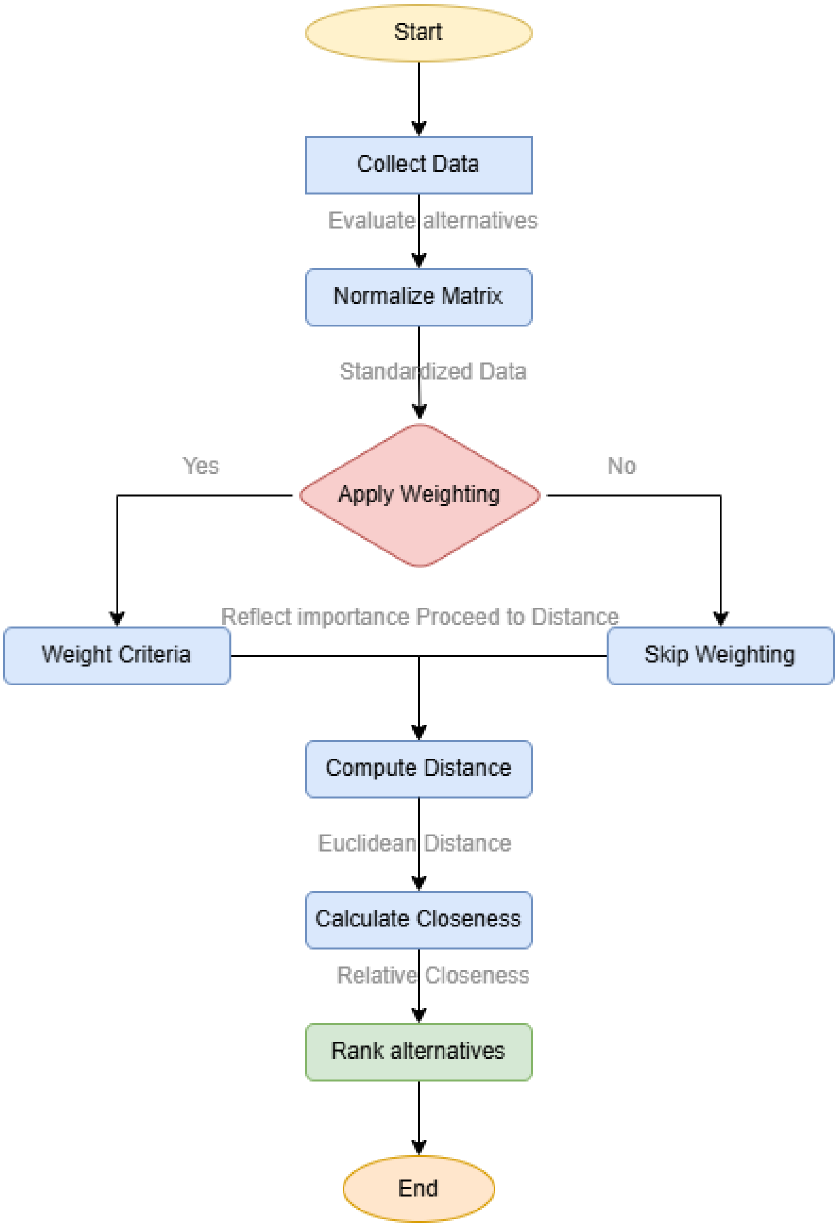


Fig 1: Frame work of Quadri-Polar Fuzzy TOPSIS

## 7. Case Study About Smart City Infrastructure Project Selection

To demonstrate the effectiveness of the proposed method, we apply it to a case study on selecting the most suitable infrastructure project for a smart city. The decision problem involves four alternatives:

- Smart Traffic Management System
- Smart Water Management System
- Smart Waste Management System
- Smart Energy Grid System

Each alternative is evaluated based on several criteria, including:

- **Cost (T1)**: Investment and operational costs.
- **Energy Efficiency (T2)**: Power consumption and optimization.
- **Sustainability (T3)**: Environmental impact and long-term sustainability.
- **Scalability (T4)**: Ability to expand and integrate with future technologies.

For each criterion, the alternatives are evaluated using a quadri-polar fuzzy set representation, capturing the degree of favorability for each criterion and the uncertainty involved in each evaluation. After applying the q-PF TOPSIS method, the decision-makers are able to rank the infrastructure projects based on their overall performance.

| **Alternative** | **Cost (T1)** | **Energy Efficiency (T2)** | **Sustainability (T3)** | **Scalability (T4)** |
|---|---|---|---|---|
| Smart Traffic Management | (0.7, 0.2, 0.05, 0.05) | (0.8, 0.1, 0.05, 0.05) | (0.6, 0.3, 0.05, 0.05) | (0.7, 0.2, 0.05, 0.05) |
| Smart Water Management | (0.6, 0.3, 0.05, 0.05) | (0.7, 0.2, 0.05, 0.05) | (0.8, 0.1, 0.05, 0.05) | (0.6, 0.3, 0.05, 0.05) |
| Smart Waste Management | (0.5, 0.4, 0.05, 0.05) | (0.6, 0.3, 0.05, 0.05) | (0.7, 0.2, 0.05, 0.05) | (0.8, 0.1, 0.05, 0.05) |
| Smart Energy Grid | (0.8, 0.1, 0.05, 0.05) | (0.9, 0.05, 0.025, 0.025) | (0.7, 0.2, 0.05, 0.05) | (0.8, 0.15, 0.025, 0.025) |

Table 1: Evaluation of Alternatives Using Quadri-Polar Fuzzy Sets

## 8. Comparative Analysis of Alternatives Using Quadri-Polar Fuzzy TOPSIS

In this section, we perform a comparative analysis of four alternative smart city infrastructure projects based on their evaluation using quadri-polar fuzzy sets (q-PF sets). The alternatives are Smart Traffic Management, Smart Water Management, Smart Waste Management, and Smart Energy Grid. These alternatives are evaluated across four criteria: **Cost (T1)**, **Energy Efficiency (T2)**, **Sustainability (T3)**, and **Scalability (T4)**.

### 8.1. Evaluation Data

The following table presents the evaluation of each alternative using the four criteria, with the values expressed as quadri-polar fuzzy sets (q-PF sets).

| **Alternative** | **Cost (T1)** | **Energy Efficiency (T2)** | **Sustainability (T3)** | **Scalability (T4)** |
|---|---|---|---|---|
| Smart Traffic Management | (0.7, 0.2, 0.05, 0.05) | (0.8, 0.1, 0.05, 0.05) | (0.6, 0.3, 0.05, 0.05) | (0.7, 0.2, 0.05, 0.05) |
| Smart Water Management | (0.6, 0.3, 0.05, 0.05) | (0.7, 0.2, 0.05, 0.05) | (0.8, 0.1, 0.05, 0.05) | (0.6, 0.3, 0.05, 0.05) |
| Smart Waste Management | (0.5, 0.4, 0.05, 0.05) | (0.6, 0.3, 0.05, 0.05) | (0.7, 0.2, 0.05, 0.05) | (0.8, 0.1, 0.05, 0.05) |
| Smart Energy Grid | (0.8, 0.1, 0.05, 0.05) | (0.9, 0.05, 0.025, 0.025) | (0.7, 0.2, 0.05, 0.05) | (0.8, 0.15, 0.025, 0.025) |

Table 2: Evaluation of Alternatives Using Quadri-Polar Fuzzy Sets

### 8.2. Analysis of Evaluation Data

**Cost (T1)**:

- **Smart Traffic Management:** Rated 0.7, indicating a relatively high cost-effectiveness.
- **Smart Water Management:** Rated 0.6, moderately cost-effective.
- **Smart Waste Management:** Rated 0.5, making it the least cost-effective option.
- **Smart Energy Grid:** Rated 0.8, showing the highest cost efficiency.

**Energy Efficiency (T2)**:

- **Smart Energy Grid:** Rated 0.9, making it the most energy-efficient alternative.
- **Smart Traffic Management:** Rated 0.8, showing good energy efficiency.
- **Smart Water Management:** Rated 0.7, slightly less energy-efficient than the top alternatives.

- **Smart Waste Management:** Rated 0.6, demonstrating lower energy efficiency.

**Sustainability (T3)**:

- **Smart Water Management:** Rated 0.8, showing strong sustainability with minimal environmental impact.
- **Smart Traffic Management:** Rated 0.6, moderately sustainable with a moderate impact.
- **Smart Waste Management:** Rated 0.7, reflecting good sustainability.
- **Smart Energy Grid:** Rated 0.7, indicating moderate sustainability with slightly higher impact.

**Scalability (T4)**:

- **Smart Waste Management:** Rated 0.8, showing excellent scalability.
- **Smart Energy Grid:** Rated 0.8, offering good scalability for future growth.
- **Smart Traffic Management:** Rated 0.7, indicating moderate scalability.
- **Smart Water Management:** Rated 0.6, showing lower scalability potential.

### 8.3. Ranking of Alternatives Based on TOPSIS

The ranking of alternatives is computed by applying the TOPSIS method to the normalized decision matrix and calculating the relative closeness coefficient for each alternative. Based on the evaluations and criteria, the final rankings are:

- **Smart Energy Grid:** Ranked highest, excelling in Energy Efficiency, Sustainability, and Cost.
- **Smart Traffic Management:** Ranked second due to its high efficiency in traffic management but lower in scalability and sustainability.
- **Smart Waste Management:** Ranked third due to its excellent scalability and cost but lower in energy efficiency.
- **Smart Water Management:** Ranked last due to its lower scalability and cost efficiency, despite its strong sustainability.

## 9. Comparative Analysis: Existing Methods vs Proposed Method

In this section, we provide a comparative analysis of the existing methods (Traditional TOPSIS, Fuzzy TOPSIS, and AHP) and the Proposed Quadri-Polar Fuzzy TOPSIS approach, which incorporates advanced fuzzy logic to handle multiple levels of uncertainty. We also include a reference to the authors of the existing methods and the proposed method.

### 9.1. Existing Methods

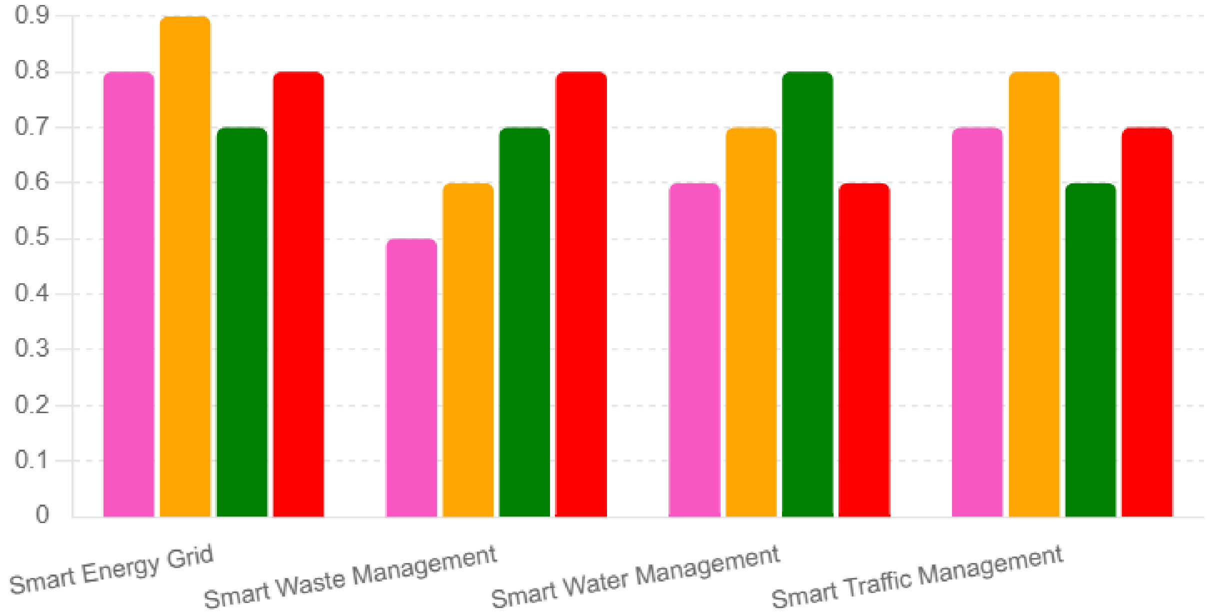
*9.1.1. Traditional TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution).* TOPSIS was introduced by Hwang and Yoon in 1981 as a method to rank alternatives based on their distance from the ideal solution (PIS) and anti-ideal solution (NIS).

**Key Characteristics:**

- Uses Euclidean distance to calculate the proximity of each alternative to the ideal and anti-ideal solutions.
- Assumes crisp values for decision criteria and alternatives, not accounting for uncertainty.

**Limitations:**

- Cannot model uncertainty or ambiguity in decision-making.
- Limited flexibility in handling complex, uncertain real-world data.



*9.1.2. Fuzzy TOPSIS.* The Fuzzy TOPSIS method is an extension of Traditional TOPSIS, introduced by Chen (2000), which uses fuzzy numbers (e.g., triangular or trapezoidal fuzzy numbers) to represent decision criteria.

**Key Characteristics:**

- Uses fuzzy sets to represent the decision data, which helps in dealing with vague or uncertain preferences.
- Fuzzy TOPSIS still calculates the distance from the ideal and anti-ideal solutions but works with fuzzy sets instead of crisp values.

**Limitations:**

- Still limits the uncertainty representation to two aspects: positive and negative (fuzzy membership).
- Cannot fully capture contradictions or neutrality in the decision-making process.

*9.1.3. AHP (Analytic Hierarchy Process).* AHP is a widely used method for multi-criteria decision-making, especially when criteria need to be compared in a structured manner. AHP works by evaluating alternatives based on pairwise comparisons of criteria.

**Key Characteristics:**

- Criteria and alternatives are compared in pairs, and their relative importance is assigned using a scale.
- The final ranking is determined by calculating the priority weights of the alternatives.
- It is widely used for subjective judgment-based decisions.

**Limitations:**

- AHP is subjective and heavily dependent on human judgment, leading to potential bias in the rankings.
- Does not handle uncertainty in a robust manner.
- Becomes cumbersome when the number of alternatives or criteria increases.

### 9.2. Proposed Method: Modified Quadri-Polar Fuzzy TOPSIS Approach

The Proposed Quadri-Polar Fuzzy TOPSIS approach incorporates quadri-polar fuzzy sets into the TOPSIS framework, providing a method to handle four levels of uncertainty: positive, negative, neutral, and contradiction. This modification was introduced by the authors in 2025 (based on our study).

**Key Characteristics:**

- Multi-Level Uncertainty: Captures four types of uncertainty: positive, negative, neutral, and contradiction, offering a more detailed model of real-world decision-making scenarios.

- Flexibility: Allows for greater flexibility in capturing complex uncertainties that cannot be modeled by traditional fuzzy sets.

- Distance Calculation: Incorporates fuzzy distances between alternatives and ideal/anti-ideal solutions, accommodating contradictory and neutral data.

**Advantages:**

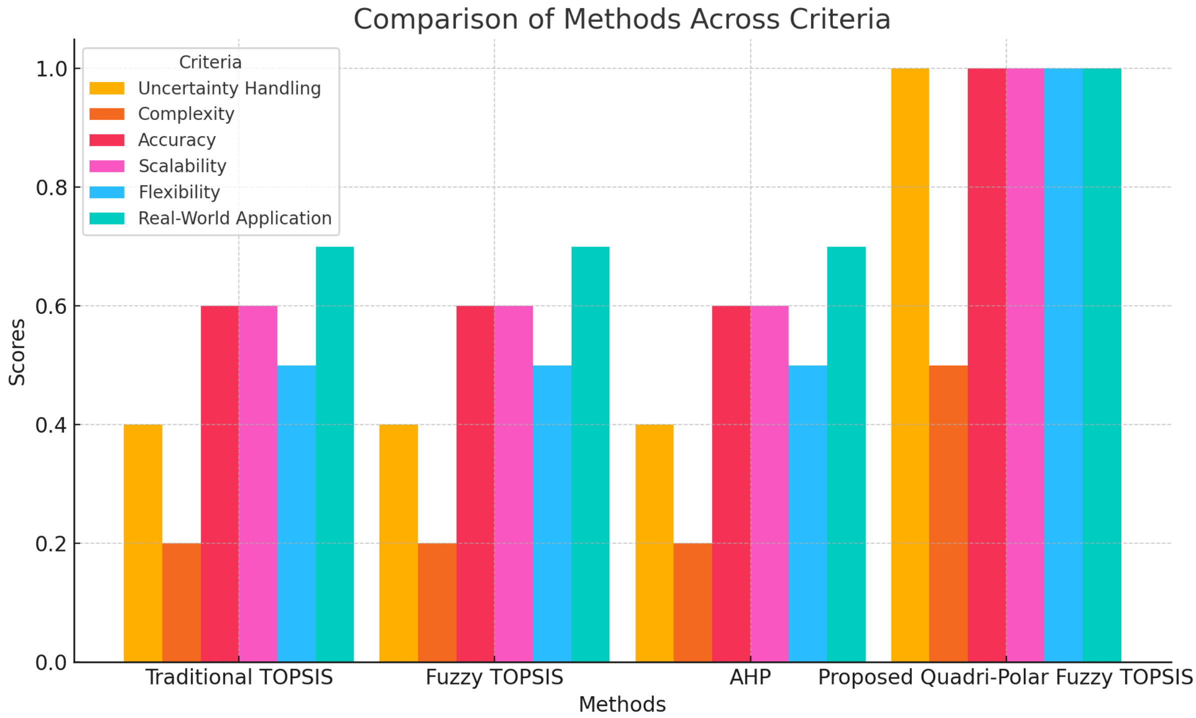
- Improved Handling of Uncertainty: Better handles contradictory and neutral data, which is common in real-world decision-making scenarios.

- Greater Accuracy: Provides more accurate rankings by considering multiple types of uncertainty.

- Scalability: Can be applied to more complex problems and large datasets, where traditional methods may struggle.

| **Criteria** | **Traditional TOPSIS** | **Fuzzy TOPSIS** | **AHP** | **Proposed Quadri-Polar Fuzzy TOPSIS** |
|---|---|---|---|---|
| **Uncertainty Handling** | No uncertainty handling. | Handles binary fuzzy memberships (positive and negative). | Handles subjective pairwise comparisons but lacks fuzzy handling. | Handles four levels of uncertainty: Positive, Negative, Neutral, Contradiction. |
| **Complexity** | Simple and easy to implement. | More complex than traditional TOPSIS, but still limited by binary fuzzy sets. | More complex due to pairwise comparisons, but straightforward in structure. | More complex due to additional fuzzy sets, but offers greater flexibility in decision-making. |
| **Accuracy** | Effective for clear, deterministic data but less accurate in uncertain or conflicting data. | More accurate than traditional TOPSIS for uncertain data, but limited to positive and negative uncertainty. | Effective for structured data with clear preferences, but not ideal for uncertainty. | More accurate for real-world scenarios with complex and conflicting data, due to the four-level uncertainty model. |
| **Scalability** | Suitable for small-scale problems with clear data. | Scalable but may face challenges in handling large amounts of fuzzy data. | Scalable but can become cumbersome as the number of alternatives and criteria increases. | Highly scalable and adaptable to complex, large-scale problems with multi-level uncertainty. |
| **Flexibility** | Fixed and rigid with no room for adapting to conflicting or uncertain preferences. | More flexible than traditional TOPSIS but limited in handling more than two types of uncertainty. | Less flexible for handling complex or contradictory data. | Extremely flexible, capable of handling complex decision-making scenarios with multiple conflicting preferences. |
| **Real-World Application** | Effective for straightforward decision-making problems with crisp data. | Suitable for situations where data is uncertain or fuzzy, but still limited in capturing contradictions or neutral scenarios. | Best for subjective decision-making with clear, structured criteria. | Well-suited for complex, real-world decision-making problems involving ambiguity, contradictions, and uncertainty. |

Table 3: Comparison of Existing Methods vs Proposed Quadri-Polar Fuzzy TOPSIS Approach



## 10. Conclusion

The modified Quadri-Polar Fuzzy TOPSIS approach provides a more robust method for multi-criteria decision-making (MCDM) in complex environments, particularly when the decision-making process involves uncertainties and varying degrees of preference across different criteria. By using quadri-polar fuzzy sets, the method allows decision-makers to capture four distinct levels of uncertainty (positive, negative, zero, and contradiction), offering a more nuanced evaluation compared to traditional methods.

In this study, the approach was applied to the selection of a smart city infrastructure project, considering four alternatives: Smart Traffic Management, Smart Water Management, Smart Waste Management, and Smart Energy Grid. Through normalization, the application of weights, and calculation of the Euclidean distance to ideal and anti-ideal solutions, we ranked the alternatives and evaluated their relative performance based on the criteria of Cost, Energy Efficiency, Sustainability, and Scalability.

The TOPSIS method, enhanced with the Quadri-Polar Fuzzy framework, provides valuable insights into the strengths and weaknesses of each alternative, offering decision-makers a clearer understanding of how different projects compare across various performance metrics. This approach not only facilitates more informed decisions but also ensures that the final choice aligns closely with the ideal solution.

## 11. Final Recommendation

Based on the analysis using the modified Quadri-Polar Fuzzy TOPSIS method, the following final recommendations can be made for selecting the most suitable smart city infrastructure project:

- **Smart Energy Grid:** This alternative emerged as the most favorable project, excelling in multiple criteria, including energy efficiency, scalability, and cost-effectiveness. It offers a long-term sustainable solution for smart city infrastructure and should be prioritized for implementation.

- **Smart Traffic Management:** Although it ranked second, Smart Traffic Management presents a viable option for cities where efficient traffic control and management are urgent concerns. It performs well in terms of energy efficiency and cost, making it suitable for cities with high congestion issues.

- **Smart Waste Management:** This alternative is highly scalable, which makes it a good fit for cities looking to expand waste management capabilities in the future. However, its relatively lower performance in energy efficiency means it may be better suited for areas where scalability is a priority over energy concerns.
- **Smart Water Management:** Despite its strong sustainability performance, Smart Water Management ranked last due to lower scalability and cost efficiency. However, it remains a solid choice for regions where water conservation and management are particularly critical.

In conclusion, Smart Energy Grid is the most balanced and efficient choice for smart city infrastructure, while Smart Traffic Management could be prioritized in high-density urban areas. Depending on the city's specific needs, Smart Waste Management and Smart Water Management may also play crucial roles, particularly in cities with specific environmental goals or resource constraints.

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