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Global Existence, Blow-Up, and Lower Bound Estimates for a Variable Exponent Parabolic Problem with High Initial Energy

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ABSTRACT: In this paper, we study an initial-boundary value problem involving the variable exponent $p(x)$-Laplacian operator under Robin boundary conditions. By combining the potential well method with the Nehari manifold and the ω-limit set, we establish results on the global existence and nonexistence of solutions when the initial energy $J(u_0)$ exceeds the mountain pass level d. Moreover, we derive both upper and lower bounds for the blow-up time of solutions. The present work complements and extends the results obtained by El Minsari and Ourraoui [*São Paulo J. Math. Sci.* **19**, 32 (2025)], where the subcritical and critical energy cases were investigated. Here, we complete the study by considering the supercritical case.

Key Words: $p(x)$-Laplacian operator, parabolic equation, global existence, blow-up, upper-lower bound.

Contents

1. Introduction and Main Results

In this paper, we deal with a solution of the following parabolic equation subject to the Robin boundary condition involving a variable exponent:

$$\begin{cases} u_t(x,t) - \Delta_{p(x)} u(x,t) = |u(x,t)|^{q(x)-2} u(x,t), \quad (x,t) \in \Omega \times (0,+\infty), \\ |\nabla u(x,t)|^{p(x)-2} \dfrac{\partial u}{\partial n} + \beta(x)|u(x,t)|^{p(x)-2} = 0, \quad (x,t) \in \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, $u_0 : \Omega \to \mathbb{R}$ is the initial data function, $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$, $\beta(x) \in L^\infty(\partial\Omega)$ with $\beta^- = \inf_{x \in \partial\Omega} \beta(x) > 0$, and $p(\cdot) \in C^+(\bar{\Omega})$, $q(\cdot) \in C^+(\Omega)$ satisfy the following hypothesis:
(H) : $2 \leq p^- \leq p^+ < q^- \leq q^+ \leq p^- \left(1 + \dfrac{2}{N}\right)$
where

$$p^- = \min_{(x,y) \in \bar{\Omega} \times \bar{\Omega}} p(x,y), \quad p^+ = \max_{(x,y) \in \bar{\Omega} \times \bar{\Omega}} p(x,y) < \infty,$$

$$q^- = \min_{x \in \bar{\Omega}} q(x), \quad q^+ = \max_{x \in \bar{\Omega}} q(x) < \infty.$$

Such problems with boundary constraints, especially the Robin condition, are central in engineering, physics, and applied mathematics. A classical example is the heat equation with Robin boundary conditions, which models temperature evolution in a medium while accounting for heat transfer through its boundary. This condition captures both the processes near the boundary and the thermal properties of materials at the interface, making it essential in modeling heat exchange and cooling phenomena. See, e.g., [13,8,2,4] and references therein.

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The notion of finite-time blow-up for solutions of (1.1) is of both mathematical and physical interest. In this context, "blow-up" refers to the situation where the solution (or some of its norms) becomes unbounded in finite time. Physically, such a phenomenon corresponds to processes like thermal runaway in heat conduction, explosive chemical reactions, or population explosions in biology. Identifying whether solutions remain globally bounded or blow up in finite time, and estimating the precise time of blow-up, is therefore crucial for both the theoretical understanding and the practical prediction of these systems.

Several works have been devoted to the study of global existence, blow-up, and asymptotic behavior of solutions to the problem:

$$u_t(x,t) - \Delta_{p(x)}u(x,t) = |u(x,t)|^{q(x)-2}u(x,t), \quad (x,t) \in \Omega \times (0,+\infty), \tag{1.2}$$

subject to homogeneous Dirichlet boundary conditions; see, for instance, [10,12,16,19]. In [12], the authors proved the existence of global weak solutions, at least for small initial data. Furthermore, they derived the decay of energy using differential inequalities and applying a non-standard approach.

Using the potential well method, the authors in [16] studied problem (1.2) and extended this method to analyze threshold results for the existence and nonexistence of global solutions when the initial energy is less than the mountain pass level d. They also examined non-global existence results for high-energy initial data.

Xiangyu Zhu et al. [19] investigated problem (1.2) and obtained global existence and blow-up results for weak solutions with arbitrarily high initial energy.

Liao in [11] obtained a non-global existence result for problem (1.2) by combining the concavity method with differential inequalities when the initial energy is positive and bounded.

Until recently, El Minsari and Ourraoui [7] dealt with the local existence and uniqueness of weak solutions and the asymptotic behavior of solutions to Eq.(1.1) with subcritical and critical energy.

As far as we know, many methods have been proposed to estimate upper bounds for blow-up times, whereas obtaining lower bounds is considerably more difficult. Pioneering work in this direction was carried out by Payne, Schaefer, and Philippin, and more recently important advances have been made on lower bounds for blow-up times under Robin boundary conditions (see [6,15,18]). Most existing results on lower bounds assume a priori that the solution blows up in finite time; see, for instance, [5] and the references therein. In contrast, in this paper we establish both upper and lower bounds for the blow-up time directly from the energy method. The key novelty lies in our derivation of the lower bound, achieved through a new combination of the interpolation inequality with Nehari functionals in variable exponent spaces.

Motivated by the aforementioned works, we employ

the potential well theory combined with the Nehari manifold and the ω-limit set, we investigate the global existence and finite-time blow-up of solutions to problem (1.1) with supercritical initial energy (i.e., $J(u_0) > d$). Finally, in cases where blow-up occurs, we derive upper and lower bounds for the blow-up time.

We now proceed with some definitions and present our main results.

Definition 1.1 *(Weak solution). A function $u(x,t)$ is called a weak solution of problem (1.1) on $\Omega \times [0,T)$, if $u \in L^\infty(0,T;W^{1,p(x)}(\Omega))$ with $u_t \in L^2(0,T;L^2(\Omega))$ satisfies (1.1) in the distribution sens, i.e., $\forall v \in W^{1,p(x)}(\Omega), t \in [0,T)$,*

$$\int_\Omega u_t v dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv dx = \int_\Omega |u|^{q(x)-2}uv dx,$$

and

$$u(x,0) = u_0(x) \in W^{1,p(x)}(\Omega).$$

let's keep the same notations as in [7]. We introduce the Nehari fonctionals associated to problem (1.1):

$$J(u) := J(t) := I_{\beta/p}(u) - \rho_{q/q}(u), \tag{1.3}$$

and

$$K(u) := K(t) := I_\beta(u) - \rho_q(u), \tag{1.4}$$

where

$$I_\beta(u) := \int_\Omega |\nabla u|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}dx,$$

$$I_{\beta/p}(u) := \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_{\partial\Omega} \beta(x)\frac{|u|^{p(x)}}{p(x)}dx,$$

and

$$\rho_{q/q}(u) := \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx, \quad \rho_q(u) := \int_\Omega |u|^{q(x)}dx.$$

Associated with J the Nehari manifold

$$\mathcal{N} := \{u \in W^{1,p(x)}(\Omega) : K(u) = 0, ||u||_{\beta(x)} > 0\},$$

where

$$||u||_{\beta(x)} = \inf\{\lambda > 0 : I_\beta(\frac{u}{\lambda}) \leq 1\},$$

which is equivalent to

$$||u||_{W^{1,p(x)}(\Omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{\nabla u}{\lambda}|^{p(x)}dx + \int_\Omega |\frac{u}{\lambda}|^{p(x)}dx \leq 1\}.$$

The montain pass level is defined as

$$d := \inf\{\sup_{\xi>0} J(\xi u) : u \in W^{1,p(x)}(\Omega), ||u||_{\beta(x)} > 0\} = \inf_{u\in\mathcal{N}} J(u). \tag{1.5}$$

In addition, we introduce

$$\mathcal{N}^+ := \{u \in W^{1,p(x)}(\Omega) : K(u) > 0\}, \quad \mathcal{N}^- := \{u \in W^{1,p(x)}(\Omega) : K(u) < 0\}.$$

Also, for all $\delta > d$, we denote

$$J^\delta := \{u \in W^{1,p(x)}(\Omega) : J(u) \leq \delta\}, \quad \mathcal{N}_\delta := \mathcal{N} \cap J^\delta.$$

Next, define

$$\lambda_\delta := \inf_{u\in\mathcal{N}_\delta} \|u\|_2^2, \quad \Lambda_\delta := \sup_{u\in\mathcal{N}_\delta} \|u\|_2^2.$$

It is easy to check that the function $\delta \longmapsto \lambda_\delta$(resp. $\delta \longmapsto \Lambda_\delta$) is non-increasing (resp. non-decreasing) with respect to δ.

Finally, let us introduce the following two sets:

$$V := \mathcal{N}^+ \cap J^\delta \cap \{u \in W^{1,p(x)}(\Omega) : \|u\|_2^2 \leq \lambda_\delta\},$$

$$W := \mathcal{N}^- \cap J^\delta \cap \{u \in W^{1,p(x)}(\Omega) : \|u\|_2^2 \geq \Lambda_\delta\}.$$

Now, we can state our main results as follows:

Theorem 1.1 *For all $\delta > d$, the following statements hold:*

1. *if $u_0 \in V$, then the local weak solution of problem (1.1) exists global in time, and vanish at $+\infty$.*
2. *if $u_0 \in W$, then the local weak solution of problem (1.1) blow up in finite time $T > 0$. Furthermore, the blow up time is given by*

$$\frac{C_3^{-1}}{(\gamma-1)\|u(t_1)\|_2^{2(\gamma-1)}} + t_1 \leq T \leq \frac{\|u(t_0)\|_2^2}{(2-q^-)J(u(t_0))} + t_0.$$

Where $C_3 > 0$ is a constant obtained by the Sobolev embedding inequality, $\gamma > 1$, and $0 < t_0 < t_1 < T$.

The rest of the paper is organized as follows: In Sect. 2, we recall some preliminary results and fundamental properties of the variable exponent Lebesgue spaces and Sobolev spaces that will be used throughout the paper. In Sect. 3, we present some results related to potential wells. Finally, in Sect. 4, we prove our main results.

2. Preliminaries

In this section, we recall some useful results. To start with, we introduce some Banach spaces of the Orlicz-Sobolev type and discuss their properties, as detailed in [1].

Set $C_+(\bar\Omega) = \{r \in C(\bar\Omega)/r(x) > 1, \forall x \in \bar\Omega\}$. For all $r \in C_+(\bar\Omega)$, we define

$$r^- = \min_{x\in\bar\Omega} r(x) \quad r^+ = \max_{x\in\bar\Omega} r(x).$$

For each $r \in C_+(\bar\Omega)$, the variable exponent Lebesgue space is defined as

$L^{r(x)}(\Omega) = \{u : u$ is a measurable real-valued function such that $\rho_{r(\cdot)}(u) < \infty\}$, endowed with the Luxemburg norm

$$||u||_{r(x)} = \inf\{\mu > 0 : \rho_{r(\cdot)}(\frac{u}{\mu}) \leq 1\},$$

where $\rho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)}$. This space is a separable reflexive Banach space.

Let $r, r' \in C_+(\bar\Omega)$ such that $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$. Then we have the following results(see [1]).

Lemma 2.1 *(Hölder inequality) If $u \in L^{r(x)}(\Omega)$ and $v \in L^{r'(x)}(\Omega)$, then*

$$|\int_\Omega uvdx| \leq \left(\frac{1}{r^-} + \frac{1}{r^+}\right) ||u||_{r(x)}||v||_{r'(x)} \leq 2||u||_{r(x)}||v||_{r'(x)}.$$

Lemma 2.2 *If $p, q \in C_+(\bar\Omega)$ such that $p(x) \leq q(x), \forall x \in \Omega$, then there exists the continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.*

The modular function $\rho_{r(\cdot)}$ satisfies the following

Proposition 2.1 *If $u \in L^{r(x)}(\Omega)$, there holds :*

1. $||u||_{r(x)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_{r(\cdot)}(u) < 1$ *(resp.* $= 1, > 1$*).*
2. $||u||_{r(x)} < 1 \Rightarrow ||u||_{r(x)}^{r^+} \leq \rho_{r(\cdot)}(u) \leq ||u||_{r(x)}^{r^-}$.
3. $||u||_{r(x)} > 1 \Rightarrow ||u||_{r(x)}^{r^-} \leq \rho_{r(\cdot)}(u) \leq ||u||_{r(x)}^{r^+}$.

Proposition 2.2 *For $n \in \mathbb{N}$, let $u_n, u \in L^{r(x)}(\Omega)$, then the following assertions are equivalent:*

1. $\lim ||u - u_n||_{r(x)} = 0$.
2. $\lim \rho_{r(\cdot)}(u - u_n) = 0$.
3. $u_n \to u$ *in measure in* Ω *and* $\lim \rho_{r(\cdot)}(u_n) = \rho_{r(\cdot)}(u)$

If $p \in C_+(\bar\Omega)$ the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, consisting of functions $u \in L^{p(x)}(\Omega))$ whose distributional gradient ∇u exists almost everywhere and belongs to $L^{p(x)}(\Omega)$ components by components, endowed with the norm $||u||_{W^{1,p(x)}(\Omega)} = \inf\{\lambda > 0; \int_\Omega |\frac{\nabla u}{\lambda}|^{p(x)}dx + \int_\Omega |\frac{u}{\lambda}|^{p(x)}dx \leq 1\}$, is a separable and reflexive Banach space.

Proposition 2.3 *If $s \in C_+(\bar\Omega)$ and $s(x) < p^\star(x)$ for all $x \in \bar\Omega$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is continuous and compact. Where*
$p^\star(x) = \frac{Np(x)}{N - p(x)}$ *if $N < p(x)$ or $p^\star(x) = \infty$ if $p(x) \geq N$.*

Proposition 2.4 *If $s \in C_+(\partial\Omega)$ and $s(x) < p_\star(x)$ for all $x \in \partial\Omega$, then the trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\partial\Omega)$ is continuous and compact. Where*
$p_\star(x) = \frac{(N-1)p(x)}{N - p(x)}$ *if $N < p(x)$ or $p^\star(x) = \infty$ if $p(x) \geq N$.*

Now, let us introduce a norm that is introduced in [1,9]

$$||u||_{\beta(x)} = \inf\{\lambda > 0; I_\beta(\frac{u}{\lambda}) \leq 1\},$$

which is equivalent to

$$||u||_{W^{1,p(x)}(\Omega)} = \inf\{\lambda > 0; \int_\Omega |\frac{\nabla u}{\lambda}|^{p(x)}dx + \int_\Omega |\frac{u}{\lambda}|^{p(x)}dx \leq 1\}.$$

The functional $I_\beta(u) = \int_\Omega |\nabla u|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}dx$ satisfies the following properties:

Proposition 2.5 *For any $u \in W^{1,p(x)}(\Omega)$ we have*

1. $\|u\|_{\beta(x)} < 1 \Rightarrow \|u\|^{p^+}_{\beta(x)} \leq I_\beta(u) \leq \|u\|^{p^-}_{\beta(x)}$,
2. $\|u\|_{\beta(x)} > 1 \Rightarrow \|u\|^{p^-}_{\beta(x)} \leq I_\beta(u) \leq \|u\|^{p^+}_{\beta(x)}$,
3. $\|u_n - u\|_{\beta(x)} \to 0 \Leftrightarrow I_\beta(u_n - u) \to 0$.

Lemma 2.3 *(See [14]) If $1 \leq p_1 < p_\theta < p_2 \leq \infty$, then*

$$\|u\|_{p_\theta} \leq \|u\|^{1-\theta}_{p_1}\|u\|^{\theta}_{p_2}, \quad \forall u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega);$$

with $\theta \in (0,1)$ defined by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

3. Potential Wells

In this section, under the hypothesis (H), we prouve the following results:

Lemma 3.1 *(i) For each $u \in W^{1,p(x)}(\Omega)$, we have*

$$\left(\frac{1}{p^+} - \frac{1}{q^-}\right) I_\beta(u) + \frac{1}{q^-}K(u) \leq J(u) \leq \left(\frac{1}{p^-} - \frac{1}{q^+}\right) I_\beta(u) + \frac{1}{q^+}K(u). \tag{3.1}$$

(ii) The number d is positive.

(iii)

$$dist(0,\mathcal{N}^-) := \inf_{u\in\mathcal{N}^-} \|u\|_{\beta(x)} > 0.$$

(iv) For any $\delta > d$, we have

$$0 < A_1 \leq \lambda_\delta \leq \Lambda_\delta \leq A_2,$$

where

$$A_1 := \begin{cases} D_3 \min\left\{\left[d\left(\frac{1}{p^-}-\frac{1}{q^+}\right)^{-1}\right]^{\frac{2(p^+-\theta q^-)}{(1-\theta)p^+q^-}}, \left[d\left(\frac{1}{p^-}-\frac{1}{q^+}\right)^{-1}\right]^{\frac{2(p^--\theta q^+)}{(1-\theta)p^-q^+}}\right\}, \; if p^- - \theta q^+ \geq 0, \\ D_3 \min\left\{\left[\delta\left(\frac{1}{p^+}-\frac{1}{q^-}\right)^{-1}\right]^{\frac{2(p^+-\theta q^-)}{(1-\theta)p^+q^-}}, \left[\delta\left(\frac{1}{p^+}-\frac{1}{q^-}\right)^{-1}\right]^{\frac{2(p^--\theta q^+)}{(1-\theta)p^-q^+}}\right\}, \; if p^+ - \theta q^- < 0, \\ D_3 \min\left\{\left[d\left(\frac{1}{p^-}-\frac{1}{q^+}\right)^{-1}\right]^{\frac{2(p^+-\theta q^-)}{(1-\theta)p^+q^-}}, \left[\delta\left(\frac{1}{p^+}-\frac{1}{q^-}\right)^{-1}\right]^{\frac{2(p^--\theta q^+)}{(1-\theta)p^-q^+}}\right\}, \\ if\, p^- - \theta q^+ < 0, \, and p^+ - \theta q^- \geq 0; \end{cases}$$

and

$$A_2 := \max\left\{\left[\delta\left(\frac{1}{p^+}-\frac{1}{q^-}\right)^{-1}\right]^{\frac{1}{p^\pm}}\right\}.$$

Proof:

(*i*) Follows from (1.3),(1.4).

(*ii*) Follows from (*i*).

(*iii*) For $u \in \mathcal{N}^-$, we have

$$\begin{aligned} I_\beta(u) &< \rho_q(u) \\ &\leq \max \|u\|_{q(x)}^{q^\pm} \\ &\leq B_1^{q^+} \max \|u\|_{\beta(x)}^{q^\pm}, \end{aligned}$$

and by Proposition 2.5, it follows that

$$\min \|u\|_{\beta(x)}^{p^\pm} \leq I_\beta(u),$$

then

$$\min \|u\|_{\beta(x)}^{p^\pm} < B_1^{q^+} \max \|u\|_{\beta(x)}^{q^\pm}.$$

Hence $dist(0, \mathcal{N}^-) > 0$.

(*iv*) For any $u \in \mathcal{N}_\delta$, we have $u \in \mathcal{N}$ and $u \in J^\delta$. By (1.5) and (3.1), we get

$$\begin{aligned} d &\leq J(u) \\ &\leq \left(\frac{1}{p^-} - \frac{1}{q^+} \right) I_\beta(u), \end{aligned}$$

then,

$$I_\beta(u) \geq d \left(\frac{1}{p^-} - \frac{1}{q^+} \right)^{-1}. \tag{3.2}$$

On the other hand, by (3.1) and the fact that $u \in J^\delta$ we have

$$\left(\frac{1}{p^+} - \frac{1}{q^-} \right) I_\beta(u) \leq \delta,$$

that is,

$$I_\beta(u) \leq \left(\frac{1}{p^+} - \frac{1}{q^-} \right)^{-1} \delta. \tag{3.3}$$

Next, using the embedding theorem and Lemma 2.3, yield

$$\begin{aligned} \|u\|_{q(x)} &\leq D_1 \|u\|_{q^+} \\ &\leq D_1 \|u\|_{p_-^\star}^\theta \|u\|_2^{1-\theta} \\ &\leq D_2 \|u\|_{\beta(x)}^\theta \|u\|_2^{1-\theta} \\ &\leq D_2 \max \left\{ (I_\beta(u))^{\frac{\theta}{p^\pm}} \right\} \|u\|_2^{1-\theta}, \end{aligned}$$

where $p_-^\star = \dfrac{Np^-}{N - p^-} < p^\star(x)$ and $\theta \in (0,1)$ defined by the formula

$$\frac{1}{q^+} = \frac{\theta}{p_-^\star} + \frac{1-\theta}{2}$$

Moreover, since $u \in \mathcal{N}$, we have $I_\beta(u) = \rho_q(u)$. Then, by

$$\|u\|_{q(x)} \geq \min\left\{(\rho_q(u))^{\frac{1}{q^\pm}}\right\} = \min\left\{(I_\beta(u))^{\frac{1}{q^\pm}}\right\},$$

and the above inequality we get that

$$\min\left\{(I_\beta(u))^{\frac{1}{q^\pm}}\right\} \leq D_2 \max\left\{(I_\beta(u))^{\frac{\theta}{p^\pm}}\right\} \|u\|_2^{1-\theta},$$

that is,

$$\|u\|_2^2 \geq D_3 \min\left\{[I_\beta(u)]^{\frac{2(p^+-\theta q^-)}{(1-\theta)p^+q^-}}, [I_\beta(u)]^{\frac{2(p^--\theta q^+)}{(1-\theta)p^-q^+}}\right\}. \tag{3.4}$$

Finally, by (3.3) we get $\Lambda_\delta \leq A_2$ and by (3.2), (3.3), and (3.4) we deduce that $\lambda_\delta \geq A_1$.

□

Note that multiplying the equation (1.1) by u_t and integrating over $\Omega \times (0,T)$ yields

$$\int_0^t ||u_\tau||_2^2 d\tau + J(u) = J(u_0), \ \forall t \in [0,T). \tag{3.5}$$

The energy equality (3.5) holds true due to the fact that the exponents $p(\cdot)$ and $q(\cdot)$ are independent of t. See [3] for more details.

4. Proof of our Main Results

In this section, we prove our main results.

Proof: [Proof of Theorem 1.1]

1. **Global existence:**

   If $u_0 \in V$, we have $u_0 \in \mathcal{N}^+$, $J(u_0) \leq \delta$, and $\|u_0\|_2^2 \leq \lambda_\delta$. By the monotonecity of λ_δ with respect to $\delta > 0$, one can see that $\lambda_\delta \leq \lambda_{J(u_0)}$.

   Now, we claim that $u \in \mathcal{N}^+$ for all $t \in [0,T)$. If it false, there exists a $t_0 \in (0,T)$ such that $K(t_0) \leq 0$. By the continuity of K with respect to t, without loss of generality, we can assume that t_0 is the first time such that $K(u(t_0)) = 0, \|u(t_0)\|_{\beta(x)} > 0$, and then $K(u) > 0, \forall t \in [0,t_0)$.

   Next, by multiplying (1.1) by u and integtaring over Ω, we get

   $$\frac{d}{dt}\|u\|_2^2 = -2K(u) < 0, \forall t \in [0,t_0),$$

   which implies that

   $$\|u(t_0)\|_2^2 < \|u_0\|_2^2 \leq \lambda_\delta. \tag{4.1}$$

   On the other hand, by the energy equality, we have $J(u(t_0)) \leq J(u_0)$, then $u(t_0) \in \mathcal{N}_\delta$, which implies from the definition of λ_δ that

   $$\|u(t_0)\|_2^2 \geq \lambda_\delta.$$

   This leads to a contradiction with (4.1). Therefore, $u \in \mathcal{N}^+$ for all $t \in [0,T)$.

   Moreover, from (3.1), (3.5) and the fact that $u \in \mathcal{N}^+$ we deduce that

   $$I_\beta(u) \leq \delta\left(\frac{1}{p^+} - \frac{1}{q^-}\right)^{-1}, \quad \forall t \in [0,T),$$

   whitch means that the local weak solution exists globally and $T = +\infty$.

   Next, we show the asymptotic behavior of the global solution as $t \to +\infty$.

   Firstly, we denote the $\omega-$limits set of $u_0 \in \mathcal{V}$ as follows

   $$\omega(u_0) := \bigcap_{t\geq 0} \overline{\{u(s) : s \geq t\}}.$$

   If $\widehat{u}_0 \in \omega(u_0)$, then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that that

(a) $\lim_{n\to+\infty} s_n = +\infty$;

(b) $\lim_{n\to+\infty} u(s_n) = \widehat{u}_0$.

Then, we have the following result:

Lemma 4.1 *Let $\widehat{u}_0 \in \omega(u_0)$ be a non-trivial function and $\widehat{u}$ be a weak solution to problem (1.1), with the initial data $\widehat{u}_0$, then $\widehat{u}_0 \in \mathcal{N}^+$ and $\widehat{u} = \widehat{u}_0$ for all $t \geq 0$.*

Proof: [Proof of Lemma 4.1.] From $u \in \mathcal{N}^+$ and (3.1) we have $J(u) > 0$, and by the energy equality, we infer that $t \mapsto J(u)$ is decreasing on $[0, \infty)$, then there exists $l \geq 0$ such that

$$\lim_{t\to\infty} J(u) = l. \tag{4.2}$$

Hence, according to the definition of $\omega-$limits set we conclude that

$$\lim_{n\to\infty} J(u(s_n)) = J(\widehat{u}_0) = l. \tag{4.3}$$

Furthermore, by the continuity of I and $u \in \mathcal{N}^+$ we deduce that

$$\lim_{n\to\infty} K(u(s_n)) = K(\widehat{u}_0) \geq 0. \tag{4.4}$$

Then, either $K(\widehat{u}_0) > 0$ or $K(\widehat{u}_0) = 0, \|\widehat{u}_0\|_{\beta(x)} > 0($ since $\|\widehat{u}_0\|_{\beta(x)} = 0$ is a trivial case). If $K(\widehat{u}_0) = 0$ and $\|\widehat{u}_0\|_{\beta(x)} > 0$, we get that $\widehat{u}_0 \in \mathcal{N}$, and for large enought n, it follows from (4.4) that $u(s_n) \in \mathcal{N}$, whitch contradict the fact that $u \in \mathcal{N}$ for all $t \geq 0$.

Thus, $K(\widehat{u}_0) > 0$ and $\widehat{u} \in \mathcal{N}^+$ for all $t \geq 0$.

According to (4.2) and (4.3), it follows that

$$\lim_{t\to\infty} J(\widehat{u}) = J(\widehat{u}_0) = l. \tag{4.5}$$

Combining (4.5) and the energy equality to get

$$\int_0^\infty \|\widehat{u}_\tau\|_2^2 d\tau = 0.$$

Hence

$$\widehat{u}_t = 0, \quad \forall t \geq 0,$$

i.e.,

$$\widehat{u} = \widehat{u}_0, \quad \forall t \geq 0.$$

□

Clearly, Lemma 4.1 says the ω-limits set of u_0 consists of all stationary solutions of (1.1) which solve the following elliptic problem:

$$\begin{cases} -\Delta_{p(x)} u(x,t) = |u(x,t)|^{q(x)-2} u(x,t), & (x,t) \in \Omega \times (0,+\infty), \\ |\nabla u(x,t)|^{p(x)-2} \dfrac{\partial u}{\partial n} + \beta(x)|u(x,t)|^{p(x)-2} = 0, & (x,t) \in \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases}$$

Now, let $\widehat{u}$ be a weak solution to (1.1) with the initial data $\widehat{u}_0 \in \omega(u_0)$. It is well know that $\widehat{u} = \widehat{u}_0$ for all $t \geq 0$, then

$$K(\widehat{u}) = \frac{d}{dt}\|\widehat{u}\|_2^2 = 0, \quad t \geq 0, \tag{4.6}$$

this implies that

$$\omega(u_0) \subset \mathcal{N} \cup \{0\}. \tag{4.7}$$

However, if $\widehat{u}_0 \in \omega(u_0)$, by the fact that $u_0 \in V$, we also have

$$J(\widehat{u}_0) \leq J(u_0) \leq \delta, \text{ and } \|\widehat{u}_0\|_2^2 < \|u_0\|_2^2 \leq \lambda_\delta.$$

Then from the definition of J^δ and $\mathcal{N}_\delta$, it follows that $\widehat{u}_0 \in J^\delta$ and $\widehat{u}_0 \notin \mathcal{N}_\delta$. Hence $\widehat{u}_0 \notin \mathcal{N}$ and $\omega(u_0) \cap \mathcal{N} = \emptyset$. Together with (4.7), we conclude that $\omega(u_0) = \{0\}$. Which means that the solution u decays to zero as $t \to +\infty$ provided $u_0 \in V$.

2. **Blow-up:**

If $u_0 \in W$, a similar contradiction argument as in part (1) of this proof shows that $u(t) \in \mathcal{N}^-$ for all $t \in [0, T)$. To prove that the solution blows up in finite time, we proceed by contradiction and assume that u exists globally, i.e., $T = +\infty$.

From the energy equality, it follows that $J(u(t))$ is non-increasing with respect to t. Therefore, only two cases may occur:

(a) there exists a constant $l \in \mathbb{R}$ such that

$$\lim_{t\to\infty} J(u(t)) = l;$$

(b)

$$\lim_{t\to\infty} J(u(t)) = -\infty.$$

We now show that both cases lead to contradictions, which will imply that the solution cannot exist globally, and thus blows up in finite time.

Case (a). Assume that $\lim\limits_{t\to\infty} J(u(t)) = l$ for some $l \in \mathbb{R}$. Using the energy equality, we have

$$\frac{d}{dt} J(u(t)) = -\|u_t(t)\|_2^2 \longrightarrow 0 \quad \text{as } t \to +\infty,$$

which implies that $u(t)$ approaches a stationary solution of problem (1.1) as $t \to +\infty$. Hence,

$$u(t) \in \mathcal{N} \cup \{0\} \quad \text{as } t \to +\infty.$$

However, since $u(t) \in \mathcal{N}^-$ for all $t \geq 0$, we necessarily have $u(t) \notin \mathcal{N}$ for any $t \geq 0$, implying that $u(t) \to 0$ as $t \to +\infty$. On the other hand, by Lemma 3.1 (iii), any function $u(t) \in \mathcal{N}^-$ satisfies $u(t) \neq 0$. This contradiction shows that case (a) cannot occur.

Case (b). Assume now that $\lim\limits_{t\to\infty} J(u(t)) = -\infty$. By the continuity of $J(u(t))$ with respect to t, there must exist a first time $t_0 > 0$ such that

$$J(u(t_0)) < 0 \quad \text{and} \quad u(t_0) \in \mathcal{N}^-.$$

Next we define

$$\phi(t) := \|u\|_2^2, \quad t \geq t_0, \tag{4.8}$$

and

$$\psi(t) := -J(u), \quad t \geq t_0. \tag{4.9}$$

Then by the monotonicity of J with respect to t, it follows that

$$\dot{\psi}(t) = \|u_t\|_2^2 \geq 0, \quad t \geq t_0. \tag{4.10}$$

By (3.1) and (4.9) we get for all $t \geq t_0$ that

$$\begin{aligned}
\dot{\phi}(t) &= -2K(u) \\
&\geq 2q^- \left[\left(\frac{1}{p^+} - \frac{1}{q^-} \right) I_\beta(u) - J(u) \right] \\
&\geq 2q^- J(u) \\
&= 2q^- \psi(t).
\end{aligned} \tag{4.11}$$

Therefore, using Schwartz's inequality, remarquable inequality, and (4.11) we obtain

$$\begin{aligned}
\phi(t)\dot{\psi}(t) &= \|u\|_2^2 \|u_t\|_2^2 \\
&\geq \left(\int_\Omega u_t u dx \right)^2 \\
&= \left(\frac{1}{2} \frac{d}{dt} \|u\|_2^2 \right)^2 \\
&= \frac{1}{4} (\dot{\phi}(t))^2 \\
&\geq \frac{q^-}{2} \dot{\phi}(t)\psi(t), \quad t \geq t_0.
\end{aligned} \tag{4.12}$$

Multiplying (4.12) by $\phi^{\frac{-q^-}{2}-1}(t)$, we obtain

$$\frac{d}{dt} \left[\phi^{\frac{-q^-}{2}}(t)\psi(t) \right] \geq 0, \quad t \geq t_0. \tag{4.13}$$

On the other hand, multiplying the inequality (4.11) by $\phi^{\frac{-q^-}{2}}(t)$ and using (4.12), we derive

$$\begin{aligned}
\phi^{\frac{-q^-}{2}}(t)\dot{\phi}(t) &\geq 2q^- \phi^{\frac{-q^-}{2}}(t)\psi(t) \\
&\geq 2q^- \phi^{\frac{-q^-}{2}}(t_0)\psi(t_0), \quad t \geq t_0,
\end{aligned}$$

that is,

$$\frac{d}{dt} \left[\phi^{1-\frac{q^-}{2}}(t) \right] \leq 2q^- \left(1 - \frac{q^-}{2} \right) \phi^{\frac{-q^-}{2}}(t_0)\psi(t_0), \quad t \geq t_0.$$

Integrating the last inequality over $[t_0, t]$, we get

$$\phi^{1-\frac{q^-}{2}}(t) \leq \phi^{1-\frac{q^-}{2}}(t_0) + q^-(2-q^-)\phi^{\frac{-q^-}{2}}(t_0)\psi(t_0)(t-t_0), \ t \geq t_0. \tag{4.14}$$

Hence,

$$\phi(t) \geq \left[\phi^{1-\frac{q^-}{2}}(t_0) - q^-(q^- - 2)\phi^{\frac{-q^-}{2}}(t_0)\psi(t_0)(t-t_0) \right]^{-\frac{2}{q^- - 2}}, \ t \geq t_0.$$

That means that the function ϕ cannot remain finite for all $t \geq t_0$, which contradicts the assumption $T = +\infty$. In other words, the weak solution u blows up in finite time.

Upper and Lower bound of blowup time estimate:

Letting $t \to T$ in (4.14), we obtaine

$$T \leq \frac{\phi(t_0)}{(q^- - 2)\psi(t_0)} + t_0.$$

Now, by $u \in \mathcal{N}^-$, we have $K(t) < 0$, then $I_\beta(u) < \rho_q(u)$. On the other hand, we know that u blows-up in finite time $T > 0$, then we may assume that $\|u\|_2^2 = \phi(t) > 1$ for all $t \in [t_1, T)$ for some $t_0 < t_1 < T$. Next, using the Sobolev embedding and Lemma 2.3, we get

$$\begin{aligned}\rho_q(u) &\leq \|u\|_{q(x)}^{q^+} \\ &\leq C_1 \|u\|_{q^+}^{q^+} \\ &\leq C_1 \|u\|_{p_-^\star}^{\theta q^+} \|u\|_2^{(1-\theta)q^+} \\ &\leq C_2 \|u\|_{\beta(x)}^{\theta q^+} [\phi(t)]^{\frac{(1-\theta)q^+}{2}} \\ &\leq C_2 \|u\|_{\beta(x)}^{\theta q^+} [\phi(t)]^{\frac{q^+}{2}} \end{aligned} \tag{4.15}$$

where $p_-^\star = \dfrac{Np^-}{N - p^-}$, $\theta \in (0,1)$ can be obtained by the formula

$$\frac{1}{q^+} = \frac{\theta}{p_-^\star} + \frac{1-\theta}{2},$$

and from the hypothesis (H) there hold $\dfrac{p^-}{\theta q^+} > 1$.

Further, we have $\|u\|_{\beta(x)}^{p^-} \leq I_\beta(u)$, thus we deduce from $I_\beta(u) < \rho_q(u)$ and (4.15) that

$$\|u\|_{\beta(x)}^{p^- - \theta q^+} \leq C_2 [\phi(t)]^{\frac{q^+}{2}}.$$

This combined with $\dfrac{d}{dt}\phi(t) = -2K(t) < 2\rho_q(u)$ and (4.15) to obtaine

$$\frac{d}{dt}\phi(t) \leq C_3 [\phi(t)]^\gamma, \tag{4.16}$$

where $2\gamma := q^+ + \dfrac{\theta (q^+)^2}{(p^- - \theta q^+)} > 2$.

Integrating (4.16) from t_1 to T, we obtaine

$$\int_{t_1}^{T} \frac{\phi'(t)}{\phi(t)^\gamma} dt \leq C_3 (T - t_1). \tag{4.17}$$

Put $\phi(t) = \xi$, then $d\xi = \phi'(t)dt$ and (4.17) becomes

$$\int_{\phi(t_1)}^{+\infty} \frac{d\xi}{\xi^\gamma} \leq C_3 (T - t_1).$$

Hence,

$$T \geq C_3^{-1} \int_{\phi(t_1)}^{+\infty} \frac{d\xi}{\xi^\gamma} + t_1.$$

This ends the proof. □

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