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Cone A_b-Metric Space over Real Banach Algebra and Some Results on Coupled Fixed Points

K. Bala Devi, Konthoujam Sangita Devi, Kumam Anthony Singh

ABSTRACT: In this paper, we expand the notion of a coupled fixed point to encompass mappings on cone A_b-metric space over real Banach algebra, and we establish some coupled fixed point theorems. Our results extend some existing coupled fixed point results on cone A_b-metric space to cone A_b-metric space over real Banach algebra. Our results are generalized enough to contain many interesting results as corollaries. We also provide an example to illustrate the validity of our result.

Keywords: Coupled fixed point, cone A_b-metric space over Banach algebra, c-sequence.

Contents

1. Introduction

The classical Banach contraction principle serves as a foundational element in metric fixed point theory, and numerous scholars have investigated and expanded upon this principle in various contexts.Huang and Zhang [2] introduced the concept of a cone metric space as an extension of a metric space,replacing the set of real numbers in a metric space with a general Banach space that is partially ordered by a cone. However, recent research has demonstrated an equivalence between cone metric spaces and metric spaces concerning the existence of fixed points of the mapping involved. To address and extend these findings, Liu and Xu [9] proposed the concept of a cone metric space using Banach algebra. This development was pivotal in fixed point theory, illustrating that cone metric spaces over Banach algebra are not equivalent to metric spaces regarding the existence of fixed points of mappings. Subsequently, Xu and Radenovic [13] expanded Liu and Xu's results without assuming the normality of cones. Jerolina Fernandez and colleagues[4] introduced A-cone metric space over Banach algebra, further generalizing A-metric spaces and cone metric spaces over Banach algebra.Very recently in 2025, as a further generalization in this direction, K. Anthony Singh et al.[5] introduced the concept of cone A_b-metric space over Banach algebra and proved some fixed point theorems. On the other hand, Bhaskar and Lakshmikantham[10] introduced the notion of a coupled fixed point of a mapping. Lakshmikantham and Ciric[11] explored additional coupled fixed point theorems in partially ordered metric spaces. Subsequently, Sabetghadam et al.[1] defined coupled fixed points for mappings on cone metric space and established some coupled fixed point theorems. K. Anthony Singh and M.R. Singh[7] further extended Sabetghadam et al.'s results to cone S_b-metric space. Additionally, K. Anthony Singh et al.[8] expanded the definition of coupled fixed points to mappings on cone A_b-metric space and proved related theorems.

This paper aims to broaden the definition of coupled fixed points to mappings on cone A_b-metric space over Banach algebra and establish some coupled fixed point theorems. Our findings extend the coupled fixed point results on cone A_b-metric spaces by K. Anthony Singh et al.[8] to cone A_b-metric spaces over Banach algebra.

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2. Preliminaries

Definition 2.1. *[2]Consider a Banach space $\mathfrak{B}$. A subset $\mathcal{C}$ of $\mathfrak{B}$ is defined as a cone if it satisfies the following:*
1. $\mathcal{C}$ is closed, nonempty and $\mathcal{C} \neq \{0\}$
2. $\alpha p+\beta q \in \mathcal{C}$, for all $p,q \in \mathcal{C}$ and for any non-negative reals α, β;
3. $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$

A partial ordering $\leq$ in $\mathfrak{B}$ can be established with respect to $\mathcal{C}$, denoted by $p \leq q$ if and only if $q - p \in \mathcal{C}$. The notation $p < q$ is used to indicate that $p \leq q$ but $p \neq q$, whereas $p << q$ signifies $q - p \in int.\mathcal{C}$, where $int.\mathcal{C}$ denotes the interior of $\mathcal{C}$. A cone $\mathcal{C}$ is characterised as normal if there exists a constant $K > 0$ such that $0 \leq p \leq q$ implies $||p|| \leq K||q||$ for all p, $q \in \mathfrak{B}$. The smallest positive number that satisfies this condition is referred to as the normal constant of $\mathcal{C}$.

A cone $\mathcal{C}$ is termed regular if every increasing sequence that is bounded above converges. Specifically, if $\{p_n\}$ is a sequence such that $p_1 \leq p_2 \leq ... \leq p_n \leq ... \leq q$ for some $q \in \mathfrak{B}$, then there exists $p \in \mathfrak{B}$ such that $||p_n - p|| \to 0$ as $n \to \infty$. Alternatively, the cone $\mathcal{C}$ is regular if and only if every decreasing sequence that is bounded below converges.It is well-established fact that a regular cone is also a normal cone.

Definition 2.2. *[2]Suppose $X \neq \phi$. Let the mapping $d : X \times X \to \mathfrak{B}$ satisfy the following:*
1. $d(\theta,\kappa) \geq 0$ and $d(\theta,\kappa) = 0$ iff $\theta = \kappa, \forall \theta,\kappa \in X$;
2. $d(\theta,\kappa) = d(\kappa,\theta), \forall \theta,\kappa \in X$;
3. $d(\theta,\kappa) \leq d(\theta,\tau) + d(\tau,\kappa), \forall \theta,\kappa,\tau \in X$.
Then, (X,d) is said to be a cone metric space or simply CMS.

Example 2.3. *[2]Let $\mathfrak{B} = \mathbb{R}^2, \mathcal{C} = \{(p,q) : p,q \geq 0\}, X = \mathbb{R}$ and $d : X \times X \to \mathfrak{B}$ such that $d(\theta,\kappa) = (|\theta-\kappa|, k|\theta-\kappa|)$, where $k \geq 0$. Then, (X,d) is a CMS.*

Definition 2.4. *[6]Consider a real Banach space $\mathfrak{B}$ and a cone $\mathcal{C}$ in $\mathfrak{B}$ with $int.\mathcal{C} \neq \phi$. Let $\leq$ be a partial ordering in $\mathfrak{B}$ generated by $\mathcal{C}$. Let $X \neq \phi$, and suppose that the function $A : X^n \to \mathfrak{B}$ satisfies the following*
1. $A(\vartheta_1,\vartheta_2,...,\vartheta_{n-1},\vartheta_n) \geq 0$;
2. $A(\vartheta_1,\vartheta_2,...,\vartheta_{n-1},\vartheta_n) = 0$ iff $\vartheta_1 = \vartheta_2 = ... = \vartheta_{n-1} = \vartheta_n$;
3. $$A(\vartheta_1,\vartheta_2,...,\vartheta_{n-1},\vartheta_n) \leq b[A(\vartheta_1,\vartheta_1,...,(\vartheta_1)_{n-1},\vartheta) + A(\vartheta_2,\vartheta_2,...,(\vartheta_2)_{n-1},\vartheta) + ... \\ +A(\vartheta_{n-1},\vartheta_{n-1},...,(\vartheta_{n-1})_{n-1},\vartheta)+A(\vartheta_n,\vartheta_n,...,(\vartheta_n)_{n-1},\vartheta)]$$
for all $\vartheta_i, \vartheta \in X$, $i = 1,2,3,...,(n-1),n$, where $b \geq 1$. Then, the function A is said to be a cone A_b-metric (CAbM) on X and the pair (X,A) is said to be a cone A_b-metric space(CAbMS).

Example 2.5. *[6]Let $\mathfrak{B} = \mathbb{R}^2$, and $\mathcal{C} = \{(p,q) \in \mathfrak{B} : p,q \geq 0\}$, a normal cone in $\mathfrak{B}$. Let $X = \mathbb{R}$ and $A : X^n \to \mathfrak{B}$ be a function satisfying $A(\vartheta_1,\vartheta_2,...,\vartheta_n) = A_*(\vartheta_1,\vartheta_2,...,\vartheta_n)(\gamma,\delta)$, where $\gamma,\delta > 0$ and A_* is an A_b-metric on X. Then, A is a CAbM on X. In particular, the function*

$$A_*(\vartheta_1,\vartheta_2,...,\vartheta_n) = \sum_{i=1}^{n}\sum_{i<j}|\vartheta_i - \vartheta_j|^2,\ \vartheta_i \in X,\ i = 1,\ 2,\ 3,\ ...,\ n,$$

is an A_b-metric on X with $b = 2$. Now, the function

$$A(\vartheta_1,\vartheta_2,...,\vartheta_n) = \left(\sum_{i=1}^{n}\sum_{i<j}|\vartheta_i - \vartheta_j|^2, \frac{1}{4}\sum_{i=1}^{n}\sum_{i<j}|\vartheta_i - \vartheta_j|^2\right),$$

is a CAbM on X with $b = 2$.

Definition 2.6. *[9]Consider a real Banach algebra $\mathcal{A}$. That is, $\mathcal{A}$ is a real Banach space in which the multiplication operation is defined, satisfying the following ($\forall \vartheta,\mu,\pi \in \mathcal{A}$, $a \in \mathbb{R}$)*
1. $(\vartheta\mu)\pi = \vartheta(\mu\pi)$
2. $\vartheta(\mu+\pi) = \vartheta\mu + \vartheta\pi$ and $(\mu+\pi)\vartheta = \mu\vartheta + \pi\vartheta$
3. $a(\mu\pi) = (a\mu)\pi = \mu(a\pi)$
4. $||\mu\pi|| \leq ||\mu||\ ||\pi||$.

In what follows, we always assume that a Banach algebra $\mathcal{A}$ *is real and has a unit* e *such that* $e\vartheta = \vartheta e = \vartheta$, $\forall \vartheta \in \mathcal{A}$.

An element $\vartheta \in \mathcal{A}$ *is said to be invertible if there is an element* $\theta \in \mathcal{A}$ *such that* $\vartheta\theta = \theta\vartheta = e$. *Here,* θ *is called the inverse of* ϑ *and is denoted by* ϑ^{-1}.

Definition 2.7. *[9]Consider a real Banach algebra* $\mathcal{A}$. *A subset* $\mathcal{C}$ *of* $\mathcal{A}$ *is defined as a cone of* $\mathcal{A}$ *if*
1. $\mathcal{C}$ *is non-empty, closed and* $\{\theta, e\} \subset \mathcal{C}$;
2. $a\mathcal{C} + b\mathcal{C} \subset \mathcal{C}$, *for all non-negative real numbers* a *and* b;
3. $\mathcal{C}^2 = \mathcal{C}\mathcal{C} \subset \mathcal{C}$;
4. $\mathcal{C} \cap (-\mathcal{C}) = \{\theta\}$;

where θ *is the null of the Banach algebra* $\mathcal{A}$. *Given a cone* $\mathcal{C}$ *of* $\mathcal{A}$, *a partial ordering* $\leq$ *is defined in* $\mathcal{A}$ *with respect to* $\mathcal{C}$ *by* $p \leq q$ *if and only if* $q - p \in \mathcal{C}$. *We write* $p < q$ *to indicate that* $p \leq q$ *but* $p \neq q$, *while the symbol* $p << q$ *stands for* $q - p \in int.\mathcal{C}$, *where* $int.\mathcal{C}$ *is the interior of* $\mathcal{C}$. *The cone* $\mathcal{C}$ *is said to be normal if there exists* $K > 0$ *such that* $\theta \leq p \leq q$ *implies* $||p|| \leq K||q||$, *for all* p, $q \in \mathcal{A}$.

The smallest such positive number K *is called the normal constant of* $\mathcal{C}$. *If* $int.\mathcal{C} \neq \phi$, *then* $\mathcal{C}$ *is said to be a solid cone.*

Definition 2.8. *[9]Let* $X \neq \phi$. *Let the mapping* $d : X \times X \to \mathcal{A}$ *satisfy the following:*

1. $\theta < d(\sigma, \tau)$ *and* $d(\sigma, \tau) = \theta$ *if and only if* $\sigma = \tau, \forall \sigma, \tau \in X$,
2. $d(\sigma, \tau) = d(\tau, \sigma), \forall \sigma, \tau \in X$,
3. $d(\sigma, \tau) \leq d(\sigma, \rho) + d(\rho, \tau), \forall \sigma, \tau, \rho \in X$.

Then, d *is said to be a cone metric on* X *and the pair* (X, d) *is called a cone metric space (CMS) over* $\mathcal{A}$.

Example 2.9. *[9]Let* $\mathcal{A} = M_m(\mathbb{R}) = \{\alpha = (\alpha_{ij})_{m\times m} | \alpha_{ij} \in \mathbb{R}, \forall 1 \leq i, j \leq m\}$ *be the algebra of all* m*-square real matrices with the norm defined as*

$$||\alpha|| = \sum_{1\leq i,j\leq m} |\alpha_{ij}|.$$

Then, $\mathcal{A}$ *is a real Banach algebra and the unit* e *is the identity matrix.*

Let $\mathcal{C} = \{\alpha \in \mathcal{A} | \alpha_{ij} \geq 0, \forall 1 \leq i, j \leq m\}$. *Then,* $\mathcal{C}$ *is a normal cone of* $\mathcal{A}$ *with normal constant 1.*

Let $X = M_m(\mathbb{R})$, *and let the metric* $d : X \times X \to \mathcal{A}$ *be defined by*

$$d(\alpha, \beta) = d\Big((\alpha_{ij})_{m\times m}, (\beta_{ij})_{m\times m}\Big) = \Big(|\alpha_{ij} - \beta_{ij}|\Big)_{m\times m} \in \mathcal{A}.$$

Then, (X, d) *is a CMS over Banach algebra* $\mathcal{A}$ *with normal cone.*

Definition 2.10. *[4]Let* $X \neq \phi$. *A function* $d : X^n \to \mathcal{A}$ *is said to be a cone A-metric on* X *if for any* $\vartheta_i, \sigma \in X, i = 1, 2, 3, ..., n$, *the following conditions hold:*
1. $d(\vartheta_1, \vartheta_2, \vartheta_3, ..., \vartheta_{n-1}, \vartheta_n) \geq \theta$,
2. $d(\vartheta_1, \vartheta_2, \vartheta_3, ..., \vartheta_{n-1}, \vartheta_n) = \theta$ *iff* $\vartheta_1 = \vartheta_2 = ... = \vartheta_{n-1} = \vartheta_n$,
3. $$\begin{aligned} d(\vartheta_1, \vartheta_2, \vartheta_3, ..., \vartheta_{n-1}, \vartheta_n) \leq{}& d(\vartheta_1, \vartheta_1, \vartheta_1, ..., (\vartheta_1)_{n-1}, \sigma) + d(\vartheta_2, \vartheta_2, \vartheta_2, ..., (\vartheta_2)_{n-1}, \sigma) \\ &+ ... + d(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_{n-1}, ..., (\vartheta_{n-1})_{n-1}, \sigma) \\ &+ d(\vartheta_n, \vartheta_n, \vartheta_n, ..., (\vartheta_n)_{n-1}, \sigma). \end{aligned}$$

The pair (X, d) *is said to be a cone A-metric space (CAMS) over Banach algebra* $\mathcal{A}$.

Lemma 2.11. *[4]For a CAMS (X,d) over Banach algebra* $\mathcal{A}$,

$d(\sigma,\sigma,...,\sigma,\tau) = d(\tau,\tau,...,\tau,\sigma)$, for all $\sigma,\tau \in X$.

Lemma 2.12. *[4]For a CAMS (X,d) over Banach algebra $\mathcal{A}$, the following hold, for all $\vartheta,\mu,\pi \in X$,*
1. $d(\vartheta,\vartheta,...,\vartheta,\pi) \leq (n-1)d(\vartheta,\vartheta,...,\vartheta,\mu) + d(\pi,\pi,...,\pi,\mu)$ and
2. $d(\vartheta,\vartheta,...,\vartheta,\pi) \leq (n-1)d(\vartheta,\vartheta,...,\vartheta,\mu) + d(\mu,\mu,...,\mu,\pi)$.

Example 2.13. *[4]Consider the Banach space $\mathcal{A} = C[p,q]$ of all continuous functions defined on the interval [p, q] with the supremum norm. Multiplication is defined in the usual way. Then, $\mathcal{A}$ is a Banach algebra with a unit 1 (identity function). We set $\mathcal{C} = \{\theta \in \mathcal{A} : \theta(s) \geq 0, s \in [p,q]\}$ and $X = \mathbb{R}$.*

A mapping $d : X^n \to \mathcal{A}$ is defined by

$$\begin{aligned} d(\vartheta_1,\vartheta_2,...,\vartheta_{n-1},\vartheta_n)(s) = \Big(&|\vartheta_1-\vartheta_2| + |\vartheta_1-\vartheta_3| + ... + |\vartheta_1-\vartheta_n| \\ &+|\vartheta_2-\vartheta_3| + |\vartheta_2-\vartheta_4| + ... + |\vartheta_2-\vartheta_n| \\ &+... + |\vartheta_{n-2}-\vartheta_{n-1}| + |\vartheta_{n-2}-\vartheta_n| \\ &+|\vartheta_{n-1}-\vartheta_n|\Big)e^s. \\ &= \sum_{i=1}^{n}\sum_{i<j}|\vartheta_i-\vartheta_j|e^s. \end{aligned}$$

Then, (X,d) becomes the usual CAMS over Banach algebra.

Definition 2.14.*[5]Consider a real Banach algebra $\mathcal{A}$, a solid cone $\mathcal{C}$ in $\mathcal{A}$ and a partial ordering $\leq$ in $\mathcal{A}$ with respect to $\mathcal{C}$. Let $X \neq \phi$, and let $A : X^n \to \mathcal{A}$ satisfy the following:*
1. $A(\vartheta_1,\vartheta_2,...,\vartheta_{n-1},\vartheta_n) \geq \theta$,
2. $A(\vartheta_1,\vartheta_2,...,\vartheta_{n-1},\vartheta_n) = \theta$ iff $\vartheta_1 = \vartheta_2 = ... = \vartheta_{n-1} = \vartheta_n$,
3. $$\begin{aligned} A(\vartheta_1,\vartheta_2,...,\vartheta_{n-1},\vartheta_n) \leq b[&A(\vartheta_1,\vartheta_1,...,(\vartheta_1)_{n-1},\vartheta) + A(\vartheta_2,\vartheta_2,...,(\vartheta_2)_{n-1},\vartheta) \\ &+... + A(\vartheta_{n-1},\vartheta_{n-1},...,(\vartheta_{n-1})_{n-1},\vartheta) + A(\vartheta_n,\vartheta_n,...,(\vartheta_n)_{n-1},\vartheta)], \\ &\forall \vartheta_i,\vartheta \in X, i = 1,2,3,...,n, \text{ where } b \geq 1. \end{aligned}$$

Then, A is called a cone A_b-metric and the pair (X,A) is said to be a cone A_b-metric space (CAbMS) over Banach algebra $\mathcal{A}$.

Definition 2.15.*[5]Consider a CAbMS (X,A) over Banach algebra $\mathcal{A}$.*

1. *A sequence $\{\omega_n\}$ in X is said to converge to $\omega \in X$ if given any $c >> \theta$ there exists a positive integer N such that $A(\omega_n,\omega_n,...,\omega_n,\omega) << c$, $\forall n \geq N$ and we write $\lim_{n\to\infty} \omega_n = \omega$ or $\omega_n \to \omega$ as $n \to \infty$.*

2. *A sequence $\{\omega_n\}$ in X is called Cauchy if given any $c >> \theta$ there exists a positive integer N such that $A(\omega_n,\omega_n,...,\omega_n,\omega_m) << c, \forall n,m \geq N$.*

3. *The space is called complete if every Cauchy sequence converges.*

Definition 2.16.*Consider a CAbMS (X,A) over Banach algebra $\mathcal{A}$. An element $(\rho,\sigma) \in X \times X$ is called a coupled fixed point (CFP) of $\mathcal{F} : X \times X \to X$ if $\mathcal{F}(\rho,\sigma) = \rho$ and $\mathcal{F}(\sigma,\rho) = \sigma$.*

Proposition 2.17.*[12]Consider a Banach algebra $\mathcal{A}$ with unit e, and let $\alpha \in \mathcal{A}$. If the spectral radius $\rho(\alpha)$ of α is less than 1, i.e.,*

$$\rho(\alpha) = \lim_{m\to\infty} ||\alpha^m||^{\frac{1}{m}} = \inf_{m\geq 1} ||\alpha^m||^{\frac{1}{m}} < 1,$$

then $e-\alpha$ has inverse. Indeed, $(e-\alpha)^{-1} = \sum_{i=0}^{\infty}\alpha^i$.

Remark 2.18.*[12]The spectral radius $\rho(\alpha)$ satisfies $\rho(\alpha) \leq ||\alpha||$, for all $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is a Banach algebra with a unit e.*

Remark 2.19.*[12]If $\rho(\alpha) < 1$, then $||\alpha^m|| \to 0 (m \to \infty)$.*

Remark 2.20.*[12]Consider a Banach algebra $\mathcal{A}$ with a unit e and let $a \in \mathcal{A}$; then, $\lim_{m\to\infty} ||a^m||^{\frac{1}{m}}$ exists and the spectral radius $\rho(a)$ satisfies*

$$\rho(a) = \lim_{m\to\infty} ||a^m||^{\frac{1}{m}}.$$

If $\rho(a) < |\gamma|$, then $(\gamma e - a)$ is invertible in $\mathcal{A}$. Moreover,

$$(\gamma e - a)^{-1} = \sum_{i=0}^{\infty} \frac{a^i}{\gamma^{i+1}}, \text{ where } \gamma \in \mathbb{C} \text{ is a constant.}$$

Remark 2.21.*[12]Consider a Banach algebra $\mathcal{A}$ with a unit e and let $r,\ s \in \mathcal{A}$. If r commutes with s, then $\rho(r + s) \leq \rho(r) + \rho(s)$, $\rho(rs) \leq \rho(r)\rho(s)$.*

Remark 2.22.*[13]Consider a Banach algebra $\mathcal{A}$ with a unit e and let $r \in \mathcal{A}$. If $\rho(r) < 1$, then $\rho(e - r)^{-1} \leq \frac{1}{1-\rho(r)}$.*

Now, we recall the definition of a c-sequence and its properties.

Definition 2.23.*[14]Consider a Banach space $\mathfrak{B}$ with a solid cone $\mathcal{C}$. A sequence $\{u_m\}$ in $\mathcal{C}$ is called a c-sequence if given any $c >> \theta$ there exists $N \in \mathbb{N}$ such that $u_m << c$ for all $m > N$.*

Proposition 2.24.*[15]Consider a Banach space $\mathfrak{B}$ with a solid cone $\mathcal{C}$. Let $\{a_m\}$ and $\{b_m\}$ be c-sequences in $\mathcal{C}$ and $\alpha, \beta > 0$. Then, $\{\alpha a_m + \beta b_m\}$ is a c-sequence.*

Lemma 2.25.*[13]Consider a Banach algebra $\mathcal{A}$ with a solid cone $\mathcal{C}$. Let $s \in \mathcal{C}$ and $\{u_m\}$ be a c-sequence in $\mathcal{C}$. Then, $\{su_m\}$ is a c-sequence.*

Lemma 2.26.*[13]Consider a real Banach space $\mathfrak{B}$ with a solid cone $\mathcal{C}$ and let $\{u_m\}$ be a sequence in $\mathcal{C}$ such that $||u_m|| \to 0 (m \to \infty)$. Then, $\{u_m\}$ is a c-sequence.*

Lemma 2.27.*[3]Let $\mathcal{A}$ be a Banach algebra with a unit e. Let $\alpha \in \mathcal{A}$ and $\rho(\alpha) < 1$. Then, $\{\alpha^n\}$ is a c-sequence.*

Lemma 2.28.*[5]Let (X, A) be a complete CAbMS over a real Banach algebra $\mathcal{A}$ with $\mathcal{C}$ as the underlying solid cone. Let $\{w_m\}$ be a sequence in X converging to w^*, then:*

1. *$\{A(w_m, ..., w_m, w^*)\}$ is a c-sequence.*
2. *For any $q \in \mathbb{N}, \{A(w_m, ..., w_m, w_{m+q})\}$ is a c-sequence.*

Lemma 2.29.*[5]Let (X, A) be a CAbMS over Banach algebra $\mathcal{A}$. Then, for all $\mu, \pi, \rho \in X$,*

1. *$A(\mu, \mu, ..., \mu, \pi) \leq bA(\pi, \pi, ..., \pi, \mu)$,*
2. *$A(\mu, \mu, ..., \mu, \pi) \leq (n-1)bA(\mu, \mu, ..., \mu, \rho) + bA(\pi, \pi, ..., \pi, \rho)$.*

3. Main Results

Theorem 3.1.*Let (X, A) be a complete CAbMS over a Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,\ z), ..., \mathcal{H}(w,\ z),\ \mathcal{H}(u, v)) \leq rA(w, ..., w, u) + sA(z, ..., z, v), \quad ...(3.1)$$

for all $w, z, u, v \in X$ and $r, s \in \mathcal{C}$ with $\rho(r + s) < \frac{1}{b^2}$. Then, $\mathcal{H}$ has a unique CFP.

Proof. Let $w_0, z_0 \in X$ and $w_1 = \mathcal{H}(w_0, z_0)$, $z_1 = \mathcal{H}(z_0, w_0)$, $w_2 = \mathcal{H}(w_1, z_1)$, $z_2 = \mathcal{H}(z_1, w_1)$, ..., $w_{m+1} = \mathcal{H}(w_m, z_m)$, $z_{m+1} = \mathcal{H}(z_m, w_m)$.
Then, using condition (3.1), we obtain

$$\begin{aligned} A(w_m, ..., w_m, w_{m+1}) &= A(\mathcal{H}(w_{m-1}, z_{m-1}), ..., \mathcal{H}(w_{m-1}, z_{m-1}),\ \mathcal{H}(w_m, z_m)) \\ &\leq rA(w_{m-1}, ..., w_{m-1}, w_m) + sA(z_{m-1}, ..., z_{m-1}, z_m) \end{aligned}$$

Also, we have

$$\begin{aligned} A(z_m, ..., z_m, z_{m+1}) &= A(\mathcal{H}(z_{m-1}, w_{m-1}), ..., \mathcal{H}(z_{m-1}, w_{m-1}), \mathcal{H}(z_m, w_m)) \\ &\leq rA(z_{m-1}, ..., z_{m-1}, z_m) + sA(w_{m-1}, ..., w_{m-1}, w_m) \end{aligned}$$

If we let $A_m = A(w_m, ..., w_m, w_{m+1}) + A(z_m, ..., z_m, z_{m+1})$, then we have

$$\begin{aligned} A_m &= A(w_m, ..., w_m, w_{m+1}) + A(z_m, ..., z_m, z_{m+1}) \\ &\leq (r + s)(A(w_{m-1}, ..., w_{m-1}, w_m) + A(z_{m-1}, ..., z_{m-1}, z_m)) \\ &= (r + s)A_{m-1} \end{aligned}$$

$\implies A_m \leq \alpha A_{m-1}$, where $\alpha = (r + s)$ and $\rho(\alpha) = \rho(r + s) < 1$.
By repeated application of the above inequality, we get

$$\theta \leq A_m \leq \alpha A_{m-1} \leq \alpha^2 A_{m-2} \leq ... \leq \alpha^m A_0.$$

If $A_0 = \theta$, then (w_0, z_0) is CFP of $\mathcal{H}$. Let us therefore suppose that $A_0 > \theta$.
Then, for $p > q$, we have

$$\begin{aligned} A(w_q, ..., w_q, w_p) &\leq b[(n-1)A(w_q, ..., w_q, w_{q+1}) + A(w_p, ..., w_p, w_{q+1})] \\ &\leq (n-1)bA(w_q, ..., w_q, w_{q+1}) + b^2A(w_{q+1}, ..., w_{q+1}, w_p) \\ &= (n-1)bA(w_q, ..., w_q, w_{q+1}) + (n-1)b^3A(w_{q+1}, ..., w_{q+1}, w_{q+2}) \\ &\quad +b^4A(w_{q+2}, ..., w_{q+2}, w_p) \\ &\leq (n-1)bA(w_q, ..., w_q, w_{q+1}) + (n-1)b^3A(w_{q+1}, ..., w_{q+1}, w_{q+2}) \\ &\quad +(n-1)b^5A(w_{q+2}, ..., w_{q+2}, w_{q+3}) + ... + b^{2(p-q-1)}A(w_{p-1}, ..., w_{p-1}, w_p) \\ &\leq (n-1)b\{A(w_q, ..., w_q, w_{q+1}) + b^2A(w_{q+1}, ..., w_{q+1}, w_{q+2}) \\ &\quad +b^4A(w_{q+2}, ..., w_{q+2}, w_{q+3}) + ... + b^{2(p-q-1)}A(w_{p-1}, ..., w_{p-1}, w_p)\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} A(z_q, ..., z_q, z_p) &\leq (n-1)b\{A(z_q, ..., z_q, z_{q+1}) + b^2A(z_{q+1}, ..., z_{q+1}, z_{q+2}) \\ &\quad +b^4A(z_{q+2}, ..., z_{q+2}, z_{q+3}) + ... + b^{2(p-q-1)}A(z_{p-1}, ..., z_{p-1}, z_p)\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &A(w_q, ..., w_q, w_p) + A(z_q, ..., z_q, z_p) \\ &\quad \leq (n-1)b\{A_q + b^2A_{q+1} + b^4A_{q+2} + ... + b^{2(p-q-1)}A_{p-1}\} \\ &\quad \leq (n-1)b\{\alpha^q + b^2\alpha^{q+1} + b^4\alpha^{q+2} + ... + b^{2(p-q-1)}\alpha^{p-1}\}A_0 \\ &\quad = (n-1)b\alpha^q\{e + b^2\alpha + b^4\alpha^2 + ... + b^{2(p-q-1)}\alpha^{p-q-1}\}A_0 \\ &\quad = (n-1)b\alpha^q\{e + b^2\alpha + (b^2\alpha)^2 + ... + (b^2\alpha)^{p-q-1}\}A_0 \\ &\quad \leq (n-1)b\alpha^q(e - b^2\alpha)^{-1}A_0. \end{aligned}$$

Here, $\rho(\alpha) < \frac{1}{b^2}$ and so $(e - b^2\alpha)^{-1}$ exists.
Also, since $\rho(\alpha) < 1$, $\{\alpha^q\}$ is a c-sequence and this implies that the sequences $\{w_m\}$ and $\{z_m\}$ are Cauchy.
Since X is complete, there exist $w^*, z^* \in X$ such that $\lim\limits_{m\to\infty} w_m = w^*$ and $\lim\limits_{m\to\infty} z_m = z^*$.
We show that (w^*, z^*) is a CFP of $\mathcal{H}$.
Now, we have

$$\begin{aligned} A(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), w^*) &\leq (n-1)bA(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), w_{m+1}) + bA(w^*, ..., w^*, w_{m+1}) \\ &\leq (n-1)bA(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), \mathcal{H}(w_m, z_m)) + bA(w^*, ..., w^*, w_{m+1}) \\ &\leq (n-1)brA(w^*, ..., w^*, w_m) + (n-1)bsA(z^*, ..., z^*, z_m) \\ &\quad + bA(w^*, ..., w^*, w_{m+1}) \\ &\leq (n-1)b^2rA(w_m, ..., w_m, w^*) + (n-1)b^2sA(z_m, ..., z_m, z^*) \\ &\quad +b^2A(w_{m+1}, ..., w_{m+}, w^*). \end{aligned}$$

Since the sequences $\{w_m\}$ and $\{z_m\}$ converge to w^* and z^* respectively, we have $A(w_m, ..., w_m, w^*)$, $A(z_m, ..., z_m, z^*)$ and $A(w_{m+1}, ..., w_{m+1}, w^*)$ form c-sequences. Therefore, for every $c \in \mathcal{A}$ with $\theta << c$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} A(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), w^*) &\leq (n-1)b^2rA(w_m, ..., w_m, w^*) + (n-1)b^2sA(z_m, ..., z_m, z^*) \\ &\quad +b^2A(w_{m+1}, ..., w_{m+1}, w^*) << c, \forall m \geq N \end{aligned}$$

$$\implies A(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), w^*) = \theta$$

$\implies \mathcal{H}(w^*, z^*) = w^*$.
Similarly, we have $\mathcal{H}(z^*, w^*) = z^*$.
Thus, (w^*, z^*) is a CFP of $\mathcal{H}$.
To show uniqueness of the CFP of $\mathcal{H}$, let (w', z') be another CFP of $\mathcal{H}$. Then we have

$$\begin{aligned} A(w', ..., w', w^*) &= A(\mathcal{H}(w', z'), ..., \mathcal{H}(w', z'), \mathcal{H}(w^*, z^*)) \\ &\leq rA(w', ..., w', w^*) + sA(z', ..., z', z^*). \end{aligned}$$

And,

$$\begin{aligned} A(z', ..., z', z^*) &= A(\mathcal{H}(z', w'), ..., \mathcal{H}(z', w'), \mathcal{H}(z^*, w^*)) \\ &\leq rA(z', ..., z', z^*) + sA(w', ..., w', w^*). \end{aligned}$$

Therefore, we have
$A(w', ..., w', w^*) + A(z', ..., z', z^*) \leq (r + s)(A(w', ..., w', w^*) + A(z', ..., z', z^*))$.
Since $\rho(r + s) < 1$, the above relation implies that
$A(w', ..., w', w^*) + A(z', ..., z', z^*) = \theta$.
This means that $A(w', ..., w', w^*) = \theta$ and $A(z', ..., z', z^*) = \theta$.
Hence, we have $(w', z') = (w^*, z^*)$.
Therefore, the CFP of $\mathcal{H}$ is unique.

Theorem 3.2. *Let (X, A) be a complete CAbMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w, z), ..., \mathcal{H}(w, z), \mathcal{H}(u, v)) \leq rA(\mathcal{H}(w, z), ..., \mathcal{H}(w, z), w) + sA(\mathcal{H}(u, v), ..., \mathcal{H}(u, v), u),$$

$\forall w, z, u, v \in X$ and $r, s \in \mathcal{C}$ with $\rho(r) + \rho(s) < \frac{1}{(n-1)b^3}, n > 1$ and r and s commute with each other. Then, $\mathcal{H}$ has a unique CFP.

Proof. Let $w_0, z_0 \in X$ and $w_1 = \mathcal{H}(w_0, z_0), z_1 = \mathcal{H}(z_0, w_0), w_2 = \mathcal{H}(w_1, z_1), z_2 = \mathcal{H}(z_1, w_1), ..., w_{m+1} = \mathcal{H}(w_m, z_m), z_{m+1} = \mathcal{H}(z_m, w_m)$.
Then, we have

$$\begin{aligned} A(w_m, ..., w_m, w_{m+1}) &= A(\mathcal{H}(w_{m-1}, z_{m-1}), ..., \mathcal{H}(w_{m-1}, z_{m-1}), \mathcal{H}(w_m, z_m)) \\ &\leq rA(\mathcal{H}(w_{m-1}, z_{m-1}), ..., \mathcal{H}(w_{m-1}, z_{m-1}), w_{m-1}) \\ &\quad + sA(\mathcal{H}(w_m, z_m), ..., \mathcal{H}(w_m, z_m), w_m) \\ &\leq rA(w_m, ..., w_m, w_{m-1}) + sA(w_{m+1}, ..., w_{m+1}, w_m) \\ &\leq rbA(w_{m-1}, ..., w_{m-1}, w_m) + sbA(w_m, ..., w_m, w_{m+1}) \end{aligned}$$

$\implies A(w_m, ..., w_m, w_{m+1}) \leq rb(e - bs)^{-1}A(w_{m-1}, ..., w_{m-1}, w_m)$
$\implies A(w_m, ..., w_m, w_{m+1}) \leq \beta A(w_{m-1}, ..., w_{m-1}, w_m)$, where $\beta = br(e - bs)^{-1}$.
Similarly, we have
$\implies A(z_m, ..., z_m, z_{m+1}) \leq \beta A(z_{m-1}, ..., z_{m-1}, z_m)$.
Now, we have r and s commute with each other and this implies that br and $(e - bs)^{-1}$ also commute with each other.

$$\begin{aligned} \text{Therefore, } \rho(\beta) &\leq \rho(br)\rho(e - bs)^{-1} \\ &\leq \tfrac{b\rho(r)}{1-b\rho(s)} < 1. \end{aligned}$$

Also, $\rho(s) < \frac{1}{(n-1)b^3} < \frac{1}{b}$ and so $(e - bs)^{-1}$ exists.
Therefore, for $p > q$, we get

$$A(w_q, ..., w_q, w_p) \leq (n - 1)b\beta^q(e - b^2\beta)^{-1}A(w_0, ..., w_0, w_1)$$

And, $A(z_q, ..., z_q, z_p) \leq (n - 1)b\beta^q(e - b^2\beta)^{-1}A(z_0, ..., z_0, z_1)$.
Here, $\rho(b^2\beta) = b^2\rho(\beta) < 1$ and so $(e - b^2\beta)^{-1}$ exists.
Also, since $\rho(\beta) < 1$, $\{\beta^q\}$ is a c-sequence. Therefore, the sequences $\{w_m\}$ and $\{z_m\}$ are Cauchy.
Since X is complete, there exist $w^*, z^* \in X$ such that $\lim\limits_{m\to\infty} w_m = w^*$ and $\lim\limits_{m\to\infty} z_m = z^*$.
We show that (w^*, z^*) is a CFP of $\mathcal{H}$.
Now, we have

$$\begin{aligned} A(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), w^*) &\leq (n - 1)bA(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), w_{m+1}) + bA(w^*, ..., w^*, w_{m+1}) \\ &= (n-1)bA(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), \mathcal{H}(w_m, z_m)) + bA(w^*, ..., w^*, w_{m+1}) \\ &\leq (n - 1)brA(\mathcal{H}(w^*, z^*), ..., \mathcal{H}(w^*, z^*), w^*) \\ &\quad + (n - 1)bsA(\mathcal{H}(w_m, z_m), ..., \mathcal{H}(w_m, z_m), w_m) + bA(w^*, ..., w^*, w_{m+1}) \end{aligned}$$

$$\begin{aligned}\implies A(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w^*) &\leq (n-1)bs\{e-(n-1)br\}^{-1}A(\mathcal{H}(w_m,z_m),...,\mathcal{H}(w_m,z_m),w_m)\\ &\quad +b\{e-(n-1)br\}^{-1}A(w^*,...,w^*,w_{m+1})\\ &\leq (n-1)bs\{e-(n-1)br\}^{-1}A(w_{m+1},...,w_{m+1},w_m)\\ &\quad +b\{e-(n-1)br\}^{-1}A(w^*,...,w^*,w_{m+1})\\ &\leq (n-1)bs\{e-(n-1)br\}^{-1}\bigg((n-1)bA(w_{m+1},...,w_{m+1},w^*)\\ &\quad +bA(w_m,...,w_m,w^*)\bigg)+b^2\{e-(n-1)br\}^{-1}A(w_{m+1},...,w_{m+1},w^*)\\ &= (n-1)^2b^2s\{e-(n-1)br\}^{-1}A(w_{m+1},...,w_{m+1},w^*)\\ &\quad +(n-1)b^2s\{e-(n-1)br\}^{-1}A(w_m,...,w_m,w^*)\\ &\quad +b^2\{e-(n-1)br\}^{-1}A(w_{m+1},...,w_{m+1},w^*).\end{aligned}$$

Here, $\rho(r)<\frac{1}{(n-1)b^3}<\frac{1}{(n-1)b}$ and so $\{e-(n-1)br\}^{-1}$ exists.
Since the sequence $\{w_m\}$ converges to w^*, the sequences $\{A(w_{m+1},...,w_{m+1},w^*)\}$ and $\{A(w_m,...,w_m,w^*)\}$ are c-sequences.
This implies that
$A(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w^*)=\theta$
$\implies \mathcal{H}(w^*,z^*)=w^*$.
Similarly, we can get $\mathcal{H}(z^*,w^*)=z^*$.
Thus, (w^*,z^*) is a CFP of $\mathcal{H}$.
Now, if (w',z') is another CFP of $\mathcal{H}$, then we have

$$\begin{aligned}A(w',...,w',w^*) &= A(\mathcal{H}(w',z'),...,\mathcal{H}(w',z'),\mathcal{H}(w^*,z^*))\\ &\leq rA(\mathcal{H}(w',z'),...,\mathcal{H}(w',z'),w')+sA(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w^*)\\ &=\theta\end{aligned}$$

Therefore, $A(w',...,w',w^*)=\theta$
So, $w'=w^*$.
Similarly, we can get $z'=z^*$.
Thus, we have $(w',z')=(w^*,z^*)$
Hence, the CFP of $\mathcal{H}$ is unique.

Theorem 3.3. *Let (X,A) be a complete CAbMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H}:X\times X\to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v))\leq rA(\mathcal{H}(w,z),...,\mathcal{H}(w,z),u)+sA(\mathcal{H}(u,v),...,\mathcal{H}(u,v),w)$$

$\forall w,z,u,v\in X$ and $r,s\in\mathcal{C}$ with $max\{\rho(r),\rho(s)\}<\frac{1}{2(n-1)^2b^3}, n>1$. Then, $\mathcal{H}$ has a unique CFP.

Proof: Let $w_0,z_0\in X$ and $w_1=\mathcal{H}(w_0,z_0), z_1=\mathcal{H}(z_0,w_0), w_2=\mathcal{H}(w_1,z_1), z_2=\mathcal{H}(z_1,w_1),...,w_{m+1}=\mathcal{H}(w_m,z_m), z_{m+1}=\mathcal{H}(z_m,w_m)$.
Then, we have

$$\begin{aligned}A(w_m,...,w_m,w_{m+1}) &= A(\mathcal{H}(w_{m-1},z_{m-1}),...,\mathcal{H}(w_{m-1},z_{m-1}),\mathcal{H}(w_m,z_m))\\ &\leq rA(\mathcal{H}(w_{m-1},z_{m-1}),...,\mathcal{H}(w_{m-1},z_{m-1}),w_m)\\ &\quad +sA(\mathcal{H}(w_m,z_m),...,\mathcal{H}(w_m,z_m),w_{m-1})\\ &= rA(w_m,...,w_m,w_m)+sA(w_{m+1},...,w_{m+1},w_{m-1})\\ &= sA(w_{m+1},...,w_{m+1},w_{m-1})\\ &\leq s\bigg((n-1)bA(w_{m+1},...,w_{m+1},w_m)+bA(w_{m-1},...,w_{m-1},w_m)\bigg)\\ &\leq s\bigg((n-1)b^2A(w_m,...,w_m,w_{m+1})+bA(w_{m-1},...,w_{m-1},w_m)\bigg)\end{aligned}$$

$\implies A(w_m,...,w_m,w_{m+1})\leq sb\{e-(n-1)b^2s\}^{-1}A(w_{m-1},...,w_{m-1},w_m)$
$\implies A(w_m,...,w_m,w_{m+1})\leq \delta A(w_{m-1},...,w_{m-1},w_m)$, where $\delta=bs\{e-(n-1)b^2s\}^{-1}$.
Similarly, we have
$A(z_m,...,z_m,z_{m+1})\leq\delta A(z_{m-1},...,z_{m-1},z_m)$.
Here, we have

$$\rho(s) < \tfrac{1}{2(n-1)^2b^3} < \tfrac{1}{(n-1)b^2}.$$

Therefore, $\{e-(n-1)b^2s\}^{-1}$ exists.
Also, $\rho(\delta) \leq \frac{b\rho(s)}{1-(n-1)b^2\rho(s)} < 1$.
Therefore, for $p > q$, we get

$$A(w_q,...,w_q,w_p) \leq (n-1)b\delta^q(e-b^2\delta)^{-1}A(w_0,...,w_0,w_1)$$

and $A(z_q,...,z_q,z_p) \leq (n-1)b\delta^q(e-b^2\delta)^{-1}A(z_0,...,z_0,z_1)$.
Here, we have

$$\rho(b^2\delta) \leq \tfrac{b^3\rho(s)}{1-(n-1)b^2\rho(s)} < 1 \text{ and so } (e-b^2\delta)^{-1} \text{ exists.}$$

Also, since $\rho(\delta) < 1$, $\{\delta^q\}$ is a c-sequence and this implies that the sequences $\{w_m\}$ and $\{z_m\}$ are Cauchy. Since X is complete, there exist $w^*, z^* \in X$ such that $\lim\limits_{m\to\infty} w_m = w^*$ and $\lim\limits_{m\to\infty} z_m = z^*$.
We show that (w^*,z^*) is a CFP of $\mathcal{H}$.
We have

$$\begin{aligned}
A(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w^*) &\leq (n-1)bA(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w_{m+1}) + bA(w^*,...,w^*,w_{m+1})\\
&= (n-1)bA(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),\mathcal{H}(w_m,z_m)) + bA(w^*,...,w^*,w_{m+1})\\
&\leq (n-1)brA(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w_m)\\
&\quad +(n-1)bsA(\mathcal{H}(w_m,z_m),...,\mathcal{H}(w_m,z_m),w^*) + bA(w^*,...,w^*,w_{m+1})\\
&\leq (n-1)brA(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w_m)\\
&\quad +(n-1)bsA(w_{m+1},...,w_{m+1},w^*) + b^2A(w_{m+1},...,w_{m+1},w^*)\\
&\leq (n-1)^2b^2rA(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w^*)\\
&\quad +(n-1)b^2rA(w_m,...,w_m,w^*) + ((n-1)bs+b^2e)A(w_{m+1},...,w_{m+1},w^*)
\end{aligned}$$

$$\begin{aligned}
\implies A(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w^*) &\leq (n-1)b^2r\{e-(n-1)^2b^2r\}^{-1}A(w_m,...,w_m,w^*)\\
&\quad +\{(n-1)bs+b^2e\}\{e-(n-1)^2b^2r\}^{-1}A(w_{m+1},...,w_{m+1},w^*).
\end{aligned}$$

Here, $\rho(r) < \frac{1}{2(n-1)^2b^3} < \frac{1}{(n-1)^2b^2}$ and so $\{e-(n-1)^2b^2\}^{-1}$ exists.
Now, $\{A(w_m,...,w_m,w^*)\}$ and $\{A(w_{m+1},...,w_{m+1},w^*)\}$ are c-sequences and therefore we have

$$A(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w^*) = \theta$$

$\implies \mathcal{H}(w^*,z^*) = w^*$.
Similarly, we can get $\mathcal{H}(z^*,w^*) = z^*$.
Thus, (w^*,z^*) is a CFP of $\mathcal{H}$.
Now, if (w',z') is another CFP of $\mathcal{H}$, then we have

$$\begin{aligned}
A(w',...,w',w^*) &= A(\mathcal{H}(w',z'),...,\mathcal{H}(w',z'),\mathcal{H}(w^*,z^*))\\
&\leq rA(\mathcal{H}(w',z'),...,\mathcal{H}(w',z'),w^*) + sA(\mathcal{H}(w^*,z^*),...,\mathcal{H}(w^*,z^*),w')\\
&= rA(w',...,w',w^*) + sA(w^*,...,w^*,w')\\
&\leq (r+sb)A(w',...,w',w^*).
\end{aligned}$$

Since $\rho(r) + b\rho(s) < 1$, the above inequality implies
$A(w',...,w',w^*) = \theta$ and so $w' = w^*$.
Similarly, we can get $z' = z^*$.
Thus, we have $(w',z') = (w^*,z^*)$ which shows that the CFP of $\mathcal{H}$ is unique.

When $r = s$ in **Theorems 3.1, 3.2 and 3.3**, we get the following corollaries.

Corollary 3.4. *Let (X,A) be a complete CAbMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H}: X\times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq r\bigg(A(w,...,w,u) + A(z,...,z,v)\bigg)$$

$\forall w,z,u,v \in X$ and $r \in \mathcal{C}$ with $\rho(r) < \frac{1}{2b^2}$. Then, $\mathcal{H}$ has a unique CFP.

Corollary 3.5.*Let (X,A) be a complete CAbMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H}: X\times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq r\bigg(A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),w) + A(\mathcal{H}(u,v),...,\mathcal{H}(u,v),u)\bigg)$$

$\forall w,z,u,v \in X$ and $r \in \mathcal{C}$ with $\rho(r) < \frac{1}{2(n-1)b^3}$. Then, $\mathcal{H}$ has a unique CFP.

Corollary 3.6.*Let (X, A) be a complete CAbMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq r\Bigg(A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),u) + A(\mathcal{H}(u,v),...,\mathcal{H}(u,v),w)\Bigg)$$

$\forall w, z, u, v \in X$ and $r \in \mathcal{C}$ with $\rho(r) < \frac{1}{2(n-1)^2 b^3}$. Then, $\mathcal{H}$ has a unique CFP.

Again if we put $b = 1$ in the **Theorems 3.1,3.2 and 3.3**, we get the following corollaries as the corresponding results in cone A-metric space (CAMS) over a real Banach algebra $\mathcal{A}$.

Corollary 3.7.*Let (X, A) be a complete CAMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq rA(w,...,w,u) + sA(z,...,z,v)$$

$\forall w, z, u, v \in X$ and $r, s \in \mathcal{C}$ with $\rho(r + s) < 1$. Then, $\mathcal{H}$ has a unique CFP.

Corollary 3.8.*Let (X, A) be a complete CAMS over Banach algebra $\mathcal{A}$ and $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq rA(\mathcal{H}(w,z),...,\mathcal{H}(w,z),w) + sA(\mathcal{H}(u,v),...,\mathcal{H}(u,v),u)$$

$\forall w, z, u, v \in X$ and $r, s \in \mathcal{C}$ with $\rho(r) + \rho(s) < \frac{1}{(n-1)}$ and r and s commute with each other. Then, $\mathcal{H}$ has a unique CFP.

Corollary 3.9.*Let (X, A) be a complete CAMS over a Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq rA(\mathcal{H}(w,z),...,\mathcal{H}(w,z),u) + sA(\mathcal{H}(u,v),...,\mathcal{H}(u,v),w)$$

$\forall w, z, u, v \in X$ and $r, s \in \mathcal{C}$ with $max\{\rho(r), \rho(s)\} < \frac{1}{2(n-1)^2}$. Then, $\mathcal{H}$ has a unique CFP.

Again, if we put $r = s$ in the **Corollaries 3.7, 3.8 and 3.9**, we get the following corollaries.

Corollary 3.10.*Let (X, A) be a complete CAMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq r\Bigg(A(w,...,w,u) + A(z,...,z,v)\Bigg)$$

$\forall w, z, u, v \in X$ and $r \in \mathcal{C}$ with $\rho(r) < \frac{1}{2}$. Then, $\mathcal{H}$ has a unique CFP.

Corollary 3.11.*Let (X, A) be a complete CAMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq r\Bigg(A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),w) + A(\mathcal{H}(u,v),...,\mathcal{H}(u,v),u)\Bigg)$$

$\forall w, z, u, v \in X$ and $r \in \mathcal{C}$ with $\rho(r) < \frac{1}{2(n-1)}$. Then, $\mathcal{H}$ has a unique CFP.

Corollary 3.12.*Let (X, A) be a complete CAMS over Banach algebra $\mathcal{A}$ and let $\mathcal{C}$ be a solid cone in $\mathcal{A}$. Let the mapping $\mathcal{H} : X \times X \to X$ satisfy the condition*

$$A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),\mathcal{H}(u,v)) \leq r\Bigg(A(\mathcal{H}(w,z),...,\mathcal{H}(w,z),u) + A(\mathcal{H}(u,v),...,\mathcal{H}(u,v),w)\Bigg)$$

$\forall w, z, u, v \in X$ and $r \in \mathcal{C}$ with $\rho(r) < \frac{1}{2(n-1)^2}$. Then, $\mathcal{H}$ has a unique CFP.

Example 3.13.*Let $\mathcal{A} = \mathcal{C}[c, d]$ be the real normed linear space of continuous functions defined on the interval $[c, d]$ with the supremum norm. Multiplication in $\mathcal{A}$ is defined as the pointwise multiplication of functions. Then, $\mathcal{A}$ is a real Banach algebra with a unit 1 (a constant function). Let $\mathcal{C} = \{u \in \mathcal{A} : u(t) \geq 0, t \in [c, d]\}$ and $X = [0, 1]$.*

Let us define a mapping $A : X^n \to \mathcal{A}$ by

$$\begin{aligned}A(v_1, v_2, ..., v_{n-1}, v_n)(t) =&\Big((v_1 - v_2)^2 + (v_1 - v_3)^2 + ... + (v_1 - v_n)^2 \\ &+(v_2 - v_3)^2 + (v_2 - v_4)^2 + ... + (v_2 - v_n)^2 \\ &+... + (v_{n-2} - v_{n-1})^2 + (v_{n-2} - v_n)^2 + (v_{n-1} - v_n)^2\Big)3^t \\ &= \sum_{i=1}^{n}\sum_{i<j}(v_i - v_j)^2 3^t.\end{aligned}$$

Then, (X, A) *is a complete CAbMS over the Banach algebra* $\mathcal{A}$ *with* $b = 2$ *and non-normal solid cone* $\mathcal{C}$. *We consider a mapping* $\mathcal{H} : X \times X \to X$ *defined by* $\mathcal{H}(\gamma, \delta) = \frac{\gamma+\delta}{10}$.
Then,

$$\begin{aligned}A(\mathcal{H}(\gamma, \delta), ..., \mathcal{H}(\gamma, \delta), \mathcal{H}(u, v))(t) &= A(\tfrac{\gamma+\delta}{10}, ..., \tfrac{\gamma+\delta}{10}, \tfrac{u+v}{10})(t) \\ &= (n-1)\Big(\tfrac{\gamma+\delta}{10} - \tfrac{u+v}{10}\Big)^2 3^t \\ &= \tfrac{(n-1)}{100}\Big((\gamma - u) + (\delta - v)\Big)^2 3^t,\end{aligned}$$

and

$$\begin{aligned}\Big(A(\gamma, ..., \gamma, u) + A(\delta, ..., \delta, v)\Big)(t) &= A(\gamma, ..., \gamma, u)(t) + A(\delta, ..., \delta, v)(t) \\ &=\Big((n-1)(\gamma - u)^2 + (n-1)(\delta - v)^2\Big)3^t \\ &= (n-1)\Big((\gamma - u)^2 + (\delta - v)^2\Big)3^t.\end{aligned}$$

We know that $(a + b)^2 \leq 2(a^2 + b^2) or \frac{1}{2}(a + b)^2 \leq (a^2 + b^2)$ *for all* $a, b \in \mathbb{R}$. *Using this inequality, we have*

$$\begin{aligned}A\Big(\mathcal{H}(\gamma, \delta), ..., \mathcal{H}(\gamma, \delta), \mathcal{H}(u, v)\Big)(t) &= \tfrac{(n-1)}{100}\Big((\gamma - u) + (\delta - v)\Big)^2 3^t \\ &= \tfrac{1}{50}\tfrac{(n-1)}{2}\Big((\gamma - u) + (\delta - v)\Big)^2 3^t \\ &\leq \tfrac{1}{50}(n-1)\Big((\gamma - u)^2 + (\delta - v)^2\Big)3^t \\ &= \tfrac{1}{50}\Big(A(\gamma, ..., \gamma, u) + A(\delta, ..., \delta, v)\Big)(t), \forall t \in [c, d].\end{aligned}$$

Therefore, we have

$$A(\mathcal{H}(\gamma, \delta), ..., \mathcal{H}(\gamma, \delta), \mathcal{H}(u, v)) \leq k\Big(A(\gamma, ..., \gamma, u) + A(\delta, ..., \delta, v)\Big)$$

for $k = \frac{1}{50}$ *(a constant function)* $\in \mathcal{C}$ *and* $\rho(k) = \frac{1}{50} < \frac{1}{2b^2} = \frac{1}{8}$.

Therefore, all the conditions of corollary 3.4 are satisfied and we see that $\mathcal{H}$ *has a unique CFP* $(0, 0)$.

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K. Bala Devi,
Department of Mathematics,
Dhanamanjuri University,
India.
E-mail address: `konj.bala23@gmail.com`

and

Konthoujam Sangita Devi,
Department of Mathematics,
Dhanamanjuri University,
India.
E-mail address: `konsangitadevi@yahoo.com`

and

Kumam Anthony Singh,
Department of Mathematics,
Manipur University,
India.
E-mail address: `anthonykumam@manipuruniv.ac.in`