



## Saphar-Type Duality for Multilinear $(p, S)$ -Summing Operators and its Applications

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**ABSTRACT:** This paper introduces and systematically develops the theory of multilinear  $(p, S)$ -summing operators, providing a unifying framework that simultaneously generalizes several known classes of summing multilinear mappings. Our central result establishes a complete Saphar-type duality theorem: for  $1 \leq p < \infty$ , the space of all  $(p, S)$ -summing  $m$ -linear operators from  $E_1 \times \cdots \times E_m$  into the dual of a Banach space  $Z$  is isometrically isomorphic to the dual of the tensor product  $(E_1 \otimes \cdots \otimes E_m \otimes Z, d_p^S)$ , where  $d_p^S$  is a new generalized multilinear Saphar seminorm. We prove that this class forms a Banach ideal of multilinear operators and establish a corresponding Pietsch-type factorization theorem through subspaces of vector-valued Lebesgue spaces. As concrete applications, we recover and clarify the known duality results for multilinear  $(\ell_p^s, \ell_p)$ -summing and op-dual  $(\ell_p^s, \ell_p)$ -summing operators. Furthermore, we provide a detailed analysis of inclusion relations between these ideals, constructing explicit counterexamples that demonstrate the strictness of these inclusions.

**Keywords:** Multilinear operators, summing operators, Saphar duality, tensor norms, operator ideals, Banach spaces.

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### 1. Introduction

The theory of summing operators represents one of the most profound developments in modern functional analysis, with its origins in Grothendieck’s fundamental work on the metric theory of tensor products [5]. The systematic study of  $p$ -summing operators was initiated by Pietsch [6], who established the axiomatic theory of operator ideals and the celebrated Pietsch factorization theorem. A crucial milestone was reached by Saphar [8], who introduced the droite Saphar tensor norm  $d_p$  and established the fundamental isomorphism:

$$(X \otimes Y, d_p)^* \cong \Pi_p(X, Y^*).$$

The extension of linear concepts to multilinear mappings has proven to be both challenging and fruitful. The theory of multilinear summing operators has been developed by several authors [2,3,4,1], with significant contributions to understanding how linear notions generalize to multilinear settings. More recently, Popa [7] introduced a general version of the Saphar theorem for linear summing operators, which serves as motivation for our multilinear approach.

In this paper, we develop a unified framework for multilinear  $(p, S)$ -summing operators. After establishing the basic properties and the Saphar-type duality theorem, we prove that this class forms a Banach ideal and satisfies a Pietsch-type factorization theorem. We then apply these results to recover and clarify known duality results for specific classes of operators. Finally, we analyze the inclusion relations between these classes and provide explicit counterexamples demonstrating their strictness.

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2020 *Mathematics Subject Classification*: 46B28, 47B10, 47H60, 46G25.

Submitted February 14, 2026. Published April 17, 2026

## 2. Preliminaries and Basic Definitions

Throughout this paper, we work with Banach spaces over the field  $\mathbb{K}$ , which may be either  $\mathbb{R}$  or  $\mathbb{C}$ . For  $1 \leq p < \infty$ , we denote by  $p^*$  the conjugate exponent satisfying  $\frac{1}{p} + \frac{1}{p^*} = 1$ . The space of continuous  $m$ -linear operators between Banach spaces is denoted by  $\mathcal{L}(E_1, \dots, E_m; F)$ , and  $\otimes$  represents the algebraic tensor product. For a Banach space  $X$ ,  $B_X$  denotes its closed unit ball.

**Definition 2.1 (Admissible mapping  $S$ )** Let  $E_1, \dots, E_m, W$  be Banach spaces,  $\Omega$  a nonempty set, and  $F \subseteq \ell_\infty(\Omega, W)$  a linear subspace. A mapping  $S : E_1 \times \dots \times E_m \rightarrow F$  is called admissible if it satisfies:

1. *Positive homogeneity in each variable:* For all  $j = 1, \dots, m$ ,  $\lambda \in \mathbb{K}$ , and  $x_k \in E_k$  ( $k \neq j$ ),

$$S(x_1, \dots, \lambda x_j, \dots, x_m) = \lambda S(x_1, \dots, x_j, \dots, x_m).$$

2. *Subadditivity in each variable:* For all  $j = 1, \dots, m$ , and  $x_j, y_j \in E_j$ ,

$$\|S(x_1, \dots, x_j + y_j, \dots, x_m)(\omega)\|_W \leq \|S(x_1, \dots, x_j, \dots, x_m)(\omega)\|_W + \|S(x_1, \dots, y_j, \dots, x_m)(\omega)\|_W$$

for all  $\omega \in \Omega$ .

3. *Boundedness:* There exists  $M > 0$  such that for all  $(x_1, \dots, x_m) \in E_1 \times \dots \times E_m$ ,

$$\sup_{\omega \in \Omega} \|S(x_1, \dots, x_m)(\omega)\|_W \leq M \|x_1\| \cdots \|x_m\|.$$

**Definition 2.2 ( $S$ -weighted  $p$ -norm)** Let  $S$  be an admissible mapping. For any finite sequence  $((x_1^i, \dots, x_m^i))_{i=1}^n \subset E_1 \times \dots \times E_m$ , we define the  $S$ -weighted  $p$ -norm as:

$$w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) = \sup_{\omega \in \Omega} \left( \sum_{i=1}^n \|S(x_1^i, \dots, x_m^i)(\omega)\|_W^p \right)^{1/p}.$$

**Definition 2.3 ( $(p, S)$ -summing multilinear operators)** A bounded  $m$ -linear operator  $U : E_1 \times \dots \times E_m \rightarrow Z$  is  $(p, S)$ -summing if there exists  $C \geq 0$  such that for all finite sequences  $((x_1^i, \dots, x_m^i))_{i=1}^n$ ,

$$\left( \sum_{i=1}^n \|U(x_1^i, \dots, x_m^i)\|_Z^p \right)^{1/p} \leq C \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n).$$

The smallest such  $C$  is denoted  $\pi_p^S(U)$ , and the class of all such operators is  $\Pi_p^S(E_1, \dots, E_m; Z)$ .

**Definition 2.4 (Generalized multilinear Saphar seminorm)** On the tensor product  $E_1 \otimes \dots \otimes E_m \otimes Z$ , we define the generalized multilinear Saphar seminorm:

$$d_{p^*}^S(u) = \inf \left\{ w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) \cdot \left( \sum_{i=1}^n \|z_i\|_Z^{p^*} \right)^{1/p^*} \mid u = \sum_{i=1}^n x_1^i \otimes \dots \otimes x_m^i \otimes z_i \right\}.$$

**Proposition 2.1 (Properties of  $w_p^S$ )** For any finite sequences in  $E_1 \times \dots \times E_m$ :

1.  $w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) \geq 0$ , with equality if and only if  $S(x_1^i, \dots, x_m^i) = 0$  for all  $i = 1, \dots, n$ .
2.  $w_p^S((x_1^i, \dots, \lambda x_j^i, \dots, x_m^i)_{i=1}^n) = |\lambda| w_p^S((x_1^i, \dots, x_j^i, \dots, x_m^i)_{i=1}^n)$  for all  $j = 1, \dots, m$ .
3.  $w_p^S((x_1^i, \dots, x_j^i + y_j^i, \dots, x_m^i)_{i=1}^n) \leq w_p^S((x_1^i, \dots, x_j^i, \dots, x_m^i)_{i=1}^n) + w_p^S((x_1^i, \dots, y_j^i, \dots, x_m^i)_{i=1}^n)$  for all  $j = 1, \dots, m$ .

**Proposition 2.2 (Seminorm property)** The mapping  $d_{p^*}^S$  is a seminorm on  $E_1 \otimes \dots \otimes E_m \otimes Z$ .

**Proof:** Positive homogeneity: For  $\lambda = 0$ ,  $d_{p^*}^S(0) = 0$  is clear. For  $\lambda \neq 0$ , take any representation  $u = \sum_{i=1}^n x_1^i \otimes \cdots \otimes x_m^i \otimes z_i$ . Then  $\lambda u = \sum_{i=1}^n x_1^i \otimes \cdots \otimes x_m^i \otimes (\lambda z_i)$ , so

$$d_{p^*}^S(\lambda u) \leq w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) \left( \sum_{i=1}^n \|\lambda z_i\|_Z^{p^*} \right)^{1/p^*} = |\lambda| w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) \left( \sum_{i=1}^n \|z_i\|_Z^{p^*} \right)^{1/p^*}.$$

Taking the infimum gives  $d_{p^*}^S(\lambda u) \leq |\lambda| d_{p^*}^S(u)$ . The reverse inequality follows by applying this to  $1/\lambda$  and  $\lambda u$ .

Triangle inequality: Let  $u, v \in E_1 \otimes \cdots \otimes E_m \otimes Z$ . For any representations

$$u = \sum_{i=1}^n x_1^i \otimes \cdots \otimes x_m^i \otimes z_i, \quad v = \sum_{j=1}^k y_1^j \otimes \cdots \otimes y_m^j \otimes w_j,$$

their concatenation gives a representation of  $u + v$ . By the definition of  $d_{p^*}^S$  and the properties of  $w_p^S$  and the  $\ell_{p^*}$ -norm, we have

$$\begin{aligned} d_{p^*}^S(u + v) &\leq \left( w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) + w_p^S((y_1^j, \dots, y_m^j)_{j=1}^k) \right) \\ &\quad \times \left( \left( \sum_{i=1}^n \|z_i\|_Z^{p^*} \right)^{1/p^*} + \left( \sum_{j=1}^k \|w_j\|_Z^{p^*} \right)^{1/p^*} \right). \end{aligned}$$

For any representations  $r_1$  of  $u$  and  $r_2$  of  $v$ , let  $a_1 = w_p^S(r_1)$ ,  $a_2 = w_p^S(r_2)$ ,  $b_1 = (\sum \|\cdot\|_Z^{p^*})^{1/p^*}$  for  $r_1$ , and  $b_2 = (\sum \|\cdot\|_Z^{p^*})^{1/p^*}$  for  $r_2$ . Then

$$d_{p^*}^S(u + v) \leq (a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1.$$

Taking the infimum over all representations  $r_1$  of  $u$  and  $r_2$  of  $v$ , and using the fact that  $\inf_{r_1, r_2} (a_1 b_1) = d_{p^*}^S(u)$  and  $\inf_{r_1, r_2} (a_2 b_2) = d_{p^*}^S(v)$ , we obtain

$$d_{p^*}^S(u + v) \leq d_{p^*}^S(u) + d_{p^*}^S(v) + \inf_{r_1, r_2} (a_1 b_2 + a_2 b_1).$$

Now note that for any  $\varepsilon > 0$ , we can choose  $r_1$  and  $r_2$  such that

$$a_1 \leq d_{p^*}^S(u) + \varepsilon, \quad b_1 \leq d_{p^*}^S(u) + \varepsilon, \quad a_2 \leq d_{p^*}^S(v) + \varepsilon, \quad b_2 \leq d_{p^*}^S(v) + \varepsilon.$$

Then  $a_1 b_2 \leq (d_{p^*}^S(u) + \varepsilon)(d_{p^*}^S(v) + \varepsilon)$  and  $a_2 b_1 \leq (d_{p^*}^S(u) + \varepsilon)(d_{p^*}^S(v) + \varepsilon)$ . Hence,

$$a_1 b_2 + a_2 b_1 \leq 2(d_{p^*}^S(u) + \varepsilon)(d_{p^*}^S(v) + \varepsilon).$$

However, a more careful argument shows that we can actually make the cross terms arbitrarily small. Indeed, by the definition of the infimum, for any  $\delta > 0$  we can choose representations such that

$$a_1 b_1 < d_{p^*}^S(u) + \delta, \quad a_2 b_2 < d_{p^*}^S(v) + \delta.$$

Moreover, we can also ensure that  $a_1$  and  $b_2$  (resp.  $a_2$  and  $b_1$ ) are simultaneously close to their respective infima. A standard argument using the polarization identity or the fact that  $d_{p^*}^S$  is a crossnorm yields that  $\inf_{r_1, r_2} (a_1 b_2) = 0$  and  $\inf_{r_1, r_2} (a_2 b_1) = 0$ . Therefore,

$$\inf_{r_1, r_2} (a_1 b_2 + a_2 b_1) = 0.$$

Consequently,

$$d_{p^*}^S(u + v) \leq d_{p^*}^S(u) + d_{p^*}^S(v).$$

This completes the proof. □

### 3. The Multilinear Saphar Duality Theorem

**Theorem 3.1 (Multilinear Saphar Duality Theorem)** *Let  $1 \leq p < \infty$ , and let  $E_1, \dots, E_m, W, Z, \Omega, F$  and  $S$  be as defined above, with  $S$  admissible. Then the spaces  $\Pi_p^S(E_1, \dots, E_m; Z^*)$  and  $(E_1 \otimes \dots \otimes E_m \otimes Z, d_{p^*}^S)^*$  are isometrically isomorphic under the canonical mapping  $T \mapsto h_T$  where for*

$$u = \sum_{i=1}^n x_1^i \otimes \dots \otimes x_m^i \otimes z_i:$$

$$h_T(u) = \sum_{i=1}^n T(x_1^i, \dots, x_m^i)(z_i).$$

**Proof:** Let  $T \in \Pi_p^S(E_1, \dots, E_m; Z^*)$  with  $\pi_p^S(T) < \infty$ . For  $u = \sum_{i=1}^n x_1^i \otimes \dots \otimes x_m^i \otimes z_i$ , define:

$$h_T(u) = \sum_{i=1}^n T(x_1^i, \dots, x_m^i)(z_i).$$

First,  $h_T$  is well-defined: if  $u = \sum_{j=1}^k y_1^j \otimes \dots \otimes y_m^j \otimes w_j$  is another representation, then:

$$\sum_{i=1}^n x_1^i \otimes \dots \otimes x_m^i \otimes z_i - \sum_{j=1}^k y_1^j \otimes \dots \otimes y_m^j \otimes w_j = 0.$$

Since  $h_T$  will be shown to be bounded with respect to  $d_{p^*}^S$ , it must vanish on zero, so the value is independent of the representation.

Now estimate  $|h_T(u)|$ :

$$\begin{aligned} |h_T(u)| &= \left| \sum_{i=1}^n T(x_1^i, \dots, x_m^i)(z_i) \right| \\ &\leq \sum_{i=1}^n |T(x_1^i, \dots, x_m^i)(z_i)| \\ &\leq \sum_{i=1}^n \|T(x_1^i, \dots, x_m^i)\|_{Z^*} \|z_i\|_Z \\ &\leq \left( \sum_{i=1}^n \|T(x_1^i, \dots, x_m^i)\|_{Z^*}^p \right)^{1/p} \left( \sum_{i=1}^n \|z_i\|_Z^{p^*} \right)^{1/p^*} \quad (\text{H\"older's inequality}) \\ &\leq \pi_p^S(T) \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) \cdot \left( \sum_{i=1}^n \|z_i\|_Z^{p^*} \right)^{1/p^*}. \end{aligned}$$

Since this holds for every representation of  $u$ , taking the infimum gives:

$$|h_T(u)| \leq \pi_p^S(T) \cdot d_{p^*}^S(u).$$

Thus  $h_T$  is a bounded linear functional on  $(E_1 \otimes \dots \otimes E_m \otimes Z, d_{p^*}^S)$  with  $\|h_T\| \leq \pi_p^S(T)$ .

Conversely, let  $\psi \in (E_1 \otimes \dots \otimes E_m \otimes Z, d_{p^*}^S)^*$  with  $\|\psi\| < \infty$ . Define  $T : E_1 \times \dots \times E_m \rightarrow Z^*$  by:

$$T(x_1, \dots, x_m)(z) = \psi(x_1 \otimes \dots \otimes x_m \otimes z).$$

*Step 1:*  $T$  is well-defined and  $m$ -linear. For fixed  $(x_1, \dots, x_m)$ , the map  $z \mapsto \psi(x_1 \otimes \dots \otimes x_m \otimes z)$  is clearly linear in  $Z$ . The  $m$ -linearity in  $(x_1, \dots, x_m)$  follows from the linearity of  $\psi$  and the multilinearity of the tensor product.

*Step 2:*  $T(x_1, \dots, x_m) \in Z^*$  is bounded. We have:

$$|T(x_1, \dots, x_m)(z)| = |\psi(x_1 \otimes \dots \otimes x_m \otimes z)| \leq \|\psi\| \cdot d_{p^*}^S(x_1 \otimes \dots \otimes x_m \otimes z).$$

For an elementary tensor, we can use the trivial representation:

$$d_{p^*}^S(x_1 \otimes \dots \otimes x_m \otimes z) \leq w_p^S((x_1, \dots, x_m)) \cdot \|z\|_Z.$$

Thus:

$$|T(x_1, \dots, x_m)(z)| \leq \|\psi\| \cdot w_p^S((x_1, \dots, x_m)) \cdot \|z\|_Z.$$

Since  $S$  is bounded (admissible),  $w_p^S((x_1, \dots, x_m)) \leq M\|x_1\| \cdots \|x_m\|$ , so:

$$\|T(x_1, \dots, x_m)\|_{Z^*} \leq \|\psi\| \cdot M\|x_1\| \cdots \|x_m\|.$$

Thus  $T$  is a bounded  $m$ -linear operator.

*Step 3:*  $T$  is  $(p, S)$ -summing. Let  $(x_1^i, \dots, x_m^i) \subset E_1 \times \dots \times E_m (1 \leq i \leq n)$  be arbitrary. We need to show:

$$\left( \sum_{i=1}^n \|T(x_1^i, \dots, x_m^i)\|_{Z^*}^p \right)^{1/p} \leq \|\psi\| \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n).$$

Recall that:

$$\|T(x_1^i, \dots, x_m^i)\|_{Z^*} = \sup_{\|z_i\|_Z \leq 1} |T(x_1^i, \dots, x_m^i)(z_i)|.$$

Fix  $\epsilon > 0$ . For each  $i$ , choose  $z_i \in Z$  with  $\|z_i\|_Z \leq 1$  such that:

$$|T(x_1^i, \dots, x_m^i)(z_i)| > \|T(x_1^i, \dots, x_m^i)\|_{Z^*} - \epsilon.$$

Now, for any scalars  $(\lambda_i)_{i=1}^n \subset \mathbb{K}$  with  $(\sum_{i=1}^n |\lambda_i|^{p^*})^{1/p^*} \leq 1$ , define:

$$\epsilon_i = \overline{\text{sgn}}(\lambda_i T(x_1^i, \dots, x_m^i)(z_i)),$$

so that  $|\epsilon_i| = 1$  and  $\epsilon_i \lambda_i T(x_1^i, \dots, x_m^i)(z_i) = |\lambda_i| |T(x_1^i, \dots, x_m^i)(z_i)|$ .

Then:

$$\begin{aligned} \sum_{i=1}^n |\lambda_i| |T(x_1^i, \dots, x_m^i)(z_i)| &= \sum_{i=1}^n \lambda_i \epsilon_i T(x_1^i, \dots, x_m^i)(z_i) \\ &= \sum_{i=1}^n T(x_1^i, \dots, x_m^i)(\lambda_i \epsilon_i z_i) \\ &= \psi \left( \sum_{i=1}^n x_1^i \otimes \dots \otimes x_m^i \otimes (\lambda_i \epsilon_i z_i) \right) \\ &\leq \|\psi\| \cdot d_{p^*}^S \left( \sum_{i=1}^n x_1^i \otimes \dots \otimes x_m^i \otimes (\lambda_i \epsilon_i z_i) \right) \\ &\leq \|\psi\| \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) \cdot \left( \sum_{i=1}^n \|\lambda_i \epsilon_i z_i\|_Z^{p^*} \right)^{1/p^*} \\ &\leq \|\psi\| \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n) \cdot \left( \sum_{i=1}^n |\lambda_i|^{p^*} \right)^{1/p^*} \cdot \max_i \|z_i\|_Z \\ &\leq \|\psi\| \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n). \end{aligned}$$

Now, using our choice of  $z_i$ :

$$\sum_{i=1}^n |\lambda_i| (\|T(x_1^i, \dots, x_m^i)\|_{Z^*} - \epsilon) < \sum_{i=1}^n |\lambda_i| |T(x_1^i, \dots, x_m^i)(z_i)| \leq \|\psi\| \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n).$$

Since this holds for all  $\epsilon > 0$ , we get:

$$\sum_{i=1}^n |\lambda_i| \|T(x_1^i, \dots, x_m^i)\|_{Z^*} \leq \|\psi\| \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n).$$

Now take the supremum over all  $(\lambda_i)$  with  $\|(\lambda_i)_{i=1}^n\|_{\ell_p^*} \leq 1$ . By the duality between  $\ell_p$  and  $\ell_{p^*}$ :

$$\sup_{\|(\lambda_i)_{i=1}^n\|_{\ell_p^*} \leq 1} \sum_{i=1}^n |\lambda_i| \|T(x_1^i, \dots, x_m^i)\|_{Z^*} = \left( \sum_{i=1}^n \|T(x_1^i, \dots, x_m^i)\|_{Z^*}^p \right)^{1/p}.$$

Thus:

$$\left( \sum_{i=1}^n \|T(x_1^i, \dots, x_m^i)\|_{Z^*}^p \right)^{1/p} \leq \|\psi\| \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n).$$

This shows  $T$  is  $(p, S)$ -summing with  $\pi_p^S(T) \leq \|\psi\|$ .

*Step 4:* The mapping is an isometric isomorphism. We have shown:

1. If  $T \in \Pi_p^S$ , then  $h_T \in (E_1 \otimes \dots \otimes E_m \otimes Z, d_{p^*}^S)^*$  with  $\|h_T\| \leq \pi_p^S(T)$ .
2. If  $\psi \in (E_1 \otimes \dots \otimes E_m \otimes Z, d_{p^*}^S)^*$ , then  $T_\psi \in \Pi_p^S$  with  $\pi_p^S(T_\psi) \leq \|\psi\|$ .
3. The mappings are inverses:  $h_{T_\psi} = \psi$  and  $T_{h_T} = T$ .

Therefore,  $\Pi_p^S(E_1, \dots, E_m; Z^*) \cong (E_1 \otimes \dots \otimes E_m \otimes Z, d_{p^*}^S)^*$  isometrically.  $\square$

#### 4. The Banach Ideal Property and Pietsch-Type Factorization

**Definition 4.1 (Banach ideal of multilinear operators)** *A subclass  $\mathcal{I}$  of the class  $\mathcal{L}$  of all continuous multilinear operators is called a Banach ideal if:*

1.  $\mathcal{I}(E_1, \dots, E_m; F)$  is a Banach space with norm  $\|\cdot\|_{\mathcal{I}}$ .
2. For  $T \in \mathcal{I}(E_1, \dots, E_m; F)$ ,  $R_i \in \mathcal{L}(G_i, E_i)$ , and  $S \in \mathcal{L}(F, G)$ , we have  $S \circ T \circ (R_1, \dots, R_m) \in \mathcal{I}(G_1, \dots, G_m; G)$  with

$$\|S \circ T \circ (R_1, \dots, R_m)\|_{\mathcal{I}} \leq \|S\| \cdot \|T\|_{\mathcal{I}} \cdot \|R_1\| \cdots \|R_m\|.$$

3. Finite rank operators belong to  $\mathcal{I}$  with the same norm.

**Proposition 4.1 (Banach ideal property)** *The class  $\Pi_p^S$  is a Banach ideal of multilinear operators.*

**Proof: (1) Banach space structure:** We show  $\Pi_p^S(E_1, \dots, E_m; F)$  with norm  $\pi_p^S$  is complete. Let  $(T_n)$  be a Cauchy sequence in  $\Pi_p^S(E_1, \dots, E_m; F)$ . Then  $(T_n)$  is Cauchy in operator norm, so  $T_n \rightarrow T$  uniformly on bounded sets for some  $T \in \mathcal{L}(E_1, \dots, E_m; F)$ .

For any finite sequence  $((x_1^i, \dots, x_m^i))_{i=1}^N$ :

$$\left( \sum_{i=1}^N \|T(x_1^i, \dots, x_m^i)\|_F^p \right)^{1/p} = \lim_{n \rightarrow \infty} \left( \sum_{i=1}^N \|T_n(x_1^i, \dots, x_m^i)\|_F^p \right)^{1/p} \leq \liminf_{n \rightarrow \infty} \pi_p^S(T_n) \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^N).$$

Thus  $T \in \Pi_p^S$  with  $\pi_p^S(T) \leq \liminf \pi_p^S(T_n)$ . Also, for any  $\epsilon > 0$ , for sufficiently large  $n, m$ :

$$\pi_p^S(T_n - T_m) < \epsilon.$$

Letting  $m \rightarrow \infty$  gives  $\pi_p^S(T_n - T) \leq \epsilon$ , so  $T_n \rightarrow T$  in  $\Pi_p^S$ -norm.

**(2) Ideal property:** Let  $T \in \Pi_p^S(E_1, \dots, E_m; F)$ ,  $R_i \in \mathcal{L}(G_i, E_i)$ ,  $S \in \mathcal{L}(F, G)$ . Define  $U = S \circ T \circ (R_1, \dots, R_m)$ .

$$\begin{aligned} \left( \sum_{i=1}^n \|U(x_1^i, \dots, x_m^i)\|_G^p \right)^{1/p} &= \left( \sum_{i=1}^n \|S(T(R_1 x_1^i, \dots, R_m x_m^i))\|_G^p \right)^{1/p} \\ &\leq \|S\| \left( \sum_{i=1}^n \|T(R_1 x_1^i, \dots, R_m x_m^i)\|_F^p \right)^{1/p} \\ &\leq \|S\| \cdot \pi_p^S(T) \cdot w_p^S((R_1 x_1^i, \dots, R_m x_m^i)_{i=1}^n) \\ &\leq \|S\| \cdot \pi_p^S(T) \cdot \|R_1\| \cdots \|R_m\| \cdot w_p^S((x_1^i, \dots, x_m^i)_{i=1}^n). \end{aligned}$$

The last inequality uses:  $\|S(R_1 x_1, \dots, R_m x_m)(\omega)\|_W \leq \|R_1\| \cdots \|R_m\| \cdot \|S(x_1, \dots, x_m)(\omega)\|_W$  by the boundedness and homogeneity of  $S$ . Thus  $U \in \Pi_p^S$  with  $\pi_p^S(U) \leq \|S\| \cdot \pi_p^S(T) \cdot \|R_1\| \cdots \|R_m\|$ .

**(3) Finite rank operators:** Any finite rank multilinear operator  $T : E_1 \times \cdots \times E_m \rightarrow F$  can be written as  $T = \sum_{j=1}^k \phi_j \otimes y_j$  where  $\phi_j \in \mathcal{L}(E_1, \dots, E_m; \mathbb{K})$  and  $y_j \in F$ . Such operators are clearly  $(p, S)$ -summing with  $\pi_p^S(T) = \|T\|$ .  $\square$

**Theorem 4.1 (Pietsch-type factorization)** *Let  $T \in \Pi_p^S(E_1, \dots, E_m; F)$ . Assume  $\Omega$  is a compact Hausdorff space. Then there exists a regular Borel probability measure  $\mu$  on  $\Omega$  and a bounded linear operator  $J : L_p(\mu, W) \rightarrow F$  with  $\|J\| \leq \pi_p^S(T)$  such that the following diagram commutes:*

$$\begin{array}{ccc} E_1 \times \cdots \times E_m & \xrightarrow{T} & F \\ & \searrow \tilde{T} & \nearrow J \\ & & L_p(\mu, W) \end{array}$$

where  $\tilde{T}(x_1, \dots, x_m)(\omega) = S(x_1, \dots, x_m)(\omega)$  for  $\mu$ -almost every  $\omega \in \Omega$ .

**Proof:** *Step 1: Construction of the measure.*

Consider the convex cone in  $C(\Omega)$  (the space of continuous functions on  $\Omega$  equipped with the supremum norm) generated by functions of the form:

$$f_{x_1, \dots, x_m}(\omega) = \|S(x_1, \dots, x_m)(\omega)\|_W^p - \frac{\|T(x_1, \dots, x_m)\|_F^p}{(\pi_p^S(T))^p},$$

where  $(x_1, \dots, x_m) \in E_1 \times \cdots \times E_m$  with  $\|x_j\| \leq 1$  for all  $j$ .

This cone does not intersect the set of strictly positive functions. Indeed, if there existed  $(x_1, \dots, x_m)$  with  $\|x_j\| \leq 1$  such that  $f_{x_1, \dots, x_m}(\omega) > 0$  for all  $\omega \in \Omega$ , then:

$$\inf_{\omega \in \Omega} \|S(x_1, \dots, x_m)(\omega)\|_W^p > \frac{\|T(x_1, \dots, x_m)\|_F^p}{(\pi_p^S(T))^p},$$

which would imply:

$$\pi_p^S(T) > \frac{\|T(x_1, \dots, x_m)\|_F}{\inf_{\omega \in \Omega} \|S(x_1, \dots, x_m)(\omega)\|_W},$$

contradicting the definition of  $\pi_p^S(T)$ .

By the Hahn-Banach separation theorem applied to the locally convex space  $C(\Omega)$  with the topology of uniform convergence, there exists a continuous linear functional  $\Lambda : C(\Omega) \rightarrow \mathbb{R}$  such that:

1.  $\Lambda(f) \geq 0$  for all  $f$  in the cone,
2.  $\Lambda(1) = 1$  (normalization).

*Step 2:* Since  $\Omega$  is compact Hausdorff, the Riesz representation theorem implies that  $\Lambda$  corresponds to a regular Borel probability measure  $\mu$  on  $\Omega$ :

$$\Lambda(f) = \int_{\Omega} f(\omega) d\mu(\omega) \quad \text{for all } f \in C(\Omega).$$

*Step 3: The key inequality.*

From  $\Lambda(f_{x_1, \dots, x_m}) \geq 0$ , we obtain for all  $(x_1, \dots, x_m)$  with  $\|x_j\| \leq 1$ :

$$\int_{\Omega} \|S(x_1, \dots, x_m)(\omega)\|_W^p d\mu(\omega) \geq \frac{\|T(x_1, \dots, x_m)\|_F^p}{(\pi_p^S(T))^p}.$$

By homogeneity, this extends to all  $(x_1, \dots, x_m)$ :

$$\left( \int_{\Omega} \|S(x_1, \dots, x_m)(\omega)\|_W^p d\mu(\omega) \right)^{1/p} \geq \frac{\|T(x_1, \dots, x_m)\|_F}{\pi_p^S(T)}.$$

*Step 4: Definition of  $\tilde{T}$  and  $J$ .*

Define  $\tilde{T} : E_1 \times \dots \times E_m \rightarrow L_p(\mu, W)$  by:

$$\tilde{T}(x_1, \dots, x_m)(\omega) = S(x_1, \dots, x_m)(\omega).$$

This is well-defined since  $S(x_1, \dots, x_m) \in \ell_{\infty}(\Omega, W) \subseteq L_p(\mu, W)$ , and  $\tilde{T}$  is  $m$ -linear. Define  $J$  initially on the range of  $\tilde{T}$  by:

$$J(\tilde{T}(x_1, \dots, x_m)) = T(x_1, \dots, x_m).$$

This is well-defined: if  $\tilde{T}(x_1, \dots, x_m) = \tilde{T}(y_1, \dots, y_m)$  in  $L_p(\mu, W)$ , then  $S(x_1, \dots, x_m) = S(y_1, \dots, y_m)$   $\mu$ -almost everywhere, so by the inequality above,  $\|T(x_1, \dots, x_m) - T(y_1, \dots, y_m)\|_F = 0$ .

Moreover,  $J$  is bounded on the range of  $\tilde{T}$ :

$$\begin{aligned} \|J(\tilde{T}(x_1, \dots, x_m))\|_F &= \|T(x_1, \dots, x_m)\|_F \\ &\leq \pi_p^S(T) \left( \int_{\Omega} \|S(x_1, \dots, x_m)(\omega)\|_W^p d\mu(\omega) \right)^{1/p} \\ &= \pi_p^S(T) \|\tilde{T}(x_1, \dots, x_m)\|_{L_p(\mu, W)}. \end{aligned}$$

Thus  $\|J\| \leq \pi_p^S(T)$  on  $\text{span}(\tilde{T}(E_1 \times \dots \times E_m))$ . Extend  $J$  by continuity to the closure of this subspace, and then by the Hahn-Banach theorem to all of  $L_p(\mu, W)$  without increasing the norm.

*Step 5: Commutativity.* By construction,  $J \circ \tilde{T} = T$ , so the diagram commutes.  $\square$

## 5. Applications to Specific Classes

**Definition 5.1 (Multilinear  $(\ell_p^s, \ell_p)$ -Summing Operators)** Let  $X_1, \dots, X_m, Y, Z$  be Banach spaces. A bounded  $m$ -linear operator  $\Phi : \mathcal{L}(X_1, \dots, X_m; Y) \rightarrow Z$  is multilinear  $(\ell_p^s, \ell_p)$ -summing if there exists  $C \geq 0$  such that for all  $n \in \mathbb{N}$  and  $(T_i)_{i=1}^n \subset \mathcal{L}(X_1, \dots, X_m; Y)$ :

$$\left( \sum_{i=1}^n \|\Phi(T_i)\|_Z^p \right)^{1/p} \leq C \sup_{\|x_1\| \leq 1, \dots, \|x_m\| \leq 1} \left( \sum_{i=1}^n \|T_i(x_1, \dots, x_m)\|_Y^p \right)^{1/p}.$$

The smallest such  $C$  is denoted  $\pi_{(\ell_p^s, \ell_p)}(\Phi)$ . We write  $\Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}X_1, \dots, X_m; Y), Z$  to denote the space of all  $(\ell_p^s, \ell_p)$ -summing operators from  $\mathcal{L}X_1, \dots, X_m; Y$  into  $Z$ .

**Theorem 5.1** *The spaces  $\Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X_1, \dots, X_m; Y), Z^*)$  and  $(\mathcal{L}(X_1, \dots, X_m; Y) \otimes Z, d_p^{op})^*$  are isometrically isomorphic.*

**Proof:** Apply Theorem 3.1 with:

$$\begin{aligned} E &= \mathcal{L}(X_1, \dots, X_m; Y), \\ \Omega &= B_{X_1} \times \dots \times B_{X_m} \text{ with the product topology,} \\ W &= Y, \\ F &= \ell_\infty(\Omega, Y), \\ S(T)(x_1, \dots, x_m) &= T(x_1, \dots, x_m). \end{aligned}$$

Then  $w_p^S((T_i)) = \sup_{\|x_j\| \leq 1} \left( \sum_{i=1}^n \|T_i(x_1, \dots, x_m)\|_Y^p \right)^{1/p}$ , and the result follows directly from Theorem 3.1.  $\square$

**Definition 5.2 (Multilinear op-dual  $(\ell_p^s, \ell_p)$ -Summing Operators)** *A bounded  $m$ -linear operator  $\Phi : \mathcal{L}(X_1, \dots, X_m; Y) \rightarrow Z$  is op-dual  $(\ell_p^s, \ell_p)$ -summing if there exists  $C \geq 0$  such that for all  $n \in \mathbb{N}$  and  $(T_i)_{i=1}^n \subset \mathcal{L}(X_1, \dots, X_m; Y)$ :*

$$\left( \sum_{i=1}^n \|\Phi(T_i)\|_Z^p \right)^{1/p} \leq C \sup_{\|y^*\| \leq 1} \left( \sum_{i=1}^n \|T_i^*(y^*)\|_{\mathcal{L}(X_1, \dots, X_{m-1}, X_m^*)}^p \right)^{1/p}.$$

Note that for a multilinear operator  $T$ , the adjoint  $T^*$  is defined by

$$T^*(y^*)(x_1, \dots, x_{m-1})(x_m) = y^*(T(x_1, \dots, x_m)),$$

The smallest such  $C$  is denoted  $\pi_{(\ell_p^s, \ell_p)}^{op-dual}(\Phi)$ . We write  $\Pi_{(\ell_p^s, \ell_p)}^{op-dual}(\mathcal{L}X_1, \dots, X_m; Y), Z)$  to denote the space of all op-dual  $(\ell_p^s, \ell_p)$ -summing operators from  $\mathcal{L}X_1, \dots, X_m; Y$  into  $Z$ .

**Theorem 5.2** *The spaces  $\Pi_{(\ell_p^s, \ell_p)}^{op-dual}(\mathcal{L}(X_1, \dots, X_m; Y), Z^*)$  and  $(\mathcal{L}(X_1, \dots, X_m; Y) \otimes Z, d_p^{op-dual})^*$  are isometrically isomorphic.*

**Proof:** Apply Theorem 3.1 with:

$$\begin{aligned} E &= \mathcal{L}(X_1, \dots, X_m; Y), \\ \Omega &= B_{Y^*} \text{ with the weak}^* \text{ topology,} \\ W &= \mathcal{L}(X_1, \dots, X_{m-1}; X_m^*), \\ F &= \ell_\infty(\Omega, W), \\ S(T)(y^*) &= T^*(y^*). \end{aligned}$$

Note that for a multilinear operator  $T$ , the adjoint  $T^*$  is defined by

$$T^*(y^*)(x_1, \dots, x_m) = y^*(T(x_1, \dots, x_m)),$$

which yields an element in  $\mathcal{L}(X_1, \dots, X_{m-1}; X_m^*)$ . The space  $\Omega$  is compact in the weak\* topology by the Banach-Alaoglu theorem. The result then follows from Theorem 3.1.  $\square$

## 6. Strict Inclusion Relations and Counterexamples

**Proposition 6.1 (Basic inclusions)** For  $1 \leq p < \infty$  and Banach spaces  $X_1, \dots, X_m, Y, Z$ ,

$$\Pi_p(\mathcal{L}(X_1, \dots, X_m; Y), Z) \subset \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X_1, \dots, X_m; Y), Z),$$

$$\Pi_p(\mathcal{L}(X_1, \dots, X_m; Y), Z) \subset \Pi_{(\ell_p^s, \ell_p)}^{op-dual}(\mathcal{L}(X_1, \dots, X_m; Y), Z).$$

**Proof:** Let  $\Phi \in \Pi_p(\mathcal{L}(X_1, \dots, X_m; Y), Z)$ . For any  $(T_i)_{i=1}^n \subset \mathcal{L}(X_1, \dots, X_m; Y)$ , and any  $(x_1, \dots, x_m) \in B_{X_1} \times \dots \times B_{X_m}$ ,  $y^* \in B_{Y^*}$ , the functional  $\phi(T) = y^*(T(x_1, \dots, x_m))$  satisfies  $\|\phi\| \leq 1$ . Thus:

$$w_p((T_i)) \geq \sup_{\substack{\|x_j\| \leq 1 \\ \|y^*\| \leq 1}} \left( \sum_{i=1}^n |y^*(T_i(x_1, \dots, x_m))|^p \right)^{1/p} \geq \sup_{\|x_j\| \leq 1} \left( \sum_{i=1}^n \|T_i(x_1, \dots, x_m)\|_Y^p \right)^{1/p}.$$

Hence  $\Phi$  is  $(\ell_p^s, \ell_p)$ -summing with  $\pi_{(\ell_p^s, \ell_p)}(\Phi) \leq \pi_p(\Phi)$ . The proof of the second inclusion is analogous.  $\square$

**Proposition 6.2 (Strictness of inclusions)** The inclusions in Proposition 6.1 are strict. Moreover:

1. There exist multilinear  $(\ell_p^s, \ell_p)$ -summing operators that are not  $p$ -summing.
2. There exist multilinear  $op$ -dual  $(\ell_p^s, \ell_p)$ -summing operators that are not  $p$ -summing.
3. The classes  $\Pi_{(\ell_p^s, \ell_p)}$  and  $\Pi_{(\ell_p^s, \ell_p)}^{op-dual}$  are incomparable.

**Proof: (1) Example:** Let  $X_1 = \dots = X_m = Y = Z = \ell_2$  and define  $\Phi : \mathcal{L}(\ell_2, \dots, \ell_2; \ell_2) \rightarrow \ell_2$  by  $\Phi(T) = T(e_1, \dots, e_1)$ , where  $e_1 = (1, 0, 0, \dots)$ .  $\Phi$  is  $(\ell_p^s, \ell_p)$ -summing because:

$$\left( \sum_{i=1}^n \|\Phi(T_i)\|^p \right)^{1/p} = \left( \sum_{i=1}^n \|T_i(e_1, \dots, e_1)\|^p \right)^{1/p} \leq \sup_{\|x_j\| \leq 1} \left( \sum_{i=1}^n \|T_i(x_1, \dots, x_m)\|^p \right)^{1/p}.$$

But  $\Phi$  is not  $p$ -summing. By the Dvoretzky-Rogers theorem,  $\ell_2$  contains a sequence  $(y_i)$  that is weakly  $p$ -summable but not absolutely  $p$ -summable. Define  $T_i(x_1, \dots, x_m) = \prod_{j=1}^m \langle x_j, e_i \rangle y_i$ , where  $(e_i)$  is an orthonormal basis. Then  $\Phi(T_i) = y_i$  is not absolutely  $p$ -summable, but for any functional  $\phi$  of the form  $\phi(T) = y^*(T(x_1, \dots, x_m))$  with  $\|y^*\| \leq 1$  and  $\|x_j\| \leq 1$ , we have

$$\sum_{i=1}^n |\phi(T_i)|^p \leq C \sum_{i=1}^n \|T_i\|^p = C \sum_{i=1}^n \|y_i\|^p < \infty$$

Thus  $\Phi$  cannot be  $p$ -summing.

**(2) Example:** Define  $\Psi : \mathcal{L}(\ell_2, \dots, \ell_2; \ell_2) \rightarrow \mathcal{L}(\ell_2, \dots, \ell_2; \ell_2)$  by  $\Psi(T) = T^*(\cdot)(e_1)$ . Similar arguments show that  $\Psi$  is  $op$ -dual  $(\ell_p^s, \ell_p)$ -summing but not  $p$ -summing.

**(3) Incomparability:** The operator  $\Phi$  from (1) is  $(\ell_p^s, \ell_p)$ -summing but not  $op$ -dual summing for appropriate choices. The operator  $\Psi$  from (2) is  $op$ -dual summing but not  $(\ell_p^s, \ell_p)$ -summing. Therefore, neither class contains the other.  $\square$

## Acknowledgments

The authors acknowledges with thanks the support of the General Direction of Scientific Research and Technological Development(DGRSDT), Algeria.

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