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Some Coupled Fixed Point Results Under Contraction Conditions using Linear and Rational Expression in Dislocated Quasi b-Metric Space

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ABSTRACT: This research paper aims to investigate a few coupled fixed point results in dislocated quasi b-metric spaces. We have established a coupled fixed point theorem for maps satisfying contraction condition using linear and rational expression in dislocated quasi b-metric space. To support our stated theorem and its corollaries, we have included an example.

Keywords: b-metric space, dislocated quasi b-metric space, fixed point, contraction map, coupled fixed point, Cauchy sequence.

Contents

1. Introduction

One of the main topics of nonlinear functional analysis is fixed point theory, which has some interesting implications to other areas of mathematics for demonstrating the existence and uniqueness of fixed points. One of the most crucial and powerful result in this field is Banach contraction principle which was established by S. Banach in 1922 [2]. Several routes have been explored for its generalization, such as, by utilizing various contraction conditions and relaxing abstract space requirements.
The concept of b-metric originated from the works of Bakhtin [4] and Bourbaki [3]. Later Czerwik [6] precisely defined b-metric space by providing an axiom that was weaker than the triangular inequality and extended the findings in b-metric space. Following Czerwik, numerous studies that included fixed point results in b-metric space were conducted. In 2016, Rahman and Sarwar [11] introduced the concept of dislocated quasi b-metric space as a generalization of b-metric space and established Banach's contraction principle and other well known fixed point theorems. In 1987, Guo and Lakshmikantham [7] first described the concept of coupled fixed point for partially ordered set. Later in 2006, Bhaskar and Lakshmikantham [5] established the existence and uniqueness of a coupled fixed point theorem in partially ordered metric space. Since then, several authors have proved coupled fixed point theorems in different generalized metric space (see [1,8,10,12,15]).
In the current research paper, we establish a coupled fixed point theorem for maps satisfying contraction condition using linear and rational expression and we prove the existence and uniqueness of a coupled fixed point in the context of dislocated quasi b-metric space.

2. Preliminaries

This section aims to provide an overview of the fundamental ideas and terminology utilised in this research study, which will facilitate comprehension of the following section. It includes example along with the definitions.

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Definition 2.1 (Bakhtin [4], Czrerwik [6]) Given that $s \geq 1$ is a real number and $\mathcal{K}$ is a nonempty set. Let $\ell : \mathcal{K} \times \mathcal{K} \to [0, \infty)$ be a function that meets the conditions listed below for all $u, v, w \in \mathcal{K}$:

(ℓ1) $\ell(u, u) = 0$;

(ℓ2) $\ell(u, v) = \ell(v, u) = 0$ implies that $u = v$;

(ℓ3) $\ell(u, v) = \ell(v, u)$;

(ℓ4) $\ell(u, v) \leq s[\ell(u, w) + \ell(w, v)]$.

A b-metric on $\mathcal{K}$ is one where ℓ satisfies the conditions from $\ell(1)$ to $\ell(4)$. It is referred to as a quasi b-metric on $\mathcal{K}$ if it meets the conditions $\ell(1), \ell(2)$ and $\ell(4)$. ℓ is referred to as a dislocated b-metric on $\mathcal{K}$ if it satisfies conditions $\ell(2)$ to $\ell(4)$ and ℓ is referred to as a dislocated quasi b-metric or in short(ℓ_q b-metric) on $\mathcal{K}$ if it satisfies conditions $\ell(2)$ and $\ell(4)$ and this notion was introduced by Rahman and Sarwar [11].

Remark 2.1

1. The definition of dislocated quasi b-metric space and the idea of dislocated quasi metric space are equivalent if $s = 1$. Every dislocated quasi metric is dislocated quasi b-metric, while the converse is false.
2. Every quasi b-metric is a dislocated quasi b-metric, while the converse is false.

Example 2.1 ([9]) Let $\mathcal{K} = \mathbb{R}$. Define $\ell : \mathcal{K} \times \mathcal{K} \to [0, \infty)$ as follows:

$$\ell(u, v) = |u - v|^2 + 3|u| + 2|v| \text{ for all } u, v \in \mathcal{K}.$$

Then $(\mathcal{K}, \ell)$ is a dislocated quasi b-metric space with the coefficient $s = 2$. $(\mathcal{K}, \ell)$ is not a b-metric space as $\ell(0, 1) \neq \ell(1, 0)$. It is obvious that $(\mathcal{K}, \ell)$ is not a dislocated quasi metric space.

Example 2.2 ([11]) Let $\mathcal{K} = \mathbb{R}$. Define $\ell : \mathcal{K} \times \mathcal{K} \to [0, \infty)$ as follows:

$$\ell(u, v) = |2u - v|^2 + |2u + v|^2 \text{ for all } u, v \in \mathcal{K}.$$

Then $(\mathcal{K}, \ell)$ is a dislocated quasi b-metric space with the coefficient $s = 2$.

Definition 2.2 ([11]) Let $\{u_j\}$ be a sequence in ℓ_q b-metric space $(\mathcal{K}, \ell)$ then

1. $\{u_j\}$ is known as convergent to $u \in \mathcal{K}$ if $\lim_{j \to +\infty} \ell(u_j, u) = 0 = \lim_{j \to +\infty} \ell(u, u_j)$.
   In this case u is called a ℓ_q b-limit of $\{u_j\}$ and is written as $\lim_{j \to +\infty} u_j = u$ or $u_j \to u$.

2. $\{u_j\}$ is known as Cauchy sequence in $(\mathcal{K}, \ell)$ if for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $i, j \geq n_0$, $\ell(u_i, u_j) < \epsilon \text{ or } \ell(u_j, u_i) < \epsilon$ that is, $\lim_{i,j \to +\infty} \ell(u_i, u_j) = 0 = \lim_{i,j \to +\infty} \ell(u_j, u_i)$.

3. A ℓ_q b-metric space $(\mathcal{K}, \ell)$ is called complete if every Cauchy sequence of points in $\mathcal{K}$ converges to a point in $\mathcal{K}$.

Lemma 2.1 *([11])* Limit of convergent sequence in ℓ_q b-metric space is unique.

Definition 2.3 *([2])* Let $(\mathcal{K}, \ell)$ be a complete metric space. Then $S : \mathcal{K} \to \mathcal{K}$ is called a contraction mapping if there exists a constant $\alpha \in [0, 1)$ such that

$$\ell(Su, Sv) \leq \alpha \ell(u, v) \, for \; all \, u, v \in \mathcal{K}.$$

Definition 2.4 Let $S : \mathcal{K} \to \mathcal{K}$ be a self map, where $\mathcal{K}$ is a nonempty set. We say that u is a fixed point of S if $Su = u$.

Theorem 2.1 *([11])* Let $(\mathcal{K}, \ell)$ be a complete dislocated quasi b-metric space with constant $s \geq 1$. Let $S : \mathcal{K} \to \mathcal{K}$ be a continuous contraction with contraction constant $\alpha \in [0, 1)$ such that $0 \leq s\alpha < 1$, then S has a unique fixed point.

Definition 2.5 *([5])* An element $(u, v) \in \mathcal{K} \times \mathcal{K}$ is called a coupled fixed point of the mapping $S : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ if $S(u, v) = u$ and $S(v, u) = v$.

3. Main Results

Theorem 3.1 Let $(\mathcal{K}, \ell)$ be a complete dislocated quasi b-metric space with coefficient $s \geq 1$. Let $S : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be a continuous mapping satisfying the following condition:

$$\begin{aligned}\ell(S(u,v),S(p,q)) \leq{}& \eta_1[\ell(u,p)+\ell(v,q)] + \eta_2[\ell(u,S(u,v))+\ell(p,S(p,q))]\\&+\frac{\eta_3}{s}[\ell(u,S(p,q))+\ell(p,S(u,v))]\\&+\frac{\eta_4}{s}\left[\frac{\ell(u,p)\ell(u,S(p,q))}{\ell(u,p)+\ell(p,S(p,q))}\right]+\frac{\eta_5}{s}\left[\frac{\ell(u,S(p,q))\ell(p,S(p,q))}{\ell(u,p)+\ell(p,S(p,q))}\right]\end{aligned} \tag{3.1}$$

for all $u, v, p, q \in \mathcal{K}$ and $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ are non-negative constants with $2s\eta_1 + (s+1)\eta_2 + (2s+2)\eta_3 + 2s\eta_4 + 2\eta_5 < 1$. Then S has a unique coupled fixed point in $\mathcal{K} \times \mathcal{K}$.

Proof: Let u_0 and v_0 be arbitrary elements in $\mathcal{K}$, let us define the iterative sequences $\{u_n\}$ and $\{v_n\}$ in $\mathcal{K}$ as follows:

$u_{n+1} = S(u_n, v_n)$ and $v_{n+1} = S(v_n, u_n)$ for $n \in \mathbb{N}$.

Consider

$$\ell(u_n, u_{n+1}) = \ell(S(u_{n-1}, v_{n-1}), S(u_n, v_n))$$

From (3.1) we have

$$\begin{aligned}\ell(u_n,u_{n+1}) \leq{}& \eta_1[\ell(u_{n-1},u_n)+\ell(v_{n-1},v_n)] + \eta_2[\ell(u_{n-1},S(u_{n-1},v_{n-1})+\ell(u_n,S(u_n,v_n)]\\&+\frac{\eta_3}{s}[\ell(u_{n-1},S(u_n,v_n))+\ell(u_n,S(u_{n-1},v_{n-1}))]\\&+\frac{\eta_4}{s}\left[\frac{\ell(u_{n-1},u_n)\ell(u_{n-1},S(u_n,v_n))}{\ell(u_{n-1},u_n)+\ell(u_n,S(u_n,v_n))}\right]+\frac{\eta_5}{s}\left[\frac{\ell(u_{n-1},S(u_n,v_n)\ell(u_n,S(u_n,v_n))}{\ell(u_{n-1},u_n)+\ell(u_n,S(u_n,v_n))}\right]\end{aligned}$$

Using the definition of the sequences $\{u_n\}$ and $\{v_n\}$ we have

$$\begin{aligned}\ell(u_n,u_{n+1}) \leq{}& \eta_1[\ell(u_{n-1},u_n)+\ell(v_{n-1},v_n)] + \eta_2[\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1}]\\&+\frac{\eta_3}{s}[\ell(u_{n-1},u_{n+1})+\ell(u_n,u_n)]\\&+\frac{\eta_4}{s}\left[\frac{\ell(u_{n-1},u_n)\ell(u_{n-1},u_{n+1})}{\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1})}\right]+\frac{\eta_5}{s}\left[\frac{\ell(u_{n-1},u_{n+1})\ell(u_n,u_{n+1})}{\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1}))}\right]\end{aligned}$$

Simplifying we get

$$\begin{aligned}\ell(u_n,u_{n+1}) &\le \eta_1[\ell(u_{n-1},u_n)+\ell(v_{n-1},v_n)]+\eta_2[\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1}]\\ &+\frac{\eta_3}{s}\left[s[\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1})]+s[\ell(u_n,u_{n+1})+\ell(u_{n-1},u_n)]\right]\\ &+\frac{\eta_4}{s}s\left[\frac{\ell(u_{n-1},u_n)[\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1}))}{\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1})}\right]\\ &+\frac{\eta_5}{s}s\left[\frac{[\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1})]\ell(u_n,u_{n+1})}{\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1}))}\right]\\ &\le \eta_1[\ell(u_{n-1},u_n)+\ell(v_{n-1},v_n)]+\eta_2[\ell(u_{n-1},u_n)+\ell(u_n,u_{n+1})]\\ &+\eta_3[2\ell(u_{n-1},u_n)+2\ell(u_n,u_{n+1})]+\eta_4\ell(u_{n-1},u_n)+\eta_5\ell(u_n,u_{n+1})\end{aligned}$$

Simplification gives

$$\ell(u_n,u_{n+1}) \le \frac{\eta_1+\eta_2+2\eta_3+\eta_4}{1-(\eta_2+2\eta_3+\eta_5)}\ell(u_{n-1},u_n)+\frac{\eta_1}{1-(\eta_2+2\eta_3+\eta_5)}\ell(v_{n-1},v_n) \tag{3.2}$$

Similarly we can show that

$$\ell(v_n,v_{n+1}) \le \frac{\eta_1+\eta_2+2\eta_3+\eta_4}{1-(\eta_2+2\eta_3+\eta_5)}\ell(v_{n-1},v_n)+\frac{\eta_1}{1-(\eta_2+2\eta_3+\eta_5)}\ell(u_{n-1},u_n) \tag{3.3}$$

Adding (3) and (3.3) we get

$$[\ell(u_n,u_{n+1})+\ell(v_n,v_{n+1})] \le \frac{2\eta_1+\eta_2+2\eta_3+\eta_4}{1-(\eta_2+2\eta_3+\eta_5)}[\ell(u_{n-1},u_n)+\ell(v_{n-1},v_n)]$$

Let $\zeta=\frac{2\eta_1+\eta_2+2\eta_3+\eta_4}{1-(\eta_2+2\eta_3+\eta_5)}$

Then $s\zeta<1$, since $2s\eta_1+(s+1)\eta_2+(2s+2)\eta_3+s\eta_4+2\eta_5<1$.

Therefore,

$$[\ell(u_n,u_{n+1})+\ell(v_n,v_{n+1})] \le \zeta[\ell(u_{n-1},u_n)+\ell(v_{n-1},v_n)]$$

Also

$$[\ell(u_n,u_{n+1})+\ell(v_n,v_{n+1})] \le \zeta^2[\ell(u_{n-2},u_{n-1})+\ell(v_{n-2},v_{n-1})]$$

Proceeding like this, we get

$$\beta_{n+1}=[\ell(u_n,u_{n+1})+\ell(v_n,v_{n+1}] \le \zeta^n[\ell(u_0,u_1)+\ell(v_0,v_1)]=\zeta^n\beta_1$$

Using the b-triangular inequality we have

$$\begin{aligned}\ell(u_n,u_m)+\ell(v_n,v_m) &\le s(\ell(u_n,u_{n+1})+\ell(u_{n+1},u_m)+\ell(v_n,v_{n+1})+\ell(v_{n+1},v_m))\\ &\le s(\ell(u_n,u_{n+1})+\ell(v_n,v_{n+1}))+s^2(\ell(u_{n+1},u_{n+2})+\ell(v_{n+1},v_{n+2}))\\ &+s^2(\ell(u_{n+2},u_m)+\ell(v_{n+2},v_m))\\ &\vdots\\ &\le s\beta_{n+1}+s^2\beta_{n+2}+\ldots+s^{m-n}\beta_m\\ &\le (s\zeta^n+s^2\zeta^{n+1}+\ldots+s^{m-n}\zeta^{m-1})\beta_1\\ &= s\zeta^n(1+s\zeta+\ldots+(s\zeta)^{m-n-1})\beta_1\\ &\le s\zeta^n\frac{1}{1-s\zeta}\beta_1\end{aligned}$$

Since $\zeta < 1$, we get $lim_{n,m\to\infty}[\ell(u_n,u_m)+\ell(v_n,v_m)] = 0$

$\Rightarrow \lim_{n,m\to\infty}\ell(u_n,u_m) = 0$ and $\lim_{n,m\to\infty}\ell(v_n,v_m) = 0$.

Similarly we can show that $\lim_{n,m\to\infty}\ell(u_m,u_n) = 0$ and $\lim_{n,m\to\infty}\ell(v_m,v_n) = 0$. Thus $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in complete dislocated quasi b-metric space $\mathcal{K}$. So there exist $a, b \in \mathcal{K}$ such that $\lim_{n\to\infty} u_n = a$ and $lim_{n\to\infty}v_n = b$. Since S is continuous and $S(u_n,v_n) = u_{n+1}$, we have

$$lim_{n\to\infty}S(u_n,v_n) = lim_{n\to\infty}u_{n+1}$$

$$S(lim_{n\to\infty}u_n, lim_{n\to\infty}v_n) = lim_{n\to\infty}u_{n+1}$$

$$\Rightarrow S(a,b) = a$$

.
Also as $S(v_n,u_n) = v_{n+1}$, we can show that $\ell(b,a) = b$. Thus $(a,b) \in \mathcal{K}\times\mathcal{K}$ is the coupled fixed point of S in $\mathcal{K}$.

Uniqueness: Let (a,b) and (a_1,b_1) be two distinct coupled fixed points of S in $\mathcal{K}\times\mathcal{K}$. From (3.1) we have

$$\begin{aligned}
\ell(a,a) &= \ell(S(a,b),S(a,b))\\
&\leq \eta_1[\ell(a,a)+\ell(b,b)] + \eta_2[\ell(a,S(a,b))+\ell(a,S(a,b))]\\
&+ \frac{\eta_3}{s}[\ell(a,S(a,b))+\ell(a,S(a,b))] + \frac{\eta_4}{s}\left[\frac{\ell(a,a)\ell(a,S(a,b))}{\ell(a,a)+\ell(a,S(a,b))}\right]\\
&+ \frac{\eta_5}{s}\left[\frac{\ell(a,S(a,b))\ell(a,S(a,b))}{\ell(a,a)+\ell(a,S(a,b))}\right]\\
&= \eta_1[\ell(a,a)+\ell(b,b)] + \eta_2[\ell(a,a)+\ell(a,a)] + \frac{\eta_3}{s}[\ell(a,a)+\ell(a,a)]\\
&+ \frac{\eta_4}{s}\left[\frac{\ell(a,a)\ell(a,a)}{\ell(a,a)+\ell(a,a)}\right] + \frac{\eta_5}{s}\left[\frac{\ell(a,a)\ell(a,a)}{\ell(a,a)+\ell(a,a)}\right]\\
&\leq \eta_1[\ell(a,a)+\ell(b,b)] + 2\eta_2\ell(a,a) + 2\frac{\eta_3}{s}\ell(a,a) + \frac{\eta_4}{s}\ell(a,a) + \frac{\eta_5}{s}\ell(a,a)\\
&\leq \eta_1[\ell(a,a)+\ell(b,b)] + 2\eta_2\ell(a,a) + (2\eta_3+\eta_4+\eta_5)[\ell(a,a)+\ell(b,b)]
\end{aligned}$$

$$\ell(a,a) \leq (\eta_1+2\eta_3+\eta_4+\eta_5)[\ell(a,a)+\ell(b,b)] + 2\eta_2\ell(a,a) \tag{3.4}$$

Similarly we can show that

$$\ell(b,b) \leq (\eta_1+2\eta_3+\eta_4+\eta_5)[\ell(a,a)+\ell(b,b)] + 2\eta_2\ell(b,b) \tag{3.5}$$

Adding (3.4) and (3.5) we have

$$[\ell(a,a)+\ell(b,b)] \leq (2\eta_1+4\eta_3+2\eta_4+2\eta_5+2\eta_2)[\ell(a,a)+\ell(b,b)]$$

Since $2\eta_1+4\eta_3+2\eta_4+2\eta_5+2\eta_2 < 1$, so above inequality is possible only if $[\ell(a,a)+\ell(b,b)] = 0$ implies

$$\ell(a,a) = \ell(b,b) = 0 \tag{3.6}$$

Following similar steps we get

$$\ell(a_1,a_1) = \ell(b_1,b_1) = 0 \tag{3.7}$$

Now consider

$$\begin{aligned}
\ell(a,a_1) &= \ell(S(a,b),S(a_1,b_1)) \\
&\leq \eta_1[\ell(a,a_1)+\ell(b,b_1)]+\eta_2[\ell(a,S(a,b))+\ell(a_1,S(a_1,b_1)] \\
&+\frac{\eta_3}{s}[\ell(a,S(a_1,b_1))+\ell(a_1,S(a,b))]+\frac{\eta_4}{s}\left[\frac{\ell(a,a_1)\ell(a,S(a_1,b_1))}{\ell(a,a_1)+\ell(a_1,S(a_1,b_1))}\right] \\
&+\frac{\eta_5}{s}\left[\frac{\ell(a,S(a_1,b_1))\ell(a_1,S(a_1,b_1))}{\ell(a,a_1)+\ell(a_1,S(a_1,b_1))}\right] \\
&= \eta_1[\ell(a,a_1)+\ell(b,b_1)]+\eta_2[\ell(a,a)+\ell(a_1,a_1)] \\
&+\frac{\eta_3}{s}[\ell(a,a_1)+\ell(a_1,a)]+\frac{eta_4}{s}\left[\frac{\ell(a,a_1)\ell(a,a_1)}{\ell(a,a_1)+\ell(a_1,a_1)}\right] \\
&+\frac{\eta_5}{s}\left[\frac{\ell(a,a_1)\ell(a_1,a_1)}{\ell(a,a_1)+\ell(a_1,a_1)}\right]
\end{aligned}$$

Using (3.6) and (3.7) we have

$$\begin{aligned}
\ell(a,a_1) &\leq \eta_1[\ell(a,a_1)+\ell(b,b_1)]+\frac{\eta_3}{s}[\ell(a,a_1)+\ell(a_1,a)]+\frac{\eta_4}{s}\ell(a,a_1) \\
&\leq \eta_1[\ell(a,a_1)+\ell(b,b_1)]+\frac{\eta_3}{s}[s(\ell(a,a_1)+\ell(a_1,a)+\ell(b,b_1))] \\
&+\frac{\eta_4}{s}[s(\ell(a,a_1)+\ell(b,b_1))]
\end{aligned}$$

$$(1-(\eta_1+\eta_3+\eta_4))\eta(a,a_1)\leq \eta_3\ell(a_1,a)+(\eta_1+\eta_3+\eta_4)\ell(b,b_1) \tag{3.8}$$

On similar steps we can get

$$(1-(\eta_1+\eta_3+\eta_4))\ell(b,b_1)\leq \eta_3\ell(b_1,b)+(\eta_1+\eta_3+\eta_4)\ell(a,a_1) \tag{3.9}$$

Adding (3.8) and (3.9) and then simplifying we get

$$[\ell(a,a_1)+\ell(b,b_1)]\leq \xi[\ell(a_1,a)+\ell(b_1,b)] \tag{3.10}$$

where $\xi=\frac{\eta_3}{1-(2\eta_1+2\eta_3+2\eta_4)}$

Similarly we can get

$$[\ell(a_1,a)+\ell(b_1,b)]\leq \xi[\ell(a,a_1)+\ell(b,b_1)] \tag{3.11}$$

Adding (3.10) and (3.11), we get

$$[\ell(a,a_1)+\ell(b,b_1)+\ell(a_1,a)+\ell(b_1,b)]\leq \xi[\ell(a,a_1)+\ell(b,b_1)+\ell(a_1,a)+\ell(b_1,b)]$$

Since $\xi<1$ so the above inequality holds only if

$$[\ell(a,a_1)+\ell(b,b_1)+\ell(a_1,a)+\ell(b_1,b)]=0$$

which implies that $\ell(a,a_1)=0,\ell(b,b_1)=0,\ell(a_1,a)=0$ and $\ell(b_1,b)=0$.
It follows that $a=a_1$ and $b=b_1$ that is, $(a,b)=(a_1,b_1)$ which contradicts our assumption.
Therefore, (a,b) is a unique coupled fixed point of S in $\mathcal{K}\times\mathcal{K}$. □

Corollary 3.1 Let $(\mathcal{K},\ell)$ be a complete dislocated quasi metric space and $S:\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ be a continuous mapping satisfying the following condition:

$$\begin{aligned}\ell(S(u,v),S(p,q)) &\leq \eta_1[\ell(u,p)+\ell(v,q)]+\eta_2[\ell(u,S(u,v))+\ell(p,S(p,q))]\\ &\quad+\eta_3[\ell(u,S(p,q))+\ell(p,S(u,v))]+\eta_4\left[\frac{\ell(u,p)\ell(u,S(p,q))}{\ell(u,p)+\ell(p,S(p,q))}\right]\\ &\quad+\eta_5\left[\frac{\ell(u,S(p,q))\ell(p,S(p,q))}{\ell(u,p)+\ell(p,S(p,q))}\right]\end{aligned}$$

for all $u,v,p,q\in\mathcal{K}$ and $\eta_1,\eta_2,\eta_3,\eta_4,\eta_5$ are non-negative constants with $2\eta_1+2\eta_2+4\eta_3+2\eta_4+2\eta_5<1$. Then S has a unique coupled fixed point in $\mathcal{K}\times\mathcal{K}$.

Proof: Let $s=1$ in Theorem 3.1, we get corollary 3.1. Thus our established theorem is a generalization of corollary 3.1. □

Corollary 3.2 Let $(\mathcal{K},\ell)$ be a complete dislocated quasi b-metric space with coefficient $s\geq 1$. $S:\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ be a continuous mapping satisfying the following contraction condition:

$$\begin{aligned}\ell(S(u,v),S(p,q)) &\leq \eta_1[\ell(u,p)+\ell(v,q)]+\eta_2[\ell(u,S(u,v))+\ell(p,S(p,q))]\\ &\quad+\frac{\eta_3}{s}[\ell(u,S(p,q))+\ell(p,S(u,v))]+\frac{\eta_4}{s}\left[\frac{\ell(u,p)\ell(u,S(p,q))}{\ell(u,p)+\ell(p,S(p,q))}\right]\end{aligned}$$

for all $u,v,p,q\in\mathcal{K}$ and $\eta_1,\eta_2,\eta_3,\eta_4$ are non-negative constants with $2s\eta_1+(s+1)\eta_2+(2s+2)\eta_3+2s\eta_4<1$. Then S has a unique coupled fixed point in $\mathcal{K}\times\mathcal{K}$.

Corollary 3.3 Let $(\mathcal{K},\ell)$ be a complete dislocated quasi b-metric space with coefficient $s\geq 1$. $S:\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ be a continuous mapping satisfying the following contraction condition:

$$\begin{aligned}\ell(S(u,v),S(p,q)) &\leq \eta_1[\ell(u,p)+\ell(v,q)]+\eta_2[\ell(u,S(u,v))+\ell(p,S(p,q))]\\ &\quad+\frac{\eta_3}{s}[\ell(u,S(p,q))+\ell(p,S(u,v))]\end{aligned}$$

for all $u,v,p,q\in\mathcal{K}$ and η_1,η_2,η_3 are non-negative constants with $2s\eta_1+(s+1)\eta_2+(2s+2)\eta_3<1$. Then S has a unique coupled fixed point in $\mathcal{K}\times\mathcal{K}$.

Corollary 3.4 Let $(\mathcal{K},\ell)$ be a complete dislocated quasi b-metric space with coefficient $s\geq 1$. $S:\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ be a continuous mapping satisfying the following contraction condition:

$$\ell(S(u,v),S(p,q)) \leq \eta_1[\ell(u,p)+\ell(v,q)]+\eta_2[\ell(u,S(u,v))+\ell(p,S(p,q))]$$

for all $u,v,p,q\in\mathcal{K}$ and η_1,η_2 are non-negative constants with $2s\eta_1+(s+1)\eta_2<1$. Then S has a unique coupled fixed point in $\mathcal{K}\times\mathcal{K}$.

Example 3.1 Let $\mathcal{K}=[0,1]$. Define $\ell:\mathcal{K}\times\mathcal{K}\to[0,\infty)$ by $\ell(u,v)=|u-v|^2+|u|$ for all $u,v\in\mathcal{K}$. Then $(\mathcal{K},\ell)$ is a complete dislocated quasi b-metric space with the coefficient $s=2$. Since $\ell(1,1)\neq 0$, hence $(\mathcal{K},\ell)$ is not a quasi b-metric space. Also it is not a dislocated b-metric space since $\ell(1,2)\neq\ell(2,1)$. Define a continuous mapping $S:\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ by

$$S(u,v)=\frac{1}{6}uv$$

for all $u, u \in \mathcal{K}$.
Since $|uv - pq|^2 \leq |u-p|^2 + |v-q|^2, |uv| \leq |u| + |v|$ holds for all $u, v, p, q \in \mathcal{K}$. Then

$$\begin{aligned}\ell(S(u,v), S(p,q)) &= \ell(\frac{1}{6}uv, \frac{1}{6}pq) \\ &= \frac{1}{36}|uv - pq|^2 + \frac{1}{6}|uv| \\ &\leq \frac{1}{36}[|u-p|^2 + |v-q|^2] + \frac{1}{6}[|u| + |v|] \\ &\leq \frac{1}{6}[|u-p|^2 + |u| + |v-q|^2 + |v|] \\ &= \frac{1}{6}[\ell(u,p) + \ell(v,q)]\end{aligned}$$

So for $\eta_1 = \frac{1}{6}$ and $\eta_2 = \eta_3 = \eta_4 = \eta_5 = 0$ all the conditions of Theorem 3.1 are satisfied and $(0,0) \in \mathcal{K} \times \mathcal{K}$ is the unique coupled fixed point of S in $\mathcal{K} \times \mathcal{K}$.

4. Conclusions

We can show that the analogous findings in the setting of dislocated quasi metric space may be inferred from our results by assuming $s = 1$. Any interested researcher can conduct their thesis work on this subject by looking for the presence and uniqueness of fixed points for maps meeting various contraction conditions in dislocated quasi extended b-metric space or any other generalisation of metric space.

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