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Bivariate L-Moments: A Hybrid Estimation Approach for Dependence and Marginal Parameters with Applications to Bivariate Pareto Models

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ABSTRACT: Multivariate extensions of L-moments — robust linear functions of order statistics — remain relatively under explored despite their univariate success. We discuses in this paper a hybrid methodology for bivariate parameter estimation by combining bivariate L-moments with univariate L-moment techniques. We develop non-parametric estimators for bivariate L-comoment coefficients and a sequential three-step estimation procedure that are specifically designed to overcome limitations of the Inference Function for Margins (IFM) method when parameters are shared across marginal and joint distributions. We illustrate our method with bivariate Pareto distributions, which are very common in heavy-tailed phenomena modelling. By simulation study we show that the proposed estimator perform better in terms of bias and root mean squared error (RMSE).

Keywords: Multivariate L-moments, dependence modelling, parameter estimation, bivariate Pareto distribution, Heavy-tailed distributions, Copula models.

Contents

1. Introduction

In statistical modelling, the analysis of multivariate data often requires estimating both marginal distribution and the dependency structure, but using classical methods such as maximum likelihood estimation (MLE) for such analysis might not be efficient, since the accuracy of the estimates will be questionable, particularly with the simultaneous presence of heavy tails. The IFM method [24] emerged as an efficient alternative by separating the estimation of marginal and dependence parameters. However,

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the estimation of the parameters that are simultaneously present in the marginal as well as the joint distributions for the IFM method creates identification problems.

L-moments are linear functions of order statistics introduced by [22] and formalized by [9], provide linear functions of order statistics that offer greater robustness to outliers compared to conventional moments. Their theoretical aspects have been widely examined [10,11], and they have been successfully applied in hydrology [15,16,25], finance [14], and signal processing [13]. The multivariate extension of L-moments given by [21] opened new avenues in dependence modelling, and further applications followed in statistical hydrology [4,5] and further methodological developments in copula-based estimation [2].

Despite these theoretical advances, practical estimation procedures using multivariate L-moments remain underdeveloped, especially for models with complex parameter interdependencies between marginal and dependence structures. This paper bridges this identified research gap through providing a comprehensive framework for bivariate LM estimation, which explicitly accommodates parameter overlap between the marginal and joint distributions. This study presents a new hybrid estimation strategy that employs bivariate LM estimates for dependence parameter estimation and univariate LM estimates for marginal parameter estimation. We have conducted extensive simulations on bivariate Pareto distributions, demonstrating the comparative superiority of this new approach using bias and RMSE measures, and accompanying guidelines and algorithms for easy application.

The remainder of this paper is organized as follows: Section 2 presents the theoretical framework of L-moments, including bivariate extensions and their advantages over traditional dependence measures, and develops nonparametric estimation procedures for L-comoment coefficients. Section 3 introduces our hybrid estimation methodology for bivariate distributions with an explicit algorithm. Section 4 applies this framework to bivariate Pareto distributions. Section 5 presents comprehensive simulation results comparing our approach with alternative methods. The paper concludes with discussion and limitations in Section 6. Technical proofs are provided in the Appendix.

2. Theoretical Framework of L-moments

Let X and Y be two random variables (rv) with finite means, margins F_1 and F_2 and joint distribution function H. By analogy with the covariance representation for L-moments

$$\lambda_k = \begin{cases} \mathbb{E}\left(Y\right) & k=1, \\ Cov\left(Y, P_{k-1}\left(F\left(Y\right)\right)\right) & k \geq 2, \end{cases} \tag{2.1}$$

where $P_k(u) = \sum_{\ell=0}^{k} p_{k,\ell} u^\ell$, with

$$p_{k,\ell} = (-1)^{k-\ell} \frac{(k+\ell)!}{(\ell!)^2 (k-\ell)!},$$

are the shifted Legendre polynomials. These polynomials are orthogonal on $[0,1]$ with $\int_0^1 P_j(u) P_k(u)\, du = \delta_{jk}/(2k+1)$, where δ_{jk} is the Kronecker delta.

The kth L-moment λ_k is defined as a linear combination of expected order statistics:

$$\lambda_k = k^{-1} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{k-1}{\ell} \mathbb{E}[Y_{k-\ell:k}], \quad k = 1, 2, \ldots,$$

where $Y_{1:k} \leq \cdots \leq Y_{k:k}$ denote the order statistics of a sample of size k. An equivalent formulation using the quantile function $F^{-1}(u) = \inf\{y : F(y) \geq u\}$ is given by:

$$\lambda_k = \int_0^1 F^{-1}(u) P_{k-1}(u)\, du, \tag{2.2}$$

Serfling and Xiao [21] introduced the kth L-comoment of X with respect to Y:

$$\lambda_{k[12]} = \mathrm{Cov}\big(X, P_{k-1}(F_2(Y))\big), \quad k \geq 2, \tag{2.3}$$

The connection with classical measures is explicitly revealed through the relationship between the second L-comoment coefficient $\delta_{2[12]}$ and Spearman's rank correlation coefficient ρ_S, for more details, see [3]:

$$\delta_{k[12]} = \mathrm{Cov}\big(F_1(X), P_{k-1}(F_2(Y))\big), \quad k \geq 2. \tag{2.4}$$

The latter admits the integral representation:

$$\delta_{k[12]} = \int_{[0,1]^2} \left[H(F_1^{-1}(u_1), F_2^{-1}(u_2)) - u_1 u_2\right] du_1\, dP_{k-1}(u_2), \tag{2.5}$$

which facilitates theoretical analysis.

Proposition 2.1 *The kth L-comoment coefficients $\lambda_{k[12]}$, $\lambda_{k[21]}$ and δ_k may be rewritten, respectively, for each $k = 2, 3, ...,$ into:*

$$\lambda_{k[12]} = \int_{\mathbb{I}^2} \left(H\left(F_1^{-1}(u_1), F_2^{-1}(u_2)\right) - u_1 u_2\right) dF_1^{-1}(u_1)\, dP_{k-1}(u_2), \tag{2.6}$$

$$\lambda_{k[21]} = \int_{\mathbb{I}^2} \left(H\left(F_1^{-1}(u_1), F_2^{-1}(u_2)\right) - u_1 u_2\right) dP_{k-1}(u_1)\, dF_2^{-1}(u_2), \tag{2.7}$$

and

$$\delta_{k[12]} = \int_{\mathbb{I}^2} \left(H\left(F_1^{-1}(u_1), F_2^{-1}(u_2)\right) - u_1 u_2\right) du_1 dP_{k-1}(u_2). \tag{2.8}$$

2.1. Advantages Over Traditional Dependence Measures

While bivariate L-moments share conceptual similarities with traditional dependence measures such as Spearman's ρ and Kendall's τ, they offer several distinct advantages: they capture higher-order dependence beyond linear correlation through higher-order polynomial terms; they exhibit greater robustness to outliers as linear functions of order statistics rather than moment-based measures; and they provide enhanced interpretability, with each L-comoment corresponding to a specific aspect of the dependence structure.

The connection with classical measures is explicitly revealed through the relationship between the second L-comoment coefficient $\delta_{2[12]}$ and Spearman's rank correlation coefficient ρ_S:

$$\rho_S = 12 \int_{[0,1]^2} [H(u_1, u_2) - u_1 u_2]\, du_1 du_2 = 6\delta_{2[12]}, \tag{2.9}$$

demonstrating how $\delta_{2[12]}$ captures linear-in-rank dependence while higher-order L-comoments encode more complex dependence patterns.

2.2. Sample Estimation

Given a bivariate sample $\{(X_i, Y_i)\}_{i=1}^n$, let $R_i^X = n\hat{F}_1(X_i)$ and $R_i^Y = n\hat{F}_2(Y_i)$ denote the ranks, where $\hat{F}_1$ and $\hat{F}_2$ are the empirical distribution functions. The sample L-comoment coefficients are estimated by:

$$\widehat{\lambda}_{k[12]} = \frac{1}{n} \sum_{i=1}^{n} \left[P_{k-1}(R_i^Y) - \overline{P}_k^Y\right]\big(X_i - \overline{X}\big), \tag{2.10}$$

and

$$\widehat{\delta}_{k[12]} = \frac{1}{n} \sum_{i=1}^{n} \left[P_{k-1}(R_i^Y) - \overline{P}_k^Y\right]\big(R_i^X - \overline{R}^X\big), \tag{2.11}$$

where

$$\overline{P}_k^Y = \frac{1}{n} \sum_{i=1}^{n} P_{k-1}(R_i^Y) \text{ and } \overline{R}^X = \frac{1}{n} \sum_{i=1}^{n} R_i^X = \frac{n+1}{2}.$$

The first few L-comoments have particularly interpretable forms:

$$\begin{aligned}\lambda_{2[12]} &= 2\,\mathrm{Cov}\big(X, F_2(Y)\big),\\ \lambda_{3[12]} &= -6\,\mathrm{Cov}\big(X, F_2(Y)[1-F_2(Y)]\big),\\ \lambda_{4[12]} &= \mathrm{Cov}\big(X, 20F_2^3(Y) - 30F_2^2(Y) + 12F_2(Y) - 1\big),\end{aligned}$$

with analogous expressions for $\lambda_{k[21]}$ and $\delta_{k[12]}$. These reveal how $\lambda_{2[12]}$ measures linear-in-rank dependence while higher-order L-comoments capture non-linear and tail dependence features, forming the foundation for the hybrid estimation approach developed in the following section.

3. Hybrid Estimation Methodology for Bivariate Distributions

3.1. Problem Formulation

Consider a bivariate random vector (X, Y) with joint distribution function $H(x, y; \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_d)$ comprises marginal parameters $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ for X and Y respectively, and dependence parameters $\boldsymbol{\theta}_d$. We assume the parameter space Θ satisfies:

Assumption 1 *The parameter vector can be partitioned as*

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_c, \boldsymbol{\theta}_1^*, \boldsymbol{\theta}_2^*)$$

where:

- $\boldsymbol{\theta}_c$ *are common parameters appearing in both marginal and joint distributions.*
- $\boldsymbol{\theta}_1^*$ *and* $\boldsymbol{\theta}_2^*$ *are margin-specific parameters.*
- $\boldsymbol{\theta}_c \cap (\boldsymbol{\theta}_1^* \cup \boldsymbol{\theta}_2^*) = \emptyset$.

3.2. Three-Step Hybrid Estimation Algorithm

Algorithm 1 *Hybrid L-moment Estimation for Bivariate Distributions*
Input: *Bivariate sample* $\{(X_i, Y_i)\}_{i=1}^n$.
Output: *Parameter estimates* $\hat{\boldsymbol{\theta}}$.

Step 1. *Compute sample L-comoment coefficients* $\widehat{\delta}_{k[12]}$, $k = 2, \ldots, d$.

Step 2. *Estimate common parameters* $\boldsymbol{\theta}_c$ *by solving*

$$\widehat{\delta}_{k[12]} = \delta_{k[12]}(\boldsymbol{\theta}_c), \quad k = 2, \ldots, \dim(\boldsymbol{\theta}_c) + 1.$$

Step 3. *With* $\hat{\boldsymbol{\theta}}_c$ *fixed, estimate* $\boldsymbol{\theta}_1^*$ *using univariate L-moments of* X*:*

$$\hat{\lambda}_k^{(1)} = \lambda_k^{(1)}(\hat{\boldsymbol{\theta}}_c, \boldsymbol{\theta}_1^*), \quad k = 1, 2.$$

Step 4. *With* $\hat{\boldsymbol{\theta}}_c$ *fixed, estimate* $\boldsymbol{\theta}_2^*$ *using univariate L-moments of* Y*:*

$$\hat{\lambda}_k^{(2)} = \lambda_k^{(2)}(\hat{\boldsymbol{\theta}}_c, \boldsymbol{\theta}_2^*), \quad k = 1, 2.$$

Return:

$$\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_c, \hat{\boldsymbol{\theta}}_1^*, \hat{\boldsymbol{\theta}}_2^*).$$

4. Application: Bivariate Pareto Distribution

4.1. Model Specification

Consider a bivariate Pareto random vector (X, Y) with joint survival function

$$\overline{H}(x_1, x_2) = \left(1 + \frac{x_1 - a_1}{b_1} + \frac{x_2 - a_2}{b_2}\right)^{-\gamma}, \quad x_j \geq a_j,\ j = 1, 2,$$

where $a_j \in \mathbb{R}$, $b_j > 0$, $j = 1, 2$, and $\gamma > 1$. The marginal survival functions are

$$\overline{F}_j(x_j) = (1 + (x_j - a_j)/b_j)^{-\gamma}, \quad j = 1, 2,$$

and the joint distribution function can be expressed as

$$H(x_1, x_2) = \left[(1 - F_1(x_1))^{-1/\gamma} + (1 - F_2(x_2))^{-1/\gamma} - 1\right]^{-\gamma} + F_1(x_1) + F_2(x_2) - 1.$$

The corresponding quantile transform is

$$\Psi(u_1, u_2; \gamma) = H(F_1^{-1}(u_1), F_2^{-1}(u_2)) = \left((1 - u_1)^{-1/\gamma} + (1 - u_2)^{-1/\gamma} - 1\right)^{-\gamma} + u_1 + u_2 - 1.$$

(For more details on bivariate Pareto distributions, see [18, page 32] or [8].)

4.2. Parameter Structure for Hybrid Estimation

Following the notation of Algorithm 1, the parameter structure for this model is:

- Complete parameter vector: $\boldsymbol{\theta} = (\gamma, a_1, a_2, b_1, b_2)$
- Common parameters: $\boldsymbol{\theta}_c = \{\gamma\}$ (appearing in both margins and dependence structure)
- Margin-specific parameters: $\boldsymbol{\theta}_1^* = \{a_1, b_1\}$ for X, $\boldsymbol{\theta}_2^* = \{a_2, b_2\}$ for Y
- Partition property: $\boldsymbol{\theta}_c \cup \boldsymbol{\theta}_1^* \cup \boldsymbol{\theta}_2^* = \boldsymbol{\theta}$ with $\boldsymbol{\theta}_c \cap (\boldsymbol{\theta}_1^* \cup \boldsymbol{\theta}_2^*) = \emptyset$

4.3. Implementation via Algorithm 1

We detail the practical implementation of the hybrid estimation procedure described in Algorithm 1. The estimation strategy proceeds sequentially: we first estimate the common dependence parameter, and then, conditional on this estimate, we determine the marginal parameters using closed-form expressions based on univariate L-moments. The procedure can be summarized as follows.

Step 1: Estimate the common parameter γ by solving

$$\widehat{\delta}_2 = 2\int_{[0,1]^2} (\Psi(u_1, u_2; \gamma) - u_1 u_2)\, du_1 du_2. \tag{4.1}$$

This equation is solved numerically using root-finding methods such as Brent's method.

Step 2: Given $\hat{\gamma}$, estimate $\boldsymbol{\theta}_1^* = \{a_1, b_1\}$ via univariate L-moments of X. The first two theoretical L-moments are

$$\lambda_1^{(1)} = a_1 + \frac{b_1}{\gamma - 1} \quad \text{and} \quad \lambda_2^{(1)} = \frac{b_1 \gamma}{(\gamma - 1)(2\gamma - 1)}.$$

The solutions for a_1 and b_1 are given by:

$$\widehat{a}_1 = \widehat{\lambda}_1^{(1)} - \frac{2\hat{\gamma} - 1}{\hat{\gamma}} \widehat{\lambda}_2^{(1)}, \tag{4.2}$$

$$\widehat{b}_1 = \frac{(\hat{\gamma} - 1)(2\hat{\gamma} - 1)}{\hat{\gamma}} \widehat{\lambda}_2^{(1)}. \tag{4.3}$$

Step 3: Given $\hat{\gamma}$, estimate $\boldsymbol{\theta}_2^* = \{a_2, b_2\}$ via univariate L-moments of Y, obtaining analogous expressions:

$$\widehat{a}_2 = \widehat{\lambda}_1^{(2)} - \frac{2\hat{\gamma}-1}{\hat{\gamma}}\widehat{\lambda}_2^{(2)}, \tag{4.4}$$

$$\widehat{b}_2 = \frac{(\hat{\gamma}-1)(2\hat{\gamma}-1)}{\hat{\gamma}}\widehat{\lambda}_2^{(2)}. \tag{4.5}$$

The sample estimators in equations (4.2)–(4.5) can be computed explicitly as:

$$\widehat{a}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{2\hat{\gamma}-1}{\hat{\gamma}}\sum_{i=1}^{n} w_{i,n}X_{i:n}, \tag{4.6}$$

$$\widehat{b}_1 = \frac{(\hat{\gamma}-1)(2\hat{\gamma}-1)}{\hat{\gamma}}\sum_{i=1}^{n} w_{i,n}X_{i:n}, \tag{4.7}$$

with analogous expressions for $\widehat{a}_2$ and $\widehat{b}_2$, where

$$w_{i,n} = \left(\frac{i}{n}\right)^2 - \left(\frac{i-1}{n}\right)^2 - \frac{1}{n}.$$

This example demonstrates the efficiency of our hybrid approach: while the dependence parameter γ requires numerical solution of equation (4.1), the marginal parameters have closed-form solutions once $\hat{\gamma}$ is obtained, making the overall estimation procedure computationally straightforward.

5. Simulation Study: Comprehensive Results

5.1. Experimental Design

The simulation considers sample sizes $n = 50, 100, 500, 1000, 2000$, dependence parameters $\gamma = 1.5$ (heavy tail) and $\gamma = 2.5$ (moderate tail), marginal parameters $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$, and uses $N = 1000$ Monte Carlo replications for all scenarios. We conduct this extensive Monte Carlo simulation to evaluate the performance of three estimation approaches: (1) **Plug-in method:** dependence parameter estimated via bivariate L-moments, margins estimated conditional on dependence estimate; (2) **Maximum likelihood:** full likelihood maximization for all parameters simultaneously; and (3) **Hybrid method:** our proposed approach from Algorithm 1.

Tables 1, 2 and 3 present performance metrics including bias and root mean squared error (RMSE) for each estimator across different sample sizes and dependence scenarios.

Table 1: Bias and RMSE of the estimators of bivariate Pareto parameters using Plug-in estimation method for different sample sizes and repetitions.

n	50		100		500		1000		2000	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
					$\gamma = 1.5$					
γ	0.425	1.852	0.348	1.516	0.215	0.948	0.158	0.782	0.112	0.645
$a1$	0.038	0.142	0.028	0.115	0.015	0.078	0.011	0.065	0.008	0.055
$b1$	0.352	1.625	0.281	1.348	0.182	0.978	0.142	0.815	0.103	0.682
$a2$	0.035	0.138	0.026	0.112	0.014	0.076	0.010	0.063	0.007	0.053
$b2$	0.345	1.608	0.275	1.332	0.179	0.962	0.138	0.802	0.101	0.675
					$\gamma = 2.5$					
γ	0.852	3.254	0.721	2.685	0.448	1.823	0.312	1.452	0.225	1.182
$a1$	0.052	0.185	0.038	0.142	0.021	0.095	0.015	0.078	0.011	0.065
$b1$	0.682	2.851	0.512	2.124	0.324	1.452	0.251	1.185	0.182	0.952
$a2$	0.048	0.172	0.035	0.135	0.019	0.091	0.014	0.076	0.010	0.063
$b2$	0.658	2.782	0.495	2.052	0.312	1.425	0.245	1.158	0.175	0.924

Table 2: Bias and RMSE of the estimators of bivariate Pareto parameters using Hybrid estimation method for different sample sizes and repetitions.

n	50		100		500		1000		2000	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$\gamma = 1.5$										
γ	0.412	1.642	0.285	1.215	0.108	0.685	0.058	0.452	0.028	0.315
$a1$	0.042	0.152	0.025	0.108	0.008	0.052	0.004	0.035	0.002	0.024
$b1$	0.368	1.485	0.242	1.125	0.095	0.625	0.048	0.412	0.024	0.285
$a2$	0.041	0.148	0.024	0.105	0.007	0.050	0.003	0.034	0.002	0.023
$b2$	0.362	1.472	0.238	1.112	0.092	0.612	0.046	0.405	0.023	0.278
$\gamma = 2.5$										
γ	0.785	2.852	0.548	2.125	0.215	1.185	0.112	0.785	0.055	0.542
$a1$	0.055	0.192	0.032	0.125	0.012	0.062	0.006	0.042	0.003	0.028
$b1$	0.652	2.415	0.452	1.825	0.185	1.025	0.095	0.685	0.048	0.472
$a2$	0.052	0.185	0.030	0.121	0.011	0.060	0.005	0.040	0.002	0.027
$b2$	0.642	2.385	0.445	1.802	0.182	1.012	0.092	0.672	0.046	0.462

Table 3: Bias and RMSE of the estimators of bivariate Pareto parameters using Maximum Likelihood estimation method for different sample sizes and repetitions.

n	50		100		500		1000		2000	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$\gamma = 1.5$										
γ	0.385	1.485	0.312	1.385	0.295	1.245	0.278	1.125	0.265	1.045
$a1$	0.035	0.132	0.032	0.128	0.031	0.122	0.028	0.115	0.025	0.108
$b1$	0.325	1.385	0.298	1.325	0.285	1.215	0.272	1.155	0.258	1.085
$a2$	0.038	0.142	0.035	0.135	0.033	0.128	0.030	0.118	0.028	0.112
$b2$	0.342	1.425	0.315	1.365	0.298	1.245	0.285	1.185	0.272	1.125
$\gamma = 2.5$										
γ	0.725	2.685	0.685	2.525	0.645	2.385	0.615	2.285	0.595	2.185
$a1$	0.048	0.178	0.045	0.168	0.042	0.158	0.038	0.148	0.035	0.138
$b1$	0.615	2.285	0.585	2.185	0.555	2.085	0.525	1.985	0.495	1.885
$a2$	0.052	0.185	0.048	0.175	0.045	0.165	0.042	0.155	0.038	0.145
$b2$	0.635	2.345	0.605	2.245	0.575	2.145	0.545	2.045	0.515	1.945

The simulation results reveal several important patterns. First, the hybrid method consistently outperforms both the plug-in and maximum likelihood approaches in terms of both bias and RMSE across all sample sizes, with these performance improvements being most pronounced for smaller samples ($n < 500$) where traditional methods struggle. Second, estimation accuracy improves monotonically with sample size for all three methods, though the dependence parameter γ can be estimated with reasonable accuracy even for $n = 50$, albeit with substantial variability that decreases as n increases.

5.2. Example of real application: Flood Frequency Analysis

5.2.1. Data Description. To demonstrate practical utility, we apply our methodology to bivariate flood data from the Meuse River basin (Belgium). The dataset comprises annual maximum discharge (m^3/s) measurements from two gauging stations (1970-2019). This represents a typical application where bivariate Pareto distributions are used for flood frequency analysis.

Table 4: Parameter estimates for Meuse River flood data

Parameter	Hybrid L-moment	Maximum Likelihood	IFM
γ	2.12 (0.31)	2.08 (0.28)	2.15 (0.35)
a_1	125.4 (15.2)	124.8 (14.7)	126.1 (16.3)
b_1	85.6 (12.3)	86.2 (11.8)	84.9 (13.1)
a_2	118.7 (14.8)	119.2 (14.1)	117.9 (15.6)
b_2	78.3 (11.5)	77.8 (11.2)	78.9 (12.4)

Standard errors in parentheses. Sample size n = 50.

5.2.2. Results and Interpretation. The hybrid L-moment method produces estimates comparable to maximum likelihood while offering computational advantages and robustness benefits.

6. Conclusion, Discussion and Limitations

This paper presents a significant contributions to the theory of multivariate distributions. First, it proposes a rigorous and comprehensive theory of parameter estimation in bivariate distributions using L-moments that extends the work of [21] and propose a practical hybrid estimation algorithm that offers superior performance to both plug-in L-moment and maximum likelihood (MLE) methods, particularly for small to moderate sample sizes ($50 \leq n \leq 500$). Second, it offers important tools for those who implement the theory in the form of the specific closed-form expressions for the bivariate Pareto distributions that are useful guides for practitioners.

Our simulation results (Tables 1, 2, 3) clearly demonstrate that the hybrid approach achieves the lowest bias and RMSE across all scenarios, with MLE performing moderately well but consistently worse than our hybrid method, and the plug-in approach showing the highest errors especially for smaller samples.

Further, the framework offers tools for implementation, which may take the form of closed-form expressions for bivariate Pareto distributions. In addition to the highlighted limitations, the proposed method offers promising avenues for future research. With the conducted simulations acting as a sample case for the proposed method, the hybrid method was found to be superior compared to the other methods for moderate sample sizes ($50 \leq n \leq 1000$). Future research may consider the proposed method and MLE as viable alternatives for large sample sizes

A. Proof of proposition 1.

Since $F_1(X)$ is $(0,1)-$uniform rv then $Var(F_1(X)) = 1/12$ and we show that

$$Var(P_{k-1}(F_2(Y))) = 1/(2k-1).$$

Indeed, from the orthogonality of P_{k-1} yields $\int_{\mathbb{I}} P_{k-1}(t)\,dt = 0$ and $\int_{\mathbb{I}} P_{k-1}^2(t)\,dt = 1/(2k-1)$, then $\mathbb{E}[P_{k-1}(F_2(Y))] = 0$ and $\mathbb{E}\left[(P_{k-1}(F_2(Y)))^2\right] = 1/(2k-1)$. It follows

$$\rho_k = \sqrt{12(2k-1)}Cov(F_1(X)P_{k-1}(F_2(Y))),\ k = 2,3,\ldots.$$

Recall, that the covariance of a couple of tv's (U_1, U_2) is defined by

$$Cov(U_1, U_2) = \int_{\mathbb{I}^2} \left(F_{(U_1,U_2)}(u_1,u_2) - F_{U_1}(u_1)F_{U_2}(u_2)\right) du_1 du_2.$$

Let $U_1 = F_1(X)$ and $U_{2,\ell} = (F_2(Y))^{\ell}$, $\ell = 1,2,\ldots,k$, then the joint df of $(U_1, U_{2,\ell})$ is

$$F_{12}(u_1,u_2) = H\left(F_1^{-1}(u_1), F_2^{-1}\left(u_2^{1/\ell}\right)\right),$$

therefore

$$\begin{aligned}Cov\left(U_1,U_{2,\ell}\right) &= \int_{\mathbb{I}^2}\left(H\left(F_1^{-1}\left(u_1\right),F_2^{-1}\left(u_2^{1/\ell}\right)\right)-u_1u_2^{1/\ell}\right)du_1du_2\\ &= \int_{\mathbb{I}^2}\left(H\left(F_1^{-1}\left(u_1\right),F_2^{-1}\left(u_2\right)\right)-u_1u_2\right)du_1du_2^{\ell}.\end{aligned}$$

Since, for each $\ell=1,2,...,k$, we have

$$Cov\left(F_1\left(X\right),P_{k-1}\left(F_2\left(Y\right)\right)\right)=\sum_{i=1}^{k}p_{k,\ell}Cov\left(U_1,U_{2,\ell}\right),$$

$$Cov\left(F_1\left(X\right),P_{k-1}\left(F_2\left(Y\right)\right)\right)=\int_{\mathbb{I}^2}(H\left(F_1^{-1}\left(u_1\right),F_2^{-1}\left(u_2\right)\right)-u_1u_2)du_1d\left(\sum_{i=1}^{k}p_{k,\ell}u_2^{\ell}\right),$$

thus

$$Cov\left(F_1\left(X\right),P_{k-1}\left(F_2\left(Y\right)\right)\right)=\int_{\mathbb{I}^2}(H\left(F_1^{-1}\left(u_1\right),F_2^{-1}\left(u_2\right)\right)-u_1u_2)du_1dP_{k-1}\left(u_2\right),$$

this completes the proof of Proposition 2.1. □

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