Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 16** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.82018

An Application to Explore the Progress of a Nation Using the Emerging Nano Topological Concepts via Multiple Ideals

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ABSTRACT: The idea of nano topology was constituted by M.L. Thivagar wherein topology was generated using an intriguing methodology of approximation of rough sets. As the approach of approximation using the multiple ideals gives a higher value of accuracy when compared to the existing notions, the objective of this paper is to propose the nearly open forms of j-multi ideal nano topology, such as j-multi ideal nano semi-open, j-multi ideal nano pre-open, j-multi ideal nano regular open and j-multi ideal nano α open sets. An interrelation of all these open forms is illustrated. Various nano continuities using multi-ideal approximation technique are introduced and their interconnection is studied. Some nano separation axioms using this ideology are defined. A real-life example of nano topology is also discussed towards the end of this paper to find the key factors that govern the development or underdevelopment of a nation.

Keywords: Nano topology, ideals, j-multi-ideal nano pre-open, j-multi ideal nano semi-open, j-multi ideal nano α open, core.

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1. Introduction

The term rough set [19] was coined by Pawlak to split a set into its upper and lower approximation using an indiscernibility relation. This study has successfully been applied to deal with intelligent systems, specified by inadequate information or data. Nano topology is a subject that has many applications in the engineering and biomedical world. Many researchers have worked on generalizing the rough set [10] and hence the nano topology. One of the techniques was approximation using multiple ideals [3,8,11], which is more reliable than the older techniques as it remarkably reduces the boundary and increases the accuracy quotient.

In the past ten years, various theories were given by mathematicians to generate a dwarf topology using neutrosophic sets [15], intuitionistic sets, Pythagorean sets [6], fuzzy sets, ideals [4], graphs [2], nano-operations [7], neighborhoods [13], etc. The intriguing fact is that the nano topology, when merged with the graphs, has successfully interpreted biological processes like blood circulation and the functioning of the heart [2,18]. Nano topology has also been used in the physical world as it can be used in the reduction of electrical transmission lines [5]. The interrelation between a digraph theory and nano-topology has been used to study the diseases of the urinary system [17]. Moreover, the approximations of a rough set have been applied to investigate the physical properties of the fractals via their nano-topological graphs [1].

2010 *Mathematics Subject Classification:* 54A05, 54C05, 54H99, 82D80.
Submitted February 16, 2026. Published May 02, 2026.

Typeset by $\mathcal{B}^{S}\mathcal{P}_{\mathcal{M}}$ style.
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In addition, many topologists have researched on the new forms of nano-open sets and their generalizations [9,14]. Kaur and Gupta introduced the multi-ideal approach of approximation [11], where the approximation of a set was done with higher accuracy and precision. Further, it was directed to induce a novel nano topology using the technique of approximating spaces via two or more ideals, which is a modified version of the basic definition of the nano topology. This method has been compared to the previous theories, which indicates that the importance of constituting this amalgam is that the computations, based on one ideal would not yield the best outcomes but a utilization of multiple ones can achieve the purpose.

This paper investigates the nearly open forms of multi-ideal nano topology, such as j- multi ideal nano pre-open, j- multi ideal nano semi-open, j- multi ideal nano α open, and j-multi-ideal nano regular open sets. The interrelation of these forms along with some results and theorems are discussed. This research is novel and has a huge significance because the multi-ideal approach is a more accurate and efficient technique in the approximation of rough sets. Therefore, the topological concepts studied in the paper are better than the existing theories. This multi-ideal ideology is an innovative method and a generalized version of the standard approaches. To emphasize the importance of this study, a real-life application of multi-ideal nano topology is illustrated where it is used to identify the most significant factors that govern the development/underdevelopment of a nation.

2. Preliminaries

In this section, we define the basic definitions and notations which will be used in this paper. Throughout, $\mathbb{U}$ is the universal set and $\mathbb{R}$ is an arbitrary relation and $j \in \{l, r, i, u\}$ unless mentioned otherwise.

Definition 2.1 *[14] If $\mathbb{U}$ is the universal set and $\mathbb{R}$ is the equivalence relation. Also, let $\mathbb{M} \subseteq \mathbb{U}$. Then*

1. $L_{\mathbb{R}}(\mathbb{M}) = \bigcup_{w \in \mathbb{U}} \left\{ \mathbb{R}(w) : \mathbb{R}(w) \subseteq \mathbb{M} \right\}$ *is called the lower approximation of $\mathbb{M}$ w.r.t $\mathbb{R}$.*
2. $U_{\mathbb{R}}(\mathbb{M}) = \bigcup_{w \in \mathbb{U}} \left\{ \mathbb{R}(w) : \mathbb{R}(w) \ \cap \ \mathbb{M} \neq \emptyset \right\}$ *is called the upper approximation space of $\mathbb{M}$ w.r.t $\mathbb{R}$.*
3. $B_{\mathbb{R}}(\mathbb{M}) = U_{\mathbb{R}}(\mathbb{M}) - L_{\mathbb{R}}(\mathbb{M})$ *is the boundary of $\mathbb{M}$ w.r.t $\mathbb{R}$.*
   Let $\tau_{\mathbb{R}}(\mathbb{M}) = \left\{ \mathbb{U},\ \emptyset,\ L_{\mathbb{R}}(\mathbb{M}),\ U_{\mathbb{R}}(\mathbb{M}),\ B_{\mathbb{R}}(\mathbb{M}) \right\}$.

This is called a nano topology w.r.t $\mathbb{M}$. The pair $(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{M}))$ is defined as a nano topological space. Elements of $(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{M}))$ are defined as nano-open $(n.o)$ sets. Their complements are defined as nano-closed $(n.c)$ sets.

Remark 2.1 [14] If $\mathbb{Y} \subseteq \mathbb{U}$, then the union of $(n.o)$ subsets of $\mathbb{Y}$ is defined as nano interior of $\mathbb{Y}$, denoted by $\eta int(\mathbb{Y})$ and the intersection of $(n.c)$ sets containing $\mathbb{Y}$ is defined as nano closure of $\mathbb{Y}$, denoted by $\eta cl(\mathbb{Y})$.

Definition 2.2 *[11] Let $< \mathbb{U}, \mathbb{R}, C_j >$ as the $\mathbb{R}_j$ - covering approximation space. $\mathbb{R}$ is the binary relation. Let $H \subseteq \mathbb{U}$. If $\mathbb{I}_1, \mathbb{I}_2, , \mathbb{I}_3 ....., \mathbb{I}_n$ are 'n' ideals on a non-empty set $\mathbb{U}$, then the $\mathbb{R}_j$- lower, $\mathbb{R}_j$- upper approximation and $\mathbb{R}_j$- boundary of a set H are mathematically represented as :*

1. $\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(H) = \{k \in H : \mathbb{R}_j(k) \cap H^c \in < \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3, ...\mathbb{I}_n >\}$.
2. $\overline{\mathbb{R}_j}^{<n\mathbb{I}>}(H) = H \cup \{k \in \mathbb{U} : \mathbb{R}_j(k) \cap H \notin < \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3, ...\mathbb{I}_n >\}$.
3. $B^{<n\mathbb{I}>}{}_{\mathbb{R}_j}(H) = \overline{\mathbb{R}_j}^{<n\mathbb{I}>}(H) - \underline{\mathbb{R}_j}^{<n\mathbb{I}>}(H)$.

Here, $< n\mathbb{I} >$ is the notation for $< \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3, ...\mathbb{I}_n >$ which is mathematically defined as $< \mathbb{I}_1, \mathbb{I}_2, ...\mathbb{I}_n > = \{\mathfrak{P}_1 \ \cup \ \mathfrak{P}_2 \ \cup \ ..... \cup \ \mathfrak{P}_n : \mathfrak{P}_1 \in \mathbb{I}_1, \mathfrak{P}_2 \in \mathbb{I}_2 .....\mathfrak{P}_n \in \mathbb{I}_n\}$.

Definition 2.3 *[11] Let $\mathbb{U}$ be the universal set, $H \subseteq \mathbb{U}$ and $\mathbb{R}_j$ be the different kind of neighbourhoods. Here,*

$$\tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(H) = \{\mathbb{U}, \emptyset, \underline{\mathbb{R}_j}^{<n\mathbb{I}>}(H),\ \overline{\mathbb{R}_j}^{<n\mathbb{I}>}(H),\ B_{\mathbb{R}_j}^{<n\mathbb{I}>}(H)\}.$$

Clearly, this forms a topology that is termed as j-multi ideal nano-topology ($\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$ - nano topology). It is abbreviated as $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - NT$. This is a nano topology of $\mathbb{U}$ w.r.t H via covering-based neighborhoods induced by multiple ideals. The pair $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(H))$ is called a j-multi ideal nano topological space ($\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$ - nano topological space). It is abbreviated as $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - NTS$.

Remark 2.2 [11] The elements of $\tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(H)$ are referred as j-multi ideal nano open ($\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$ -$(n.o)$) sets. Their complements are referred as j-multi ideal nano closed ($\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$ -$(n.c)$) sets. The j-multi ideal nano open sets are abbreviated as $j - m.i.n.o$ sets and the j-multi ideal nano closed sets are abbreviated as $j - m.i.n.c$ sets.

Remark 2.3 [11] If $K \subseteq \mathbb{U}$, then the union of all $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$ -$(n.o)$ subsets of K is referred as j-multi ideal nano interior ($\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$ - nano interior) of K, written as $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} int(K)$. Intersection of all $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$ - $(n.c)$ sets containing K is referred as j-multi ideal nano closure ($\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$ - nano closure) of K, written as $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} cl(K)$.

Theorem 2.1 *[11] Let $(\mathbb{U}, \tau_{\mathbb{R}_j}{}^{<n\mathbb{I}>}(H))$ be an $\eta_{\mathbb{R}_j}{}^{<n\mathbb{I}>} - NTS$ w.r.t H. Here, $H \subseteq \mathbb{U}$. Let $L \subseteq \mathbb{U}$ and $M \subseteq \mathbb{U}$. Then the given results hold:*

1. $\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(L) \subseteq L \subseteq \overline{\mathbb{R}_j}^{<n\mathbb{I}>}(L)$.
2. $\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(\mathbb{U}) = \overline{\mathbb{R}_j}^{<n\mathbb{I}>}(\mathbb{U}) = \mathbb{U}$ *and* $\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(\emptyset)) = \overline{\mathbb{R}_j}^{<n\mathbb{I}>}(\emptyset) = \emptyset$.
3. *If* $L \subseteq M$, *then* $\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(L) \subseteq \underline{\mathbb{R}_j}^{<n\mathbb{I}>}(M)$ *and* $\overline{\mathbb{R}_j}^{<n\mathbb{I}>}(L) \subseteq \overline{\mathbb{R}_j}^{<n\mathbb{I}>}(M)$.
4. $\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(L) = (\overline{\mathbb{R}_j}^{<n\mathbb{I}>}(L^c))^c$ *and* $\overline{\mathbb{R}_j}^{<n\mathbb{I}>}(L) = (\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(L^c))^c$
5. $\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(L)) \subseteq \underline{\mathbb{R}_j}^{<n\mathbb{I}>}(L)$ *and* $\overline{\mathbb{R}_j}^{<n\mathbb{I}>} \subseteq \overline{\mathbb{R}_j}^{<n\mathbb{I}>}(\overline{\mathbb{R}_j}^{<n\mathbb{I}>}(L))$.
6. *If* $L \in < n\mathbb{I} >$, *then* $\overline{\mathbb{R}_j}^{<n\mathbb{I}>}(L) = L$.
7. *If* $L^c \in < n\mathbb{I} >$, *then* $\underline{\mathbb{R}_j}^{<n\mathbb{I}>}(L) = L$.

3. Nearly Open Forms of Multi Ideal Nano Topology

In this section, we will investigate the nearly open forms of j-multi ideal nano topology. Note that $j \in \{l, r, i, u\}$, unless mentioned otherwise.

Definition 3.1 *Let $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(H)$ be a multi -ideal N.T.S where $j \in \{l, r, i, u\}$. Let $E \subseteq \mathbb{U}$.*

1. *If $E \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} int(E))$, then E is defined as j-multi ideal nano semi-open $(j - m.i.n.s.o)$ set .*
2. *If $E \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} cl(E))$, then E is said to be j-multi ideal nano pre-open $(j - m.i.n.p.o)$ set .*
3. *$E \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} int(E)))$, then E is said to be j-multi ideal nano α open $(j - m.i.n.\alpha.o)$ set .*
4. *If $E = \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} cl(E))$, then E is defined as j-multi ideal nano regular open $(j - m.i.n.r.o)$ set .*

Let $j-m.i.n.s.o(\mathbb{U},H)$ represents the set of j-multi ideal nano semi-open sets, $j-m.i.n.p.o(\mathbb{U},H)$ represents the collection of all j-multi ideal nano pre-open sets, $j-m.i.n.\alpha.o(\mathbb{U},H)$ or $\tau^{j\alpha}_{<n\mathbb{I}>}(H)$ represents the collection of all j-multi ideal nano alpha-open sets and $j-m.i.n.r.o(\mathbb{U},H)$ represents the collection of all j-multi ideal nano regular open sets.

Remark 3.1 If the relation is symmetric or equivalence, then :

1. $r-m.i.n.s.o(\mathbb{U},H)=l-m.i.n.s.o(\mathbb{U},H)=i-m.i.n.s.o(\mathbb{U},H)=u-m.i.n.s.o(\mathbb{U},H)$.
2. $r-m.i.n.p.o(\mathbb{U},H)=l-m.i.n.p.o(\mathbb{U},H)=i-m.i.n.p.o(\mathbb{U},H)=u-m.i.n.p.o(\mathbb{U},H)$.
3. $r-m.i.n.r.o(\mathbb{U},H)=l-m.i.n.r.o(\mathbb{U},H)=i-m.i.n.r.o(\mathbb{U},H)=u-m.i.n.r.o(\mathbb{U},H)$.
4. $r-m.i.n.\alpha.o(\mathbb{U},H)=l-m.i.n.\alpha.o(\mathbb{U},H)=i-m.i.n.\alpha.o(\mathbb{U},H)=u-m.i.n.\alpha.o(\mathbb{U},H)$.

Remark 3.2 If the relation is symmetric or equivalence and $\mathbb{I}_1=\mathbb{I}_2=.....=\mathbb{I}_n=\mathbb{I}$, then $\forall\ j\in\{l,r,i,u\}$, then these nearly open forms coincide with the weak forms of ideal nano topological space, given by Kandil [4].

Remark 3.3 If the relation is symmetric or equivalence and $\mathbb{I}_1=\mathbb{I}_2=.....=\mathbb{I}_n=\phi$, then $\forall\ j\in\{l,r,i,u\}$, then these nearly open forms coincide with the weak forms of standard nano topological space, given by Thivagar [14].

Example 3.1 Let $\mathbb{U}=\{s,t,u,v\}$ and H be $\{v,t\}$. Also, let $\mathbb{I}_1=\{\emptyset\},\mathbb{I}_2=\{\emptyset,\{v\}\}$ and $\mathbb{I}_3=\{\emptyset,\{u\}\}$.
Then, $<\mathbb{I}_1,\mathbb{I}_2,\mathbb{I}_3>=\{\emptyset,\{v\},\{u\},\{v,u\}\}$.
If $\mathbb{R}=\{\{s,s\},\{t,t\},\{u,u\},\{v,v\},\{s,t\},\{t,s\}\}$,
then $\tau^{<3\mathbb{I}>}_{\mathbb{R}_j}(H)=\big\{\emptyset,\mathbb{U},\{t,s\},\{s,t,v\},\{v\}\big\}$.
$(\mathbb{U},\tau^{<3\mathbb{I}>}_{\mathbb{R}_j}(H)$ is a j-multi -ideal $N.T.S$.
$j-m.i.n.s.o\ (\mathbb{U},H)=\big\{\emptyset,\mathbb{U},\{s,t\},\{s,t,v\},\{v\}\big\}$.
$j-m.i.n.p.o\ (\mathbb{U},H)=\big\{\emptyset,\mathbb{U},\{s\},\{t\},\{v\},\{v,t\},\{s,v\},\{t,s\},\{v,t,u\},\{s,t,v\},\{s,u,v\}\big\}$.
$j-m.i.n.\alpha.o\ (\mathbb{U},H)$ or $\tau^{j\alpha}_{<3\mathbb{I}>}(H)=\big\{\emptyset,\mathbb{U},\{s,t\},\{s,t,v\},\{v\}\big\}$.
$j-m.i.n.r.o\ (\mathbb{U},H)=\big\{\emptyset,\mathbb{U},\{t,s\}\big\}$.

Remark 3.4

Note that $j-m.i.n.s.o\ (\mathbb{U},H)$, $j-m.i.n.p.o\ (\mathbb{U},H)$ and $j-m.i.n.r.o\ (\mathbb{U},H)$ doesn't form a topology but $j-m.i.n.\alpha.o\ (\mathbb{U},H)$ or $\tau^{j\alpha}_{<n\mathbb{I}>}(H)$ always forms a topology.

Theorem 3.1 *If $\mathcal{Q}$ is $j-m.i.n.o$ in $(\mathbb{U},\tau^{<n\mathbb{I}>}_{\mathbb{R}_j}(H)$, then it is $j-m.i.n.\alpha.o$ in $(\mathbb{U},\tau^{<n\mathbb{I}>}_{\mathbb{R}_j}(H)$.*
Proof Since $\mathcal{Q}$ is $j-m.i.n.o$ in $\mathbb{U}$, $\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\mathcal{Q})=\mathcal{Q}$.
So, $\mathcal{Q}\subseteq\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}cl(\mathcal{Q})\subseteq\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}cl(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\mathcal{Q})$.
$\implies \eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\mathcal{Q})\subseteq\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}cl(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\mathcal{Q}))))$.
$\implies \mathcal{Q}$ *is* $j-m.i.n.\alpha.o$.
$\implies \mathcal{Q}\in\tau^{j\alpha}_{<n\mathbb{I}>}(H)$.

Theorem 3.2 $\tau^{j\alpha}_{<n\mathbb{I}>}(H)\subseteq j-m.i.n.s.o\ (\mathbb{U},H)$ *in* $(\mathbb{U},\tau^{<n\mathbb{I}>}_{\mathbb{R}_j}(H)$. *Proof If* $\mathcal{Q}\in\tau^{j\alpha}_{<n\mathbb{I}>}(H)\implies\mathcal{Q}\subseteq\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}cl(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\mathcal{Q}))))$.
For any set K, we know that $\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(K)\subseteq K$.
Therefore, $\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}cl(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\mathcal{Q})))\subseteq\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}cl(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\mathcal{Q}))$.
$\implies \mathcal{Q}\in j-m.i.n.s.o\ (\mathbb{U},H)$.
So, $\tau^{j\alpha}_{<n\mathbb{I}>}(H)\subseteq j-m.i.n.s.o\ (\mathbb{U},H)$.

Theorem 3.3 $\tau^{j\alpha}_{<n\mathbb{I}>}(H)\subseteq j-m.i.n.p.o\ (\mathbb{U},H)$ *in* $(\mathbb{U},\tau^{<n\mathbb{I}>}_{\mathbb{R}_j}(H)$.
Proof Let $\mathcal{Q}\in\tau^{j\alpha}_{<n\mathbb{I}>}(H)$.
Then, $\mathcal{Q}\subseteq\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}cl(\eta^{<n\mathbb{I}>}_{\mathbb{R}_j}int(\mathcal{Q}))))$.

Since $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\mathcal{Q}) \subseteq \mathcal{Q}, (\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\mathcal{Q})) \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\mathcal{Q})$.
$\implies \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\mathcal{Q})) \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\mathcal{Q}))$.
$\implies \mathcal{Q} \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\mathcal{Q}))$.
Thus, $\mathcal{Q} \in j - m.i.n.p.o\ (\mathbb{U}, H)$.
So, $\tau_{<n\mathbb{I}>}^{j\alpha}(H) \subseteq j - m.i.n.p.o\ (\mathbb{U}, H)$ *in* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(H))$.

Remark 3.5 Note that $\mathbb{U}$ and $\emptyset$ are always j-multi ideal nano α-open ($j - m.i.n.\alpha.o$) since $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\mathbb{U}))) = \mathbb{U}$ and $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\emptyset))) = \emptyset$.

Theorem 3.4 $\tau_{<n\mathbb{I}>}^{j\alpha}(H) = (j - m.i.n.p.o\ (\mathbb{U}, H)) \cap (j - m.i.n.s.o\ (\mathbb{U}, H))$ *in* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(H))$.
Proof Let $\mathcal{Q} \in \tau_{<n\mathbb{I}>}^{\alpha}(H)$.
Then, $\mathcal{Q} \in j - m.i.n.p.o\ (\mathbb{U}, H)$ *and also,* $\mathcal{Q} \in j - m.i.n.s.o\ (\mathbb{U}, H)$.
Thus, $\mathcal{Q} \in j - m.i.n.p.o\ (\mathbb{U}, H)) \cap (j - m.i.n.s.o\ (\mathbb{U}, H)$.
Hence, $\tau_{<n\mathbb{I}>}^{j\alpha}(H) \subseteq (j - m.i.n.p.o\ (\mathbb{U}, H) \cap (j - m.i.n.s.o\ (\mathbb{U}, H))$.
For the converse statement, if $\mathcal{Q} \in j - m.i.n.p.o\ (\mathbb{U}, H) \cap j - m.i.n.s.o\ (\mathbb{U}, H)$, *then* $\mathcal{Q} \in j - m.i.n.s.o\ (\mathbb{U}, H) \implies \mathcal{Q} \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\mathcal{Q}))$. *[i]*
Also, if $\mathcal{Q} \in j - m.i.n.p.o\ (\mathbb{U}, H) \implies \mathcal{Q} \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\mathcal{Q}))$. *[ii]*
From [i], $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\mathcal{Q}) \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\mathcal{Q})))$.
$\implies \mathcal{Q} \in \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\mathcal{Q})$.
Thus, $j - m.i.n.s.o\ (\mathbb{U}, H) \subseteq \tau_{<n\mathbb{I}>}^{j\alpha}(H)$. *[iii]*
From [ii] , $\mathcal{Q} \subseteq (\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\mathcal{Q}))$.
$\implies \mathcal{Q} \subseteq \eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}cl(\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}int(\mathcal{Q}))$.
Thus, $j - m.i.n.p.o\ (\mathbb{U}, H) \subseteq \tau_{<n\mathbb{I}>}^{j\alpha}(H)$. *[iv]*
Equations [iii] and [iv] imply that
$(j - m.i.n.p.o\ (\mathbb{U}, H)) \cap (j - m.i.n.s.o\ (\mathbb{U}, H)) \subseteq \tau_{<n\mathbb{I}>}^{j\alpha}(H)$.
Hence, $\tau_{<n\mathbb{I}>}^{j\alpha}(H) = (j - m.i.n.p.o\ (\mathbb{U}, H)) \cap (j - m.i.n.s.o\ (\mathbb{U}, H))$.

Remark 3.6

The following implications hold $\forall\ j \in \{l, r, i, u\}$ as shown in Fig 1:

$j - m.i.n.r.o \implies j - m.i.n.o \implies j - m.i.n.\alpha.o \implies j - m.i.n.s.o.$
$j - m.i.n.r.o \implies j - m.i.n.o \implies j - m.i.n.\alpha.o \implies j - m.i.n.p.o$.

However, the reverse implications don't hold always as illustrated by the counterexample below:

(Counterexample): Let $\mathbb{U} = \{e, f, g, h\}$ and H be $\{f, h\}$.
If $\mathbb{I}_1 = \{\emptyset\}$, $\mathbb{I}_2 = \{\emptyset, \{h\}\}$, $\mathbb{I}_3 = \{\emptyset, \{g\}\}$,
then $< \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3 >= \{\emptyset, \{h\}, \{g\}, \{h, g\}\}$.
If $\mathbb{R} = \{\{e, e\}, \{f, f\}, \{g, g\}, \{h, h\}, \{f, e\}, \{e, f\}\}$, then
$\tau_{\mathbb{R}_j}^{<3\mathbb{I}>}(H) = \big\{\emptyset, U, \{f, e\}, \{e, f, h\}, \{h\}\big\}$.
So, $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<3\mathbb{I}>}(H))$ is a $\eta_{\mathbb{R}_j}^{<3\mathbb{I}>} - NTS$ w.r.t. H.
Here, $\big\{e, f, h\big\}$ is $j - m.i.n.o$ but not $j - m.i.n.r.o$.
$\{h, f, g\}$ is $j - m.i.n.p.o$ but not $j - m.i.n.\alpha.o$.
$\{e, h\}$ is $j - m.i.n.s.o$ but not $j - m.i.n.\alpha.o$.

4. Multi-ideal Nano Continuities

In this section, we define various j- multi-ideal nano continuities. Also, $j \in \{l, r, i, u\}$, unless mentioned otherwise. Throughout, $functn.$, $cts - fn$ and $inv - img$ are the abbreviations for function, continuous function, and inverse image respectively.

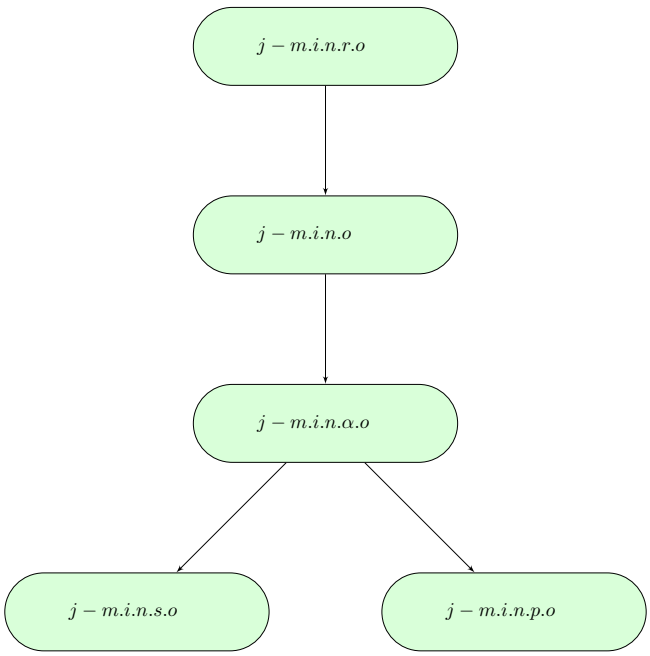

Figure 1: Flowchart depicting the interrelation of different open forms of multi-ideal nano topology

Definition 4.1 *Let* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *and* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *be two* j*- multi -ideal N.T.Ss, then the functn.* $g : (\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X)) \longrightarrow (\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is defined as* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$*-cts* $- fn$ *if the inv* $-$ *img of every* $j - m.i.n.o.$ *set in* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is* $j - m.i.n.o.$ *set in* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$.

Remark 4.1 Let $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ and $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y)$ be two j - multi -ideal $N.T.S$s. Also, let $g : (\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X)) \longrightarrow (\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y)$ be $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$*-cts* $- fn$, then

- $g^{-1}(\mathfrak{Q}') = (g^{-1}(\mathfrak{Q}))'$ for any $j - m.i.n.o.$ set $\mathfrak{Q}$ of $\mathbb{V}$.
- $g^{-1}(\mathfrak{G}') \subset (g^{-1}(\mathfrak{G}))'$ for any $j - m.i.n.o.$ set $\mathfrak{G}$ of $\mathbb{U}$.
- If $\mathfrak{Q}_1 \subset \mathfrak{Q}_2$, then $g^{-1}(\mathfrak{Q}_1) \subset g^{-1}(\mathfrak{Q}_2)$ where $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are $j - m.i.n.o.$ sets of $\mathbb{V}$.
- If $\mathfrak{G}_1 \subset \mathfrak{G}_2$, then $g^{-1}(\mathfrak{G}_1) \subset g^{-1}(\mathfrak{G}_2)$ where $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are $j - m.i.n.o.$ sets of $\mathbb{U}$.
- $g(g^{-1}(\mathfrak{Q})) \subset \mathfrak{Q}$ for any $j - m.i.n.o.$ set $\mathfrak{Q}$ of $\mathbb{V}$.
- $\mathfrak{G} \subset g(g^{-1}(\mathfrak{G}))$ for any $j - m.i.n.o.$ set $\mathfrak{G}$ of $\mathbb{U}$.

Definition 4.2 *Let* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *and* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *be two* j*- multi ideal N.T.Ss, then the functn.* $g : (\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X)) \longrightarrow (\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is defined as* j*-multi ideal nano open function if the image of every* $j - m.i.n.o.$ *set in* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *is* $j - m.i.n.o.$ *set in* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$.

Definition 4.3 *Let* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *and* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *be two* j*- multi ideal N.T.Ss, then the functn.* $g : (\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X)) \longrightarrow (\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is defined as* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$*-pre-cts*$-fn$ *if the inv*$-$*img of every* $j-m.i.n.o.$ *set in* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is* $j - m.i.n.p.o.$ *set in* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$.

Definition 4.4 *Let* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *and* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *be two* j*- multi ideal N.T.Ss, then the functn.* $g : (\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X)) \longrightarrow (\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is defined as* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$*-semi-cts*$-fn$ *if the inv*$-$*img of every* $j-m.i.n.o.$ *set in* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is* $j - m.i.n.s.o.$ *set in* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$.

Definition 4.5 *Let* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *and* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *be two* j*- multi ideal N.T.Ss, then the functn.* $g : (\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X)) \longrightarrow (\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is defined as* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$*-alpha-cts* $- fn$ *if the* $inv - img$ *of every* $j - m.i.n.o.$ *set in* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is* $j - m.i.n.\alpha.o.$ *set in* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$.

Definition 4.6 *Let* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *and* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *be two* j*- multi ideal N.T.Ss, then the functn.* $g : (\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X)) \longrightarrow (\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is defined as* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>}$*-regular-cts* $- fn$ *if the* $inv - img$ *of every* $j - m.i.n.o.$ *set in* $(\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y))$ *is* $j - m.i.n.r.o.$ *set in* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$.

Remark 4.2

The following implications hold for every $j \in \{l, r, i, u\}$ as shown in Fig 2:

j-multi ideal nano regular continuity $\implies$ j-multi ideal nano continuity $\implies$ j-multi ideal nano α continuity $\implies$ j-multi ideal nano semi-continuity.
j-multi ideal nano regular continuity $\implies$ j-multi ideal nano continuity $\implies$ j-multi ideal nano α continuity $\implies$ j-multi ideal nano pre-continuity.

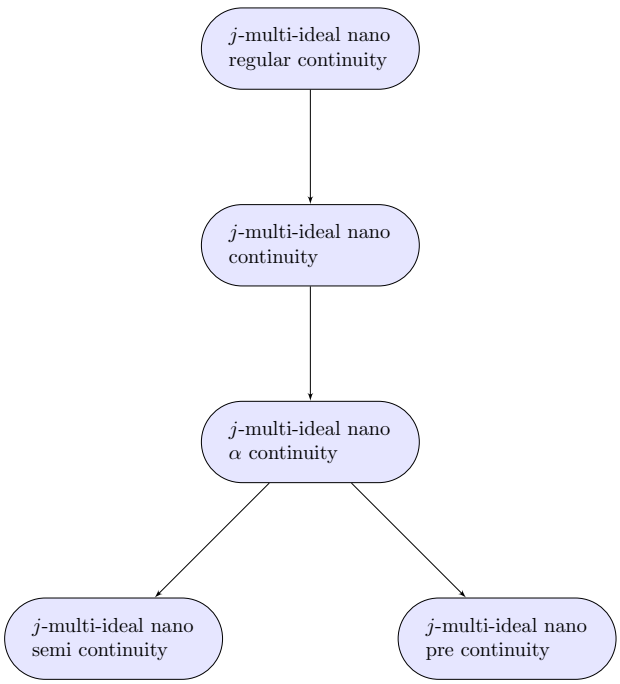

Figure 2: Flowchart depicting the interrelation of different multi-ideal nano continuities

5. Multi-ideal Nano Separation Axioms

In this section, we define multi-ideal nano separation axioms. Throughout, $j \in \{l, r, i, u\}$, unless mentioned otherwise.

Definition 5.1 *A space* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *is defined as* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_0$ *if* $\forall \;\; u \neq w$, $\exists$ *two* $j - m.i.n.o.$ *sets* Y *and* W *such that* $u \in Y, w \notin W$.

Definition 5.2 *A space* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *is defined as* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *if* $\forall \;\; u \neq w$, $\exists$ *two* $j - m.i.n.o.$ *sets* Y *and* W *such that* $u \in Y, w \notin W$ *and* $u \notin Y, w \in W$.

Definition 5.3 *A space* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *is defined as* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_2$ *if* $\forall\ u \neq w$, $\exists$ *two* $j-m.i.n.o.$ *sets* Y *and* W , $Y \cap W = \emptyset$ *such that* $u \in Y, w \notin W$ *and* $u \notin Y, w \in W$.

Theorem 5.1 *For a space* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$, $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_2 \implies \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1 \implies \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_0$.
Proof Firstly, let $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *be* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_2$ *space.*
To prove $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_2 \implies \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$, *assume* $u, w \in \mathbb{U}, u \neq w$.
Since $\mathbb{U}$ *is* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_2$, $\exists$ *two* $j-m.i.n.o.$ *sets* Y *and* W *in* $\mathbb{U}$, $Y \cap W = \emptyset$ *such that* $u \in Y, w \in W$. *It implies* $u \in Y$ *and* $V \cap W = \emptyset \implies w \notin Y$ *and* $w \in W$ *and* $Y \cap W = \emptyset \implies u \notin W$. *So,* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_2 \implies \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$.
Secondly, let $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *be* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *space. To prove* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1 \implies \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_0$, *by the definition of* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$, $\exists$ $j-m.i.n.o.$ *sets* Y *such that* $u \in Y, w \notin Y$. *It gives* $u \in Y$ *and doesn't include another element. So,* $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *is* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_0$. *Hence,* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1 \implies \eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_0$.

Remark 5.1 The converse of the above implication doesn't hold in general.

Theorem 5.2 *A multi-ideal NTS is* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *iff the singletons are* $j-m.i.n.c.$ *sets.*
Proof Let $(\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X))$ *be* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *and* $k \in W$. *Let* $m \in W - \{k\}$. *Then for* $k \neq m$, $\exists$ $j-m.i.n.o$ *set* W_m *such that* $m \in W_m \subseteq \mathbb{U} - \{k\}$. $U - \{k\}$ *is* $j-m.i.n.o$ *set, which implies that* $\{k\}$ *is* $j-m.i.n.c$ *set. The singletons are* $j-m.i.n.c.$ *sets.*
Conversely, suppose $\{k\}$ *is* $j-m.i.n.c$ *set* $\forall\ k \in \mathbb{U}$. *Let* k *and* $m \in \mathbb{U}$ *with* $k \neq m$. *Now* $k \neq m$ *implies* $m \in W - \{k\}$. *Hence,* $W - \{k\}$ *is* $j-m.i.n.o$ *set containing* m *but not* k. *Therefore,* W *is* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *space.*

Theorem 5.3 *The property being* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *is preserved under bijection and* j *- multi-ideal nano open functn..*
Proof Let $f : (\mathbb{U}, \tau_{\mathbb{R}_j}^{<n\mathbb{I}>}(X)) \longrightarrow (\mathbb{V}, \tau_{\mathfrak{R}_j}^{<m\mathbb{J}>}(Y)$ *be a bijection and* j *- multi-ideal nano open functn.. Let* $\mathbb{U}$ *be a* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *space and* m_1, m_2 *be two points of* $\mathbb{V}$ *such that* $m_1 \neq m_2$. *Since* f *is bijective,* $\exists$ *distinct points* k_1, k_2 *of* $\mathbb{U}$ *such that* $m_1 = f(x_1)$ *and* $m_2 = f(x_2)$. *Now,* $\mathbb{U}$ *being a* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *space,* $\exists$ $j-m.i.n.o$ *sets* G *and* H *such that* $m_1 \in G, m_2 \notin G$ *and* $m_1 \notin H, m_2 \in H$. *Therefore,* $m_1 = f(x_1) \in f(G)$ *and* $m_2 = f(x_2) \notin f(G)$ *and* $m_2 = f(x_2) \in f(H)$ *and* $m_1 = f(x_1) \notin f(H)$. *Now,* f *being* j*- multi-ideal nano open functn.,* $f(G)$ *and* $f(H)$ *are* $j-m.i.n.o$ *subsets of* $\mathbb{V}$ *such that* $m_1 \in f(G)$ *but* $m_2 \notin f(G)$ *and* $m_2 \in f(H)$ *and* $m_1 \notin f(H)$. *Hence,* $\mathbb{V}$ *is* $\eta_{\mathbb{R}_j}^{<n\mathbb{I}>} - T_1$ *space.*

6. An Application: Development of a Nation

In this example, we find the most significant factors that govern the development of a country depending upon the various conditional attributes. Consider Table 1 :

Countries	Pop. rate	Corruption	Poverty	Infrastructure	Unemployment	Literacy rate	Decision
∇_1	low	moderate	medium	good	low	high	developed
∇_2	high	high	poor	bad	low	low	underdeveloped
∇_3	low	low	rich	good	low	high	underdeveloped
∇_4	high	low	medium	good	high	average	developed
∇_5	high	high	poor	bad	high	low	underdeveloped
∇_6	low	low	rich	good	low	high	underdeveloped
∇_7	low	low	medium	good	low	high	developed
∇_8	high	high	rich	bad	high	low	underdeveloped
∇_9	high	high	poor	bad	high	low	developed
∇_{10}	low	low	rich	good	low	high	developed

Table 1: Illustration of countries w.r.t. various parameters and reports.

Let $\mathbb{U} = \{\nabla_1, \nabla_2, \nabla_3, \nabla_4, \nabla_5, \nabla_6, \nabla_7, \nabla_8, \nabla_9, \nabla_{10}\}$ be the set of 10 countries. It is the universal set. The following table depicts the information of ten countries, w.r.t the attributes namely, pop. rate, corruption, poverty, infrastructure, unemployment, literacy rate, abbreviated as (L.R). The set of attributes

is W.

Also, let $H = \{\nabla_1, \nabla_4, \nabla_7, \nabla_9, \nabla_{10}\}$ be the five countries which are considered as developed by a report and $H^C = \{\nabla_2, \nabla_3, \nabla_5, \nabla_6, \nabla_8\}$ be the set of countries which are underdeveloped by the report.

Here, $\mathbb{I}_1 = \{\phi, \{\nabla_1\}, \{\nabla_3\}, \{\nabla_1, \nabla_3\}\}$, $\mathbb{I}_2 = \{\phi, \{\nabla_3\}, \{\nabla_5\}, \{\nabla_3, \nabla_5\}\}$, $\mathbb{I}_3 = \{\phi, \{\nabla_9\}\}$, and $\mathbb{I}_4 = \{\phi, \{\nabla_1\}, \{\nabla_9\}, \{\nabla_1, \nabla_9\}\}$.
$< \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3, \mathbb{I}_4 > = \{\phi, \{\nabla_1\}, \{\nabla_3\}, \{\nabla_5\}, \{\nabla_9\}, \{\nabla_3, \nabla_5\}, \{\nabla_5, \nabla_9\}, \{\nabla_3, \nabla_9\}, \{\nabla_1, \nabla_3\},$
$\{\nabla_1, \nabla_9\}, \{\nabla_1, \nabla_5\}, \{\nabla_1, \nabla_3, \nabla_5\}, \{\nabla_1, \nabla_3, \nabla_9\}, \{\nabla_1, \nabla_5, \nabla_9\}, \{\nabla_3, \nabla_5, \nabla_9\}, \{\nabla_1, \nabla_3, \nabla_5, \nabla_9\}\}$.
Here $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_3$ and $\mathbb{I}_4$ represent the lists of developed countries by external reports 1, 2, 3 and 4 respectively:
$< \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3, \mathbb{I}_4 >$ i.e. an ideal generated by $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_3$ and $\mathbb{I}_4$ means collaborative report of four external experts.
Here, the decision attributes are {developed, underdeveloped} depending upon whether recommended or not.
The domains of attributes, namely population rate, corruption, poverty, infrastructure, unemployment, and literacy rate are {low, high}, {moderate, high, low}, {medium, rich, poor}, {good, bad}, {low, high}, {high, low, average} respectively.
Now, if $\mathbb{R}$ is the indiscernibility relation, the two nations are related to each other if those are similar w.r.t W. $\mathbb{U}/\mathbb{R}$ represents the class w.r.t relation $\mathbb{R}$.
Here, when all the attributes are taken together,
$\mathbb{U}/\mathbb{R} = \big\{\{\nabla_1\}, \{\nabla_2\}, \{\nabla_3, \nabla_6, \nabla_{10}\}, \{\nabla_4\}, \{\nabla_5, \nabla_9\}, \{\nabla_7\}, \{\nabla_8\}\big\}$.
If $\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3, \mathbb{I}_4$ given and $H \subseteq \mathbb{U}$, then multi-ideal nano topology can be given by $\tau_{\mathbb{R}_j}^{<4\mathbb{I}>}(H) = \{\phi, \mathbb{U},$
$\underline{\mathbb{R}_j}^{<4\mathbb{I}>}(H), \underline{\mathbb{R}_j}^{<4\mathbb{I}>}(H), B^{<4\mathbb{I}>}{}_{\mathbb{R}_j}(H)\}$
$= \{\emptyset, \mathbb{U}, \{\nabla_1, \nabla_4, \nabla_7, \nabla_9\}, \{\nabla_1, \nabla_4, \nabla_3, \nabla_6, \nabla_7, \nabla_9, \nabla_{10}\}, \{\nabla_3, \nabla_6, \nabla_{10}\}$.

CASE-I: Firstly, when we neglect an attribute "pop. rate" from the H, then we have $\mathbb{R}' = \mathbb{R} - \{$pop. rate$\}$. So, $\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R} - \{$pop. rate$\}$ which implies that $\mathbb{U}/\mathbb{R}' = \big\{\{\nabla_1\}, \{\nabla_2\},$
$\{\nabla_3, \nabla_6, \nabla_{10}\}, \{\nabla_4\}, \{\nabla_5, \nabla_9\}, \{\nabla_7\}, \{\nabla_8\}\big\} = \mathbb{U}/\mathbb{R}$. Hence, $\tau_{\mathbb{R}_j}^{'\,<4\mathbb{I}>}(H)$ is induced by $\mathbb{R}' = \mathbb{R} - \{$pop. rate$\}$ is equal to $\tau_{\mathbb{R}_j}^{<4\mathbb{I}>}(H)$, i.e., $\tau_{\mathbb{R}_j}^{'\,<4\mathbb{I}>}(H) = \{\phi, \mathbb{U}, \underline{\mathbb{R}'_j}^{<4\mathbb{I}>}(H), \underline{\mathbb{R}'_j}^{<4\mathbb{I}>}(H),$
$B^{<4\mathbb{I}>}{}_{\mathbb{R}'_j}(H)\} = \tau_{\mathbb{R}_j}^{<4\mathbb{I}>}(H)$ as $\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R}$.

CASE-II: Secondly, neglecting "corruption" from the H, then we have $\mathbb{R}' = \mathbb{R} - \{$corruption$\}$. So, $\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R} - \{$corruption$\}$ which implies that $\mathbb{U}/\mathbb{R}' = \big\{\{\nabla_1, \nabla_7\}, \{\nabla_2\}, \{\nabla_3, \nabla_6, \nabla_{10}\},$
$\{\nabla_4\}, \{\nabla_5, \nabla_9\}, \{\nabla_8\}\big\} \neq \mathbb{U}/\mathbb{R}$. But, $\tau_{\mathbb{R}_j}^{'\,<4\mathbb{I}>}(H)$ (w.r.t $\mathbb{R}'$) is given by $\tau_{\mathbb{R}_j}^{'\,<4\mathbb{I}>}(H) = \{\emptyset, \mathbb{U}, \{\nabla_1, \nabla_4, \nabla_7, \nabla_9\},$
$\{\nabla_1, \nabla_4, \nabla_3, \nabla_6, \nabla_7, \nabla_9, \nabla_{10}\}, \{\nabla_3, \nabla_6, \nabla_{10}\} = \tau_{\mathbb{R}_j}^{<4\mathbb{I}>}(H)$.

CASE-III: Then, neglecting "poverty" from the H, then we have $\mathbb{R}' = \mathbb{R} - \{poverty\}$. So, $\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R} - \{$poverty$\}$ which implies that $\mathbb{U}/\mathbb{R}' = \big\{\{\nabla_1\}, \{\nabla_2\}, \{\nabla_3, \nabla_6, \nabla_7, \nabla_{10}\}, \{\nabla_4\},$
$\{\nabla_5, \nabla_8, \nabla_9\}\big\} \neq U/\mathbb{R}$.
Hence, $\tau_{\mathbb{R}_j}^{'\,<4\mathbb{I}>}(H) = \{\phi, \mathbb{U}, \{\nabla_1\}, \{\nabla_1, \nabla_3, \nabla_4, \nabla_6, \nabla_7, \nabla_9, \nabla_{10}\}, \{\nabla_3, \nabla_4, \nabla_6, \nabla_7, \nabla_9, \nabla_{10}\}\} \neq \tau_{\mathbb{R}_j}^{<4\mathbb{I}>}(H)$.

CASE-IV: On removing factor "infrastructure" from the H, then we have $\mathbb{R}' = \mathbb{R} - \{$infrastructure$\}$. So, $\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R} - \{$infrastructure$\}$ which implies that $\mathbb{U}/\mathbb{R}' = \big\{\{\nabla_1\}, \{\nabla_2\}, \{\nabla_3, \nabla_6, \nabla_{10}\}\},$
$\{\nabla_4\}, \{\nabla_5, \nabla_9\}, \{\nabla_7\}, \{\nabla_8\}\big\} = \mathbb{U}/\mathbb{R}$. Hence, $\tau_{\mathbb{R}_j}^{'\,<4\mathbb{I}>}(H)$ is induced by $\mathbb{R}' = \mathbb{R} - \{infrastructure\}$ is equal to $\tau_{\mathbb{R}_j}^{<4\mathbb{I}>}(H)$, i.e., $\tau_{\mathbb{R}_j}^{'\,<4\mathbb{I}>}(H) = \tau_{\mathbb{R}_j}^{<4\mathbb{I}>}(H)$ as $\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R}$.

CASE-V: Neglecting "unemployment" from the H, then we have $\mathbb{R}' = \mathbb{R} - \{$unemployment$\}$. So,

$\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R} - \{\text{unemployment}\}$ which implies that $\mathbb{U}/\mathbb{R}' = \big\{\{\nabla_1\}, \{\nabla_2, \nabla_5, \nabla_9\}, \{\nabla_3, \nabla_6, \nabla_{10}\}, \{\nabla_4\}, \{\nabla_7\}, \{\nabla_8\}\big\} \neq \mathbb{U}/\mathbb{R}$.
So, $\tau'^{<4\mathbb{I}>}_{\mathbb{R}_j}(H) = \{\phi, \mathbb{U}, \{\nabla_1, \nabla_4, \nabla_7\}, \{\nabla_1, \nabla_3, \nabla_4, \nabla_6, \nabla_7, \nabla_9, \nabla_{10}\}, \{\nabla_3, \nabla_6, \nabla_9, \nabla_{10}\}\} \neq \tau^{<4\mathbb{I}>}_{\mathbb{R}_j}(H)$.

CASE-VI: Lastly, neglecting "literacy rate" (L.R) from the H, then we have $\mathbb{R}' = \mathbb{R} - \{(\text{L.R})\}$. So, $\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R} - \{(\text{L.R})\}$ which implies that $\mathbb{U}/\mathbb{R}' = \big\{\{\nabla_1\}, \{\nabla_2\}, \{\nabla_3, \nabla_6, \nabla_{10}\}, \{\nabla_4\}, \{\nabla_5, \nabla_9\}, \{\nabla_7\}, \{\nabla_8\}\big\} = \mathbb{U}/\mathbb{R}$. Hence, $\tau'^{<4\mathbb{I}>}_{\mathbb{R}_j}(H)$ is induced by $\mathbb{R}' = \mathbb{R} - \{(\text{L.R.})\}$ is equal to $\tau^{<4\mathbb{I}>}_{\mathbb{R}_j}(H)$, i.e., $\tau'^{<4\mathbb{I}>}_{\mathbb{R}_j}(H) = \tau^{<4\mathbb{I}>}_{\mathbb{R}_j}(H)$ as $\mathbb{U}/\mathbb{R}' = \mathbb{U}/\mathbb{R}$.
So, we conclude that the collection of attributes for which $\tau'^{<4\mathbb{I}>}_{\mathbb{R}_j}(H) \neq \tau^{<4\mathbb{I}>}_{\mathbb{R}_j}(H)$ is {poverty, unemployment}. Hence, the above-listed attributes are the most imperative to decide whether the country is developed or not.

CORE(R)= {Poverty, Unemployment}.

Hence, poverty and unemployment are the most imperative factors responsible for the underdevelopment of a country.

7. Future Scope and Conclusion

The notion of nano topology was introduced by M.L. Thivagar [14]. The collaboration of nano topology with rough set theory and graph theory has served in the areas of medicine, information science, research, and technology. In the last decade, numerous theories were given and deeply explored by active topologists where nano topology was generated via an approximation technique using different tools of mathematics. The technique of inducing nano topology by approximating via multiple ideals has proven to be more efficient and reliable than the others, which implies that multi-ideal nano topology is more accurate than the classic term of nano topology. This paper investigates the nearly open forms of multi-ideal nano topology and their interrelation. Further, this theory holds scope in the future as the multi-ideal notion helps to study the inter-dependency of condition attributes and decision attributes through a collaborated point of view. Research work on its applications is currently in progress.

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