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On Some Results of Projectors on $\mathbb{C}^{(n)}$

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ABSTRACT: In this paper, we have obtained some results on projectors on the vector space $\mathbb{C}^{(n)}(\mathbb{C})$, $n \geq 1$ and its subspaces. We have also established one similarity result on projective matrix.

Keywords: Vector Spaces, orthonormal basis, idempotent and Hermitian matrices, projectors.

Contents

1. Introduction

Projectors are important linear operators in finite-dimensional linear algebra, particularly in the study of matrix structure and singularity. A projector on $\mathbb{C}^{(n)}(\mathbb{C})$ is a linear transformation P satisfying the idempotent condition $P^2 = P$ [1]. This property immediately restricts the possible eigenvalues of P to only 0 and 1. As a consequence, a nontrivial projector is always a singular matrix, since its determinant must be zero. The rank of a projector equals the dimension of its range, while its nullity equals the dimension of its kernel. These two subspaces together yield a direct sum decomposition of $\mathbb{C}^{(n)}$ [1]. Projectors therefore provide natural examples connecting rank, nullity, and singularity. Their simple algebraic form makes them useful for illustrating fundamental result such as the rank–nullity theorem.

2. Preliminaries

In this section we discuss some preliminaries need for this paper.

Definition 2.1 *[2] Let V be a vector space over $\mathbb{C}$ of dimension d and W be a subspace of V of dimension k. Then a projector P on W is defined as*

$$\sum_{j=1}^{k} v_j . v_j^* = P,$$

where, $\{v_1, v_2, ..., v_k\}$ is an orthonormal basis for W and v_j^* is the conjugate transpose of column vector v_j.

Example 2.2 *Let $V = \mathbb{C}^2$ be a vector space over $\mathbb{C}$ and W be a subspace of V such that $W = V$. To find a projector P on W, let us consider an orthonormal basis of W as $B =$ Let $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. If $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_1^* = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $v_2^* = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Therefore the projector P on W with respect to the basis B is, $P = v_1 v_1^* + v_2 v_2^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$.*

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Definition 2.3 *[2] Let V be a finite dimensional vector space over $\mathbb{C}$ and W be a subspace of V . Then the projector P on the subspace W with respect to an orthonormal basis of W is known as projective matrix.*

Definition 2.4 *[3] Let V be a vector space over F, where F is a field of real or complex number. An inner product $(,)$ on V is a function from $V \times V$ to F which satisfies the following conditions:*

1. $(u, u) \geq 0$ *and* $(u, u) = 0$ *if and only if* $u = 0, \forall u \in V$
2. $\overline{(u, v)} = (v, u), \forall u, v \in V$
3. $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w), \forall u, v, w \in V$ *and* $\forall \alpha, \beta \in F$.

- If $V = \mathbb{C}^{(n)}$, $F = \mathbb{C}$ then $<u, v> = u^* v$ is known as standard inner product in $\mathbb{C}^{(n)}$, where u^* is the conjugate transpose of column vector u.

Remark: Throughout this paper we use standard inner product in $\mathbb{C}^{(n)}(\mathbb{C})$.

3. Some Results on Projectors

Theorem 3.1 *[2] If V be a vector space over $\mathbb{C}$ of dimension d and W be a subspace of V of dimension k, then a projector P on W is idempotent and hermitian matrix.*

Proof: Let $\{v_1, v_2, ..., v_k\}$ is an orthonormal basis for W. By definition of a projector P,

$$P = v_1 v_1^* + v_2 v_2^* + ... + v_k v_k^* \tag{3.1}$$

Now, let $v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{d1} \end{pmatrix}$, where $a_{11}, a_{21}, ..., a_{d1} \in \mathbb{C}$.

Since, v_1 is an element of orthonormal basis, we have $||v_1|| = 1$.

$$i.e; a_{11}\bar{a_{11}} + a_{21}\bar{a_{21}} + ... + a_{d1}\bar{a_{d1}} = |a_{11}|^2 + |a_{21}|^2 + ... + |a_{d1}|^2 = 1 \tag{3.2}$$

Now, $v_1 v_1^* = \begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{d1} \end{pmatrix} \begin{pmatrix} \bar{a_{11}} & \bar{a_{21}} & ... & \bar{a_{d1}} \end{pmatrix} = \begin{pmatrix} |a_{11}|^2 & a_{11}\bar{a_{21}} & ... & a_{11}\bar{a_{d1}} \\ a_{21}\bar{a_{11}} & |a_{21}|^2 & ... & a_{21}\bar{a_{d1}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{d1}\bar{a_{11}} & a_{d1}\bar{a_{21}} & ... & |a_{d1}|^2 \end{pmatrix}$

$$i.e; v_1 v_1^* = \begin{pmatrix} |a_{11}|^2 & a_{11}\bar{a_{21}} & ... & a_{11}\bar{a_{d1}} \\ a_{21}\bar{a_{11}} & |a_{21}|^2 & ... & a_{21}\bar{a_{d1}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{d1}\bar{a_{11}} & a_{d1}\bar{a_{21}} & ... & |a_{d1}|^2 \end{pmatrix} \tag{3.3}$$

Now, $(v_1 v_1^*)(v_1 v_1^*) = \begin{pmatrix} |a_{11}|^2 & a_{11}\bar{a_{21}} & ... & a_{11}\bar{a_{d1}} \\ a_{21}\bar{a_{11}} & |a_{21}|^2 & ... & a_{21}\bar{a_{d1}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{d1}\bar{a_{11}} & a_{d1}\bar{a_{21}} & ... & |a_{d1}|^2 \end{pmatrix} \begin{pmatrix} |a_{11}|^2 & a_{11}\bar{a_{21}} & ... & a_{11}\bar{a_{d1}} \\ a_{21}\bar{a_{11}} & |a_{21}|^2 & ... & a_{21}\bar{a_{d1}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{d1}\bar{a_{11}} & a_{d1}\bar{a_{21}} & ... & |a_{d1}|^2 \end{pmatrix}$

$$= \begin{pmatrix} |a_{11}|^2 & a_{11}\bar{a_{21}} & ... & a_{11}\bar{a_{d1}} \\ a_{21}\bar{a_{11}} & |a_{21}|^2 & ... & a_{21}\bar{a_{d1}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{d1}\bar{a_{11}} & a_{d1}\bar{a_{21}} & ... & |a_{d1}|^2 \end{pmatrix}$$

Therefore,

$$(v_1v_1^*)(v_1v_1^*) = v_1v_1^* \tag{3.4}$$

Similarly, $(v_2v_2^*)(v_2v_2^*) = v_2v_2^*$ and so on. If $v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{d2} \end{pmatrix}$,

where, $a_{12}, a_{22}, ..., a_{d2} \in \mathbb{C}$, then by orthogonality of v_1 and v_2,

$$a_{12}\bar{a_{11}} + ... + a_{d2}\bar{a_{d1}} = 0. \tag{3.5}$$

Also,

$$v_2v_2^* = \begin{pmatrix} |a_{12}|^2 & ... & ... & a_{12}\bar{a_{d2}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{d2}\bar{a_{12}} & ... & ... & |a_{d2}|^2 \end{pmatrix} \tag{3.6}$$

Multiplying (3.3) and (3.6) using (3.5) we get, $(v_1v_1^*)(v_2v_2^*) = \mathbf{0}$.

In general, $(v_mv_m^*)(v_nv_n^*) = \begin{cases} v_mv_m^*, & \text{if } m = n \\ \mathbf{0}, & \text{if } m \neq n \end{cases}$

Hence, $P^2 = P$. i.e; P is an idempotent matrix.

Again, clearly $(v_1v_1^*)^* = v_1v_1^*, (v_2v_2^*)^* = v_2v_2^*$ and so on.
Thus, we conclude that, $P^* = P$.
$\implies$ P is hermitian matrix.

□

Theorem 3.2 *If $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, then any projector P on V is identity matrix.*
i.e; If $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, then $P = I_n$.

Proof: Let $B = \left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{pmatrix}, ..., \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{nn} \end{pmatrix} \right\}$ is an orthonormal basis of $V = \mathbb{C}^n$ $(n \geq 1)$.

By definition of projector P,

$$P = \begin{pmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{pmatrix} \begin{pmatrix} \bar{a_{11}} & \bar{a_{21}} & ... & \bar{a_{n1}} \end{pmatrix} + \begin{pmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{pmatrix} \begin{pmatrix} \bar{a_{12}} & \bar{a_{22}} & ... & \bar{a_{n2}} \end{pmatrix} + ... + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{nn} \end{pmatrix} \begin{pmatrix} \bar{a_{1n}} & \bar{a_{2n}} & ... & \bar{a_{nn}} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\bar{a}_{11} + ... + a_{1n}\bar{a}_{1n} & ... & a_{11}\bar{a}_{n1} + ... + a_{1n}\bar{a}_{nn} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{n1}\bar{a}_{11} + ... + a_{nn}\bar{a}_{1n} & ... & a_{n1}\bar{a}_{n1} + ... + a_{nn}\bar{a}_{nn} \end{pmatrix}.$$

Now, B is an orthonormal basis implies that all the vectors of B are linearly independent. So, $k =$ $\det \begin{pmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{n1} & ... & a_{nn} \end{pmatrix} \neq 0$. Therefore, $detP = |k|^2$. Since $k \neq 0$, we have $detP \neq 0$ and hence, P is non-singular.

Also, we have projector matrix is always idempotent [Theorem 3.1] and identity is only non-singular idempotent matrix.
Hence, $P = I_n$. □

Corollary 3.3 *If $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, then the projector P on V is non-singular.*

Theorem 3.4 *If $W \neq \{0\}$ be a subspace of $\mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, and P be a projector on W with respect to some orthonormal basis of W such that $P = I_n$, then $W = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$.*

Proof: Let W be of dimension x. Then, $1 \leq x \leq n$.

Let $B=\left\{ \begin{pmatrix} a_{1k} \\ a_{2k} \\ . \\ . \\ . \\ a_{nk} \end{pmatrix} \middle| 1 \leq k \leq x \right\}$ be an orthonormal basis of W.

Therefore,

$$|a_{1k}|^2 + ... + |a_{nk}|^2 = 1; 1 \leq k \leq x \tag{3.7}$$

By definition of the projector,

$$P = \begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{n1} \end{pmatrix} \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & ... & \bar{a}_{n1} \end{pmatrix} + ... + \begin{pmatrix} a_{1x} \\ a_{2x} \\ . \\ . \\ . \\ a_{nx} \end{pmatrix} \begin{pmatrix} \bar{a}_{1x} & \bar{a}_{2x} & ... & \bar{a}_{nx} \end{pmatrix}.$$

$$= \begin{pmatrix} |a_{11}|^2 + ... + |a_{1x}|^2 & ... & ... \\ ... & |a_{21}|^2 + ... + |a_{2x}|^2 & ... \\ . & . & . \\ . & . & . \\ . & . & . \\ ... & ... & |a_{n1}|^2 + ... + |a_{nx}|^2 \end{pmatrix}.$$

Here, trace of $P = x = n$ (as $P = I_n$).
Thus, $dim\ W = dim\ \mathbb{C}^n(\mathbb{C})$.
Hence, $W = \mathbb{C}^n(\mathbb{C})$. □

Theorem 3.5 *If P is a projector on a subspace $W \neq \{0\}$ of a vector space $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, then $P \neq 0_n$ $\forall n \geq 1$, where 0_n is null matrix of order $n \times n$.*

Proof: Case-1: If $W = V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, then by Theorem (3.2) the projector P on W is $I_n \neq 0_n$.
Case-2: Let W be a proper subspace of $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$. For the case $V = \mathbb{C}$, $\nexists$ any proper subspace of V, therefore we compelled to take $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 2)$.
Let *dim* $W = x$.
Then $1 \leq x \leq (n-1)$.

Let $B = \left\{ \begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{nk} \end{pmatrix} \middle| 1 \leq k \leq x \right\}$ be an orthonormal basis of W

Therefore,

$$|a_{1k}|^2 + ... + |a_{nk}|^2 = 1; 1 \leq k \leq x \tag{3.8}$$

We prove the theorem by contradiction.
Suppose $P = 0_n$. By definition of the projector,

$$P = \begin{pmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{pmatrix} \begin{pmatrix} \bar{a_{11}} & \bar{a_{21}} & ... & \bar{a_{n1}} \end{pmatrix} + ... + \begin{pmatrix} a_{1x} \\ a_{2x} \\ \cdot \\ \cdot \\ \cdot \\ a_{nx} \end{pmatrix} \begin{pmatrix} \bar{a_{1x}} & \bar{a_{2x}} & ... & \bar{a_{nx}} \end{pmatrix}.$$

$$= \begin{pmatrix} |a_{11}|^2 + ... + |a_{1x}|^2 & ... & ... \\ ... & |a_{21}|^2 + ... + |a_{2x}|^2 & ... \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ ... & ... & |a_{n1}|^2 + ... + |a_{nx}|^2 \end{pmatrix}.$$

Here, trace of $P = x$ [Using equation 3.8], since B is orthonormal. Which is a contradiction to the assumption, that $x \neq 0$.
Hence, $P \neq 0_n$. □

Proposition 3.6 *If P is a projector on a subspace $W \neq \{0\}$ of a vector space $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, then $PX \neq 0$, for some $n \times 1$ non-zero vector X.*

Proof: Since P is projector on $W \neq \{0\}$, we have $P \neq \mathbf{0}_n$ [Theorem 3.5]. We know that the rank of any $n \times n$ non-zero matrix P is at least 1 and at most n. Since rank of P is at least 1, range space of P with respect to some linear transformation contains a non-zero vector.
i.e; $\exists$ a $n \times 1$ non-zero vector X such that $PX \neq 0$. □

Theorem 3.7 *If $W_1, W_2, ..., W_m$ are non-zero d-dimensional subspaces of $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, then projectors P_1, P_2, ..., P_m on $W_1, W_2, ..., W_m$ with respect to orthonormal basis of $W_1, W_2, ..., W_m$ respectively are similar.*

Proof: Here, $d \neq 0$, as $W_1, W_2, ..., W_m$ are non-zero subspaces of V.
If $d = n$, then $W_1 = W_2 = ... = W_m = V = \mathbb{C}^{(n)}(\mathbb{C})$. Therefore, projectors P_1, P_2, ..., P_m on $W_1, W_2, ..., W_m$ are $P_1 = P_2 = ... = P_m = I_n$ [Theorem 3.2].
Now, let $1 \leq d \leq (n-1)$, $n \geq 2$. Then $W_1, W_2, ..., W_m$ are proper subspaces of $V = \mathbb{C}^{(n)}(\mathbb{C})$.

Let $B_i = \left\{ \begin{pmatrix} a_{1k(i)} \\ a_{2k(i)} \\ \cdot \\ \cdot \\ \cdot \\ a_{nk(i)} \end{pmatrix} \middle| 1 \leq k \leq d \right\}$ be the orthonormal basis of W_i, $(1 \leq i \leq m)$.

Then B_1 is the orthonormal basis of W_1, B_2 is the orthonormal basis of W_2 and so on. Let P_i be the projector on W_i with respect to orthonormal basis B_i $(1 \leq i \leq m)$. For orthonormal basis B_1, we have

$$|a_{1k(1)}|^2 + ... + |a_{nk(1)}|^2 = 1, 1 \leq k \leq d \tag{3.9}$$

The projector P_1 on W_1 with respect to orthonormal basis B_1 is given by,

$$P_1 = \begin{pmatrix} a_{11(1)} \\ a_{21(1)} \\ . \\ . \\ . \\ a_{n1(1)} \end{pmatrix} \begin{pmatrix} a_{1\bar{1}(1)} & a_{2\bar{1}(1)} & ... & a_{n\bar{1}(1)} \end{pmatrix} + ... + \begin{pmatrix} a_{1d(1)} \\ a_{2d(1)} \\ . \\ . \\ . \\ a_{nd(1)} \end{pmatrix} \begin{pmatrix} a_{1\bar{d}(1)} & a_{2\bar{d}(1)} & ... & a_{n\bar{d}(1)} \end{pmatrix}.$$

$$= \begin{pmatrix} |a_{11(1)}|^2 + ... + |a_{1d(1)}|^2 & ... & ... \\ ... & |a_{21(1)}|^2 + ... + |a_{2d(1)}|^2 & ... \\ . & . & . \\ . & . & . \\ . & . & . \\ ... & ... & |a_{n1(1)}|^2 + ... + |a_{nd(1)}|^2 \end{pmatrix}.$$

Then trace of $P_1 = d$ [By equation (3.9)]
Therefore, Rank of P_1 = d (as P_1 is idempotent matrix). Also we have, P_1 is diagonizable over $\mathbb{C}$ and the only possible eigen values of P_1 are 0 & 1.
Hence, rank of P_1 is the number of non-zero eigen values of P_1 which is equal to d. Therefore, 1 is the eigen value of P_1 with multiplicity d.
Then Characteristic polynomial of P_1 is $x^{n-d}(x-1)^d$ and minimal polynomial of P_1 is $x(x-1)$. When we extend similar argument for P_2, ..., P_m we get, $x^{n-d}(x-1)^d$ is the characteristic polynomial for P_2, ..., P_m and $x(x-1)$ is the minimal polynomial for P_2, ..., P_m. Since P_1, P_2, ..., P_m all are diagonizable matrices with same characteristic polynomial, P_1,P_2, ..., P_m have same Jordan canonical form.
Hence, P_1, P_2, ..., P_m are similar matrices.

□

Theorem 3.8 *If $W \neq \{0\}$ is a subspace of $V = \mathbb{C}^2(\mathbb{C})$, then projectors on W with respect to different orthonormal bases of W are identical.*

Proof: If $W = V = \mathbb{C}^2(\mathbb{C})$. Then projector P on W with respect to any orthonormal basis of W is I_2 [Theorem 3.2].
Let W be a proper subspace of V. Let $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid ax + by = 0,\ a, b \in \mathbb{C} \text{ \& either } a \neq 0 \text{ or } b \neq 0 \right\}$.
Let $\left\{ \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right\}$, $\left\{ \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right\}$,...,$\left\{ \begin{pmatrix} a_k \\ b_k \end{pmatrix} \right\}$ be k different orthonormal bases of W.
Let P_m be the projector on W with respect to orthonormal basis B_m, where $B_m = \left\{ \begin{pmatrix} a_m \\ b_m \end{pmatrix} \right\}$, for all $m = 1, 2, ..., k$.
Then P_1 is the projector on W with respect to orthonormal basis $B_1 = \left\{ \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right\}$, P_2 is the projector on W with respect to orthonormal basis B_2=$\left\{ \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right\}$ and so on. By definition of projector, we have
$P_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \begin{pmatrix} \bar{a_1} & \bar{b_1} \end{pmatrix} = \begin{pmatrix} |a_1|^2 & a_1\bar{b_1} \\ b_1\bar{a_1} & |b_1|^2 \end{pmatrix}$
Similarly, $P_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \begin{pmatrix} \bar{a_2} & \bar{b_2} \end{pmatrix} = \begin{pmatrix} |a_2|^2 & a_2\bar{b_2} \\ b_2\bar{a_2} & |b_2|^2 \end{pmatrix}$ and when we continue, we get $P_k = \begin{pmatrix} a_k \\ b_k \end{pmatrix} \begin{pmatrix} \bar{a_k} & \bar{b_k} \end{pmatrix} = \begin{pmatrix} |a_k|^2 & a_k\bar{b_k} \\ b_k\bar{a_k} & |b_k|^2 \end{pmatrix}$. We have to show $P_1 = P_2 = ... = P_k$.
Case-1: If $a = 0$, $b \neq 0$, then $b_1 = b_2 = ... = b_k = b_1\bar{b_1} = b_2\bar{b_2} = ... = b_k\bar{b_k} = a_1\bar{b_1} = a_2\bar{b_2} = ... =$

$a_k\bar{b_k}= b_1\bar{a_1}= b_2\bar{a_2}= ... = b_k\bar{a_k}= 0$. Also, $a_1\bar{a_1}= a_2\bar{a_2}= ... = a_k\bar{a_k}= 1$ (as P_1, P_2, ... , P_k are non-zero idempotent matrices). In this case $P_1 = P_2 = ... = P_k$.
Case-2: If $a \neq 0$, $b = 0$, then similar as in case-1, we have $P_1 = P_2 = ... = P_k$.
Case-3: If $a \neq 0$, $b \neq 0$. Then $|a_1|^2 + |b_1|^2 = |a_2|^2 + |b_2|^2 = ... = |a_k|^2 + |b_k|^2 = 1$ and $aa_1 + bb_1 = aa_2 + bb_2 = ... = aa_k + bb_k = 0$.
Since, $aa_1 + bb_1 = aa_2 + bb_2 = 0$, we have $a(a_1 - a_2) = b(b_2 - b_1) = 0$.
Thus, $\frac{a}{b}(a_1 - a_2) = b_2 - b_1 = 0$.
So, $a_1 - a_2 = b_1 - b_2 = 0$ as $\frac{a}{b} \neq 0$. Therefore, $a_1 = a_2$ and $b_1 = b_2$.
Thus, $a_1\bar{a_1} = a_2\bar{a_2}$, $a_1\bar{a_1} = a_2\bar{a_2}$, $a_1\bar{b_1} = a_2\bar{b_2}$ and $b_1\bar{a_1} = b_2\bar{a_2}$. This proves $P_1 = P_2$. If we use the relation $aa_2 + bb_2 = aa_3 + bb_3 = 0$, then by above process, we get, $P_2 = P_3$ i.e; $P_1 = P_2 = P_3$. Further, when we extend the similar argument we get, $P_3 = P_4$, $P_4 = P_5$,...,$P_{(k-1)} = P_k$.
Thus, $P_1 = P_2 = ... = P_k$.
Hence, all the projectors on W are identical . □

Theorem 3.9 *If W is a proper subspace of $V = \mathbb{C}^{(n)}(\mathbb{C})(n \geq 2)$, then any projector P on W is singular.*

Proof: Let $dimW = x$. Then $1 \leq x \leq (n-1)$.

Let $B = \left\{ \begin{pmatrix} a_{1k} \\ a_{2k} \\ . \\ . \\ . \\ a_{nk} \end{pmatrix} \middle| 1 \leq k \leq x \right\}$ be an orthonormal basis of W.

Therefore,

$$|a_{1k}|^2 + ... + |a_{nk}|^2 = 1; 1 \leq k \leq x \tag{3.10}$$

By definition of the Projector,

$$P = \begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{n1} \end{pmatrix} \begin{pmatrix} \bar{a_{11}} & \bar{a_{21}} & ... & \bar{a_{n1}} \end{pmatrix} + ... + \begin{pmatrix} a_{1x} \\ a_{2x} \\ . \\ . \\ . \\ a_{nx} \end{pmatrix} \begin{pmatrix} \bar{a_{1x}} & \bar{a_{2x}} & ... & \bar{a_{nx}} \end{pmatrix}$$

$$= \begin{pmatrix} |a_{11}|^2 + ... + |a_{1x}|^2 & ... & ... \\ ... & |a_{21}|^2 + ... + |a_{2x}|^2 & ... \\ . & . & . \\ . & . & . \\ . & . & . \\ ... & ... & |a_{n1}|^2 + ... + |a_{nx}|^2 \end{pmatrix}.$$

Here, trace of $P = (|a_{11}|^2 + |a_{21}|^2 + ... + |a_{n1}|^2) + ... + (|a_{1x}|^2 + |a_{2x}|^2 + ... + |a_{nx}|^2) = x$ [Using equation 3.10].
Thus, trace of $P = x < n$.
Now, P is an idempotent matrix. For any idempotent matrix P, we have rank of P = trace of P.
Thus, rank of $P = x < n$.
Hence, P is singular, as any $n \times n$ matrix P is non-singular if an only if rank of $P = n$. □

Proposition 3.10 *If P is a projector on a subspace W of a vector space $V = \mathbb{C}^{(n)}(\mathbb{C})$ $(n \geq 1)$, then P^k is also a projector on W, $\forall k \in N$.*

Proof: Here, P is a projector on W. Therefore, P is idempotent and hermitian [Theorem 3.1].
So,

$$P = P^2 \tag{3.11}$$

$$P = P^* \tag{3.12}$$

We prove the theorem using mathematical induction.
Step 1: For $k = 2$, $P^2 = P$ is trivial.
The statement is true for $k = 2$.
Step 2: For $k = m$, Let $P^m = P$. Then we show that the result is true for $k = m + 1$.
Step 3: For $k = m + 1$, $P^{m+1} = P^m P = PP = P^2 = P$.
The statement is true for $k = m + 1$.

Hence, P^k is also a projector on W, $\forall k \in N$. □

4. Conclusion

In this paper we try to visualize projectors on $\mathbb{C}^{(n)}(\mathbb{C}), n \geq 1$ and its subspaces. It is established that projectors on any subspace $W \neq \{0\}$ of $\mathbb{C}^{(n)}(\mathbb{C})$ are all non-zero.

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