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A New Class of Bernstein Operators with Shifted Knots and Statistical Approximation

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ABSTRACT: This article introduces a generalized Bernstein operators with two shifted nodes. We first establish fundamental approximation tools, including estimates for moments and central moments. Using these, we prove a Korovkin-type convergence theorem adapted to the scaling behavior induced by the fractional integral. Further, we discuss convergence theorems and order of approximation in terms of first order modulus of smoothness. Next, we study pointwise approximation results in terms of Peetre's K-functional, second order modulus of smoothness, Lipschitz type space and r^{th} order Lipschitz type maximal function. Lastly, weighted approximation results and statistical approximation theorems are proved.

Key Words: Bernstein–Kantorovich operators, modulus of continuity, rate of convergence, statistical convergence, Korovkin-type theorem.

Contents

1. Introduction and Preliminaries

Approximation theory provides a foundational framework for representing complex functions with simpler mathematical constructs, making it indispensable across a wide range of disciplines. In mathematics and engineering, it underpins computational techniques for solving differential equations, modeling geometric shapes, and designing efficient numerical algorithms. Beyond classical applications, it plays a pivotal role in modern fields such as computer graphics, where it enables realistic rendering of curves and surfaces, and computer algebra systems, which rely on approximation for symbolic computation.

The influence of approximation theory extends into applied domains such as control theory, where concepts like control points and control nets are essential for analyzing parametric curves and surfaces-fundamental tools in the design of engineering systems ([1], [2]). In recent years, the rise of artificial intelligence, data science, and machine learning has further expanded its relevance. Approximation-theoretic methods are now integral to developing algorithms for data analysis, pattern recognition, and predictive modeling, enabling the construction of models that capture intricate relationships within datasets. Ongoing research in medical science and related fields continues to explore and extend these techniques ([3], [4], [5]), underscoring the enduring and expanding role of approximation theory in both theoretical and applied contexts.

The first sequence of operators to support above application part was introduced by Bernstein [6]. Although, his motive was to provide a short and elegant proof of the approximation theorem of Weierstrass with the assistance of binomial distribution as:

$$B_l(g;u)=\sum_{\nu=0}^{l}\left(\begin{array}{c} l\\ \nu \end{array}\right)u^{\nu}(1-u)^{l-\nu}g\left(\frac{\nu}{l}\right),\quad u\in[0,1], \tag{1.1}$$

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where g belongs to $C[0,1]$. He proved that these operators approximate uniformly on $[0,1]$ to every continuous function $g \in C[0,1]$. The Bernstein operators have been one of the most extensively examined positive linear operators in the area of approximation theory. However, these operator are not applicable for discontinuous functions. Further, Kantorovich [7] introduced a sequence of operators which is a generalization of the operators (1.1) over the space of Lebesgue integrable functions $L[0,1]$ as:

$$K_l(g;u) = (l+1)\sum_{\nu=0}^{l}\left(\begin{array}{c} l \\ \nu \end{array}\right) u^{\nu}(1-u)^{l-\nu}\int_{\frac{u}{l+1}}^{\frac{\nu+1}{l+1}} g(t)dt, \quad u \in [0,1].$$

Over the past decade, many generalizations, as well as modifications of Bernstein and Kantorovich operators are presented by many authors and researchers, e.g., Alotaibi et al. [8], [9], Raza et al. [10], Arif et al. [11], Mohiuddine et al. ([12], [13]), Aslan et al. [14], [15], [16], Nasiruzzaman et al. [17,21], Nasiruzzaman [18], Özger et al. [19], [20], Ayman-Mursaleen et al. [22], Rao et al. [23], Berwal et al. [24], Alotaibi [25], Kajla et al. [26], Alotaibi et al. [27], Savaş and Patterson [28], etc.

Recently, Usta ([29]) presented a new sequence of Bernstein operators for the function g, which are continuous and defined on $(0,1)$ with $u \in (0,1)$ as follows:

$$P_l^*(g;u) = \frac{1}{l}\sum_{\nu=0}^{l}\left(\begin{array}{c} l \\ \nu \end{array}\right)(\nu - lu)^2 u^{\nu-1}(1-u)^{l-\nu-1} g\left(\frac{\nu}{l}\right), \quad l \in \mathbb{N}. \tag{1.2}$$

Remark 1.1 *These operators given in (1.2) are restricted for the space of continuous functions only.*

In 2025, Jha et al. [30] introduced stancu variant of the operators defined in (1.2) to achieve better flexibility in approximation by using two shifted notde $0 \leq \alpha \leq \beta$ as:

$$P_l^*(g;u) = \frac{1}{l}\sum_{\nu=0}^{l}\left(\begin{array}{c} l \\ \nu \end{array}\right)(\nu - lu)^2 u^{\nu-1}(1-u)^{l-\nu-1} g\left(\frac{\nu+\alpha}{l+\beta}\right), \quad l \in \mathbb{N}. \tag{1.3}$$

In addition of the above literature and to discuss the more flexibility in the approximations results given by operators (1.3). we define generalized Bernstein type operators in Schurer sense as: For $u \in [0, 1+p]$ and $0 < p < \infty$

$$G_{l+p}^{\alpha,\beta}(g;u) = \frac{1}{l+p}\sum_{\nu=0}^{l+p}\left(\begin{array}{c} l \\ \nu \end{array}\right)(\nu - (l+p)u)^2 u^{\nu-1}(1-u)^{l+p-\nu-1} g\left(\frac{\nu+\alpha}{l+p+\beta}\right), \quad l \in \mathbb{N}. \tag{1.4}$$

Remark 1.2 *For any $g, h \in C[0,1+p]$ and $a_1, a_2 \in \mathbb{R}$, we have*

$$G_{l+p}^{\alpha,\beta}(a_1 g + a_2 h; u) = a_1 G_{l+p}^{\alpha,\beta}(g;u) + a_2 G_{l+p}^{\alpha,\beta}(h;u).$$

Which implies that the operator $G_{l+p}^{\alpha,\beta}(.;.)$ is linear operator.

Remark 1.3 *Also for any $g \geq 0$, we must have $G_{l+p}^{\alpha,\beta}(g;u) \geq 0$, which shows that the sequence of operators are positive.*

The structure of our research work is organized as follows: Section 1 compute some estimates for the operators 1.4 in terms of test functions and central moments. In section 2, we study the uniform convergence theorem and approximation order via of Korovkin theorem and first order modulus of continuity respectively. In section 3, we discuss the local and global approximation results using first and second order modulus of continuity, Peetre's K-functional and weight functions in several functional spaces.

To discuss the existence and convergence of operators (1.4), we Consider $e_i(t) = t^i$, $i = 0,1,2$. Then, in the following Lemmas (1.2) and (1.3), we estimate the operators introduced in terms of central moments and test functions.

Lemma 1.1 *[30] For the operators $P_n^*(.;.)$, we have following identities are obtained*

$$\begin{aligned} P_l^*(e_0;u) &= 1, \\ P_l^*(e_1;u) &= \frac{1}{l+\beta}\left\{(l-2)u+1+\alpha\right\}, \\ P_l^*(e_2;u) &= \frac{1}{(l+\beta)^2}\left\{(l^2-7l+6)u^2+(5l+2l\alpha-6-4\alpha)u+1+2\alpha+\alpha^2\right\}. \end{aligned}$$

Lemma 1.2 *Let $G_{l+p}^{\alpha,\beta}(.;.)$ be presented in (1.4). Then, the following identities are acquired*

$$\begin{aligned} G_{l+p}^{\alpha,\beta}(e_0;u) &= 1, \\ G_{l+p}^{\alpha,\beta}(e_1;u) &= \frac{1}{l+p+\beta}\left\{(l+p-2)u+1+\alpha\right\}, \\ G_{l+p}^{\alpha,\beta}(e_2;u) &= \frac{1}{(l+p+\beta)^2}\Bigg\{((l+p)^2-7l+6)u^2+(5(l+p)+2(l+p)\alpha-6-4\alpha)u \\ &+ 1+2\alpha+\alpha^2\Bigg\}. \end{aligned}$$

Proof: From the result of Lemma 1.1, we prove above results of lemma 1.2. □

Lemma 1.3 *Let $\psi_u^i(t)=(t-u)^i$, $i=0,1,2$. Then, we have the central moments of generalized Bernstein Stancu operators (1.4) as follows:*

$$\begin{aligned} G_{l+p}^{\alpha,\beta}(\psi_u^0(t);u) &= 1, \\ G_{l+p}^{\alpha,\beta}(\psi_u^1(t);u) &= \frac{1}{l+p+\beta}\left\{-(2+\beta)u1+\alpha\right\}, \\ G_{l+p}^{\alpha,\beta}(\psi_u^2(t);u) &= \frac{1}{(l+p+\beta)^2}\Bigg\{(6-3(l+p)+4\beta+\beta^2)u^2 \\ &+ (3(l+p)-6-4\alpha-2\beta-2\alpha\beta)u+(1+2\alpha+\alpha^2)\Bigg\}. \end{aligned}$$

Proof: In the light of Lemma 1.2 and linearity properties, we can easily prove Lemma 1.3. □

Remark 1.4 *If we take $p=0$, then the estimates of the operators (1.4) reduces to the estimates of the operators defined by Jha et al. ([30]).*

Remark 1.5 *It is also noticed that $G_{l+p}^{\alpha,\beta}(\psi_u^2(t);u)\leq P_l^*(\psi_u^2(t);u)$, which shows that the rate of convergence of operators (1.4) is faster than operators ([30]).*

2. Rapidity of Convergence and Order of Approximation

Definition 2.1 *Let $g\in C[0,1+p]$. Then, the modulus of continuity is defined as:*

$$\omega(g;\tilde{\eta})=\sup_{|u_1-u_2|\leq\tilde{\eta}}|g(u_1)-g(u_2)|, \qquad u_1,u_2\in[0,1+p].$$

Theorem 2.1 *Let $G_{l+p}^{\alpha,\beta}(.;.)$ be given in (1.4). Then, for all $g\in C_B[0,1+p]$, $G_{l+p}^{\alpha,\beta}(g;u)\rightrightarrows g$ on each bounded and closed subset of $[0,1+p]$, where $\rightrightarrows$ symbol denotes uniform convergence.*

Proof: Using Korovkin result which implies the convergence is uniform, it is adequate to see that

$$\lim_{l\to\infty} G^{\alpha,\beta}_{l+p}(t^i;u) = u^i,\ i = 0,1,2,$$

uniformly on $(0,1)$. In view of Lemma 1.2, we can arrive at the desired result. □

In the light of Shisha et al. [39], one can show the order of approximation via Ditzian-Totik modulus of continuity.

Theorem 2.2 *Let $g \in C_B(0,1)$. Then, operators $G^{\alpha,\beta}_{l+p}(.;.)$ given in (1.4), we have*

$$|G^{\alpha,\beta}_{l+p}(g;u) - g(u)| \leq 2\omega(g;\tilde{\eta}),$$

where $\tilde{\eta} = \sqrt{G^{\alpha,\beta}_{l+p}((t-u)^2;u)}$.

3. Direct Results

Here, we recall a functional space as: $C_B(0,1)$, where $C_B(0,1)$ denotes a space of continuous and bounded functions and Peetre's K-functional is as:

$$K_2(g,\tilde{\eta}) = \inf_{h\in C^2_B[0,1+p]} \left\{ \|g-h\|_{C_B[0,1+p]} + \tilde{\eta}\|h''\|_{C^2_B[0,1+p]} \right\},$$

where $C^2_B[0,1+p] = \{h \in C_B[0,1+p] : h', h'' \in C_B[0,1+p]\}$ endowed with the norm

$$\|g\| = \sup_{0<u<1} |g(u)|.$$

Further, we recall second order Ditzian-Totik modulus of continuity is as:

$$\omega_2(g;\sqrt{\tilde{\eta}}) = \sup_{0<\nu\leq\sqrt{\tilde{\eta}}}\ \sup_{u\in[0,1+p]} |g(u+2\nu) - 2g(u+\nu) + g(u)|.$$

We also have a relation from [31] page no. 177, Theorem 2.4 as follows:

$$K_2(g;\tilde{\eta}) \leq \tilde{C}\omega_2(g;\sqrt{\tilde{\eta}}), \tag{3.1}$$

where $\tilde{C}$ is an absolute constant. Next, in order to discuss the approximation result, we consider the auxiliary sequence of operator as:

$$\widehat{G}^{\alpha,\beta}_{l+p}(g;u) = G^{\alpha,\beta}_{l+p}(g;u) + g(u) - g\left(\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}\right), \tag{3.2}$$

where $g \in C_B[0,1+p]$, $u \geq 0$ and $l > 2$. From (3.2), we get

$$\widehat{G}^{\alpha,\beta}_{l+p}(\psi^0_u(t);u) = 1,\ \widehat{G}^{\alpha,\beta}_{l+p}(\psi^1_u(t);u) = 0 \text{ and } |\widehat{G}^{\alpha,\beta}_{l+p}(g;u)| \leq 3\|g\|. \tag{3.3}$$

Lemma 3.1 *For $u \geq 0$ and $l > 2$, we have*

$$|\widehat{G}^{\alpha,\beta}_{l+p}(g;u) - g(u)| \leq \theta(u)\|g''\|,$$

where $g \in C^2_B[0,1+p]$ and $\theta(u) = \widehat{G}^{\alpha,\beta}_{l+p}(\psi^2_u(t);u) + (\widehat{G}^{\alpha,\beta}_{l+p}(\psi^1_u(t);u))^2$.

Proof: For $h \in C^2_B[0,1+p]$ and in the direction of Taylor's theorem, we have

$$g(t) = g(u) + (t-u)g'(u) + \int_u^t (t-v)g''(v)dv. \tag{3.4}$$

Now, by using the auxiliary operators $\widehat{G}_{l+p}^{\alpha,\beta}(.;.)$ introduced in (3.2) on both the sides in the above (3.4), we have

$$\widehat{G}_{l+p}^{\alpha,\beta}(g;u)-g(u) \quad = \quad g'(u)\widehat{G}_{l+p}^{\alpha,\beta}(\psi_u^1(t);u)+\widehat{G}_{l+p}^{\alpha,\beta}\Big(\int\limits_u^t(t-v)g''(v)dv;u\Big).$$

In the light of (3.3) and (3.4), we have

$$\begin{aligned}\widehat{G}_{l+p}^{\alpha,\beta}(g;u)-g(u) &= \widehat{G}_{l+p}^{\alpha,\beta}\Big(\int\limits_u^t(t-v)g''(v)dv;u\Big)\\ &= G_{l+p}^{\alpha,\beta}\Big(\int\limits_u^t(t-v)g''(v)dv;u\Big)\\ &- \int\limits_u^{\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}}\left(\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}-v\right)g''(v)dv,\end{aligned}$$

$$\begin{aligned}|\widehat{G}_{l+p}^{\alpha,\beta}(g;u)-g(u)| &\leq \left|G_{l+p}^{\alpha,\beta}\Big(\int\limits_u^t(t-v)g''(v)dv;u\Big)\right|\\ &+\left|\int\limits_u^{\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}}\left(\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}-v\right)g''(v)dv\right|.\end{aligned} \tag{3.5}$$

Since,

$$\left|\int\limits_u^t(t-v)g''(v)dv\right|\leq(t-u)^2\parallel g''\parallel, \tag{3.6}$$

therefore

$$\left|\int\limits_u^{\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}}\left(\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}-v\right)g''(v)dv\right|\leq\left(\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}-u\right)^2\parallel g''\parallel. \tag{3.7}$$

In account of (3.5), (3.6) and (3.7), we find

$$\begin{aligned}|\widehat{G}_{l+p}^{\alpha,\beta}(g;u)-g(u)| &\leq \left\{\widehat{G}_{l+p}^{\alpha,\beta}(\psi_u^2(t);u)+\left(\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}-u\right)^2\right\}\|g''\|\\ &= \theta(u)\|g''\|.\end{aligned}$$

We arrive at the required result. □

Theorem 3.1 *For $g\in C_B^2[0,1+p]$, there exists a constant $\tilde{C}>0$ such that*

$$\mid G_{l+p}^{\alpha,\beta}(g;u)-g(u)\mid\leq\tilde{C}\omega_2\big(g;\sqrt{\theta(u)}\big)+\omega(g;G_{l+p}^{\alpha,\beta}(\psi_u^1(t);u)),$$

where $\theta(u)$ is in Lemma 3.1.

Proof: For $g \in C_B[0,1+p]$ and $h \in C_B^2[0,1+p]$ and in account of the definition of $\widehat{G}_{l+p}^{\alpha,\beta}(.;.)$, we yield

$$|G_{l+p}^{\alpha,\beta}(g;u) - g(u)| \leq |\widehat{G}_{l+p}^{\alpha,\beta}(g-h;u)| + |(g-h)(u)| + |\widehat{G}_{l+p}^{\alpha,\beta}(h;u) - h(u)| + \left| g\Big(\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}\Big) - g(u)\right|.$$

In the direction of Lemma 3.1 and inequalities in (3.3), we yield

$$\begin{aligned}|G_{l+p}^{\alpha,\beta}(g;u)-g(u)| &\leq 4\|g-h\|+|\widehat{G}_{l+p}^{\alpha,\beta}(h;u)-h(u)|+\left|g\left(\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}\right)-g(u)\right| \\ &\leq 4\|g-h\| + \theta(u)\|h''\| + \omega\Big(g; G_{l+p}^{\alpha,\beta}((t-u);u)\Big).\end{aligned}$$

In view of (3.1), we arrived at required result. □

Here, we recall the next result in Lipschitz type space [38], which is given as:

$$Lip_{\tilde{M}}^{\zeta_1,\zeta_2}(\gamma) := \Big\{ g \in C_B[0,1+p] : |g(t)-g(u)| \leq \tilde{M}\frac{|t-u|^\gamma}{(t+\zeta_1 u+\zeta_2 u^2)^{\frac{\gamma}{2}}} : u,t \in (0,1)\Big\},$$

where $0 < \gamma \leq 1$, $\tilde{M} > 0$ and $\zeta_1, \zeta_2 > 0$.

Theorem 3.2 *Let $g \in Lip_M^{\zeta_1,\zeta_2}(\gamma)$ and the operators $G_{l+p}^{\alpha,\beta}(.;.)$ given in (1.4). Then, we have*

$$|G_{l+p}^{\alpha,\beta}(g;u) - g(u)| \leq \tilde{M}\left(\frac{\lambda(u)}{\zeta_1 u + \zeta_2 u^2}\right)^{\frac{\gamma}{2}}, \tag{3.8}$$

where $0 < \gamma \leq 1$, $\zeta_1, \zeta_2 \in (0,1)$ and $\lambda(u) = G_{l+p}^{\alpha,\beta}(\psi_u^2(t);u)$.

Proof: For $\gamma = 1$ and $u > 0$, we yield

$$\begin{aligned}|G_{l+p}^{\alpha,\beta}(g;u) - g(u)| &\leq G_{l+p}^{\alpha,\beta}(|g(t)-g(u)|;u) \\ &\leq \tilde{M} G_{l+p}^{\alpha,\beta}\left(\frac{|t-u|}{(t+\zeta_1 u+\zeta_2 u^2)^{\frac{1}{2}}};u\right).\end{aligned}$$

Since $\dfrac{1}{t+\zeta_1 u+\zeta_2 u^2} < \dfrac{1}{\zeta_1 u+\zeta_2 u^2}$, for all $u \in (0,1)$, we get

$$\begin{aligned}|G_{l+p}^{\alpha,\beta}(g;u) - g(u)| &\leq \frac{\tilde{M}}{(\zeta_1 u+\zeta_2 u^2)^{\frac{1}{2}}}(G_{l+p}^{\alpha,\beta}(\psi_u^2(t);u))^{\frac{1}{2}} \\ &\leq \tilde{M}\left(\frac{\lambda(u)}{\zeta_1 u+\zeta_2 u^2}\right)^{\frac{1}{2}},\end{aligned}$$

which proves the result of Theorem 3.2 holds for $\gamma = 1$. Further, we consider for $\gamma \in (0,1)$ with the account of Hölder's inequality via $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, one get

$$\begin{aligned}|G_{l+p}^{\alpha,\beta}(g;u) - g(u)| &\leq \big(G_{l+p}^{\alpha,\beta}(|g(t)-g(u)|^{\frac{2}{\gamma}};u)\big)^{\frac{\gamma}{2}} \\ &\leq \tilde{M}\left(G_{l+p}^{\alpha,\beta}\left(\frac{|t-u|^2}{(t+\zeta_1 u+\zeta_2 u^2)};u\right)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Since $\dfrac{1}{t+\zeta_1 u+\zeta_2 u^2} < \dfrac{1}{\zeta_1 u+\zeta_2 u^2}$, for all $u \in (0,1)$, we have

$$|G^{\alpha,\beta}_{l+p}(g;u) - g(u)| \leq \tilde{M}\left(\frac{G^{\alpha,\beta}_{l+p}(|t-u|^2;u)}{\zeta_1 u+\zeta_2 u^2}\right)^{\frac{\gamma}{2}} \leq \tilde{M}\Big(\frac{\lambda(u)}{\zeta_1 u+\zeta_2 u^2}\Big)^{\frac{\gamma}{2}}.$$

Hence, we get the proof of Theorem 3.2. □

Moreover, we investigate approximation result locally in the direction of r^{th} order modulus of continuity. Lenze [37] gave the Lipschitz-type maximal function as:

$$\widetilde{\omega}_r(g;u) = \sup_{t\neq u, t\in(0,\infty)} \frac{|g(t)-g(u)|}{|t-u|^r}, \ u \in [0,1+p] \text{ and } r \in (0,1]. \tag{3.9}$$

Theorem 3.3 *For $g \in C_B[0,1+p]$ and $r \in (0,1]$. and for all $u \in [0,1+p]$, one get*

$$|G^{\alpha,\beta}_{l+p}(g;u) - g(u)| \leq \widetilde{\omega}_r(g;u)\Big(\lambda(u)\Big)^{\frac{r}{2}}.$$

Proof: It is found that

$$|G^{\alpha,\beta}_{l+p}(g;u) - g(u)| \leq G^{\alpha,\beta}_{l+p}(|g(t)-g(u)|;u).$$

In view of (3.9), we have

$$|G^{\alpha,\beta}_{l+p}(g;u) - g(u)| \leq \widetilde{\omega}_s(g;u) G^{\alpha,\beta}_{l+p}(|t-u|^r;u).$$

In account of Hölder's inequality witth the aid of $p=\frac{2}{r}$ and $q=\frac{2}{2-r}$, we have

$$|G^{\alpha,\beta}_{l+p}(g;u) - g(u)| \leq \widetilde{\omega}_r(g;u)\big(G^{\alpha,\beta}_{l+p}(|t-u|^2;u)\big)^{\frac{r}{2}}.$$

Which completes the required result. □

4. Approximation Properties Globally

Suppose that $\nu(u) = 1+u^2, 0<u<1$ is a weight function. Then, $B_\nu(0,1) = \{g(u) : |g(u)| \leq \tilde{M}_g(1+u^2)$, here $\tilde{M}_g$ is a constant based on g and $C_\nu(0,1)$ represents the space of continuous function in $B_\nu(0,1)$ equipped with $\|g(u)\|_\nu = \sup\limits_{u\in(0,1)} \frac{|g(u)|}{\nu(u)}$ and $C^{\tilde{k}}_\nu(0,1) = \{g \in C_\nu(0,1) : \lim\limits_{u\to\infty} \frac{g(u)}{\nu(u)} = \tilde{k}$, where constant $\tilde{k}$ is depending on $g\}$.

Ditzian-Totik modulus of smoothness for the function g defined on the closed interval $[a,b]$ with $a,b>0$ is defined by

$$\omega_b(g,\tilde{\eta}) = \sup_{|t-u|\leq\tilde{\eta}} \sup_{u,t\in[a,b]} |g(t)-g(u)|. \tag{4.1}$$

One can easily note that for any $g \in C_\nu(0,1)$, the modulus of smoothness given in (4.1) tends to zero.

Theorem 4.1 *Let $g \in C_\nu(0,1)$ and the modulus of continuity $\omega_{b+1}(g;\tilde{\eta})$ be given on $[a,b+1] \subset (0,1)$. Then, for any $u \in [a,b]$, we get*

$$\|G^{\alpha,\beta}_{l+p}(.;.) - g\|_{C[a,b]} \leq 4\tilde{M}_g(1+b^2)\tilde{\eta}_s(b) + 2\omega_{b+1}(g;\sqrt{\tilde{\eta}_s(b)}),$$

where $\tilde{\eta}_s(b) = \max\limits_{u\in[a,b]} G^{\alpha,\beta}_{l+p}(\psi^2_u(t);u)$.

Proof: For $u \in [a,b]$ and $t \in (0,1)$, we get

$$|g(t)-g(u)| \leq 4\tilde{M}_g(1+b^2)(t-u)^2 + \left(1+\frac{|t-u|}{\tilde{\eta}}\right)\omega_{b+1}(g;\tilde{\eta}). \tag{4.2}$$

Using operator $G_{l+p}^{\alpha,\beta}(.;.)$ on in 4.2, we get

$$|G_{l+p}^{\alpha,\beta}(g;u)-g(u)| \leq 4\tilde{M}_g(1+b^2)G_{l+p}^{\alpha,\beta}((\psi_u^2(t);u) + \left(1+\frac{G_{l+p}^{\alpha,\beta}(|t-u|;u)}{\tilde{\eta}}\right)\omega_{b+1}(g;\tilde{\eta}).$$

Further, in the light of Lemma 1.3 and $u \in [a,b]$, we have

$$|G_{l+p}^{\alpha,\beta}(.;.)-g| \leq 4\tilde{M}_g(1+b^2)\tilde{\eta}_l(b) + \left(1+\frac{\sqrt{\tilde{\eta}_l(b)}}{\tilde{\eta}}\right)\omega_{b+1}(g;\tilde{\eta}).$$

We take $\tilde{\eta} = \tilde{\eta}_l(b)$. Then, one can easily arrive at the desired result. □

Remark 4.1 *Throughout the paper, we consider test function as $e_i(t)=t^i$, $i=0,1,2$.*

Theorem 4.2 *([35], [36]) Suppose that the sequence of positive linear operators $(L_n)_{n\geq 1}$ acting from $C_\nu(0,1)$ to $B_\nu(0,1)$ satisfies the conditions*

$$\lim_{n\to\infty} ||L_n(e_i;.)-e_i||_\nu = 0, \quad where\ i=0,1,2,$$

then, for $g \in C_\nu^{\tilde{k}}(0,1)$, we have

$$\lim_{n\to\infty} ||L_n g - g||_\nu = 0.$$

Theorem 4.3 *For $g \in C_\nu^{\tilde{k}}(0,1)$. Then, we have*

$$\lim_{l\to\infty} \|G_{l+p}^{\alpha,\beta}(g;u)-g\|_\nu = 0.$$

Proof: To prove the result of above Theorem, it is adequate to show

$$\lim_{l\to\infty} \|G_{l+p}^{\alpha,\beta}(e_i;.)-e_i\|_\nu = 0, for\ i=0,1,2.$$

In account of Lemma 1.2, it is obvious $\|G_{l+p}^{\alpha,\beta}(e_0;.)-1\|_\nu = 0$, also

$$\begin{aligned}\|G_{l+p}^{\alpha,\beta}(e_1;.)-e_1\|_{\nu(u)} &= \sup_{u\in(0,1)} \frac{1}{\nu(u)}\left|\frac{2(l+p-2)u+3+2\alpha}{l+p+\beta+1}-u\right| \\ &= \frac{1}{l+\beta+1}\sup_{u\in(0,1)}\frac{(l+p-5-\beta)u}{1+u^2} + \frac{1}{l+p+\beta+1}\sup_{u\in(0,1)}\frac{3+2\alpha}{1+u^2}.\end{aligned}$$

For a large value of l, we get $\|G_{l+p}^{\alpha,\beta}(e_1;.)-e_1\|_\nu \to 0$.
Also,

$$\begin{aligned}\|G_{l+p}^{\alpha,\beta}(e_2;.)-e_2\|_\nu \leq& \left(\frac{1}{(l+p+\beta+1)^2}\right)\sup_{u\in(0,1)}\frac{((l+p)^2-7(l+p)-6)u^2}{1+u^2} \\ &+ \left(\frac{1}{(l+p+\beta+1)^2}\right)\sup_{u\in(0,1)}\frac{\{6(l+p)-8+(2(l+p)-4)\alpha\}u}{1+u^2} \\ &+ \left(\frac{1}{(l+\beta+1)^2}\right)\sup_{u\in(0,1)}\frac{7+9\alpha+3\alpha^2}{1+u^2}.\end{aligned}$$

This implies that $\|G_{l+p}^{\alpha,\beta}(e_2;.)-e_2\|_\nu \to 0$ as $l \to \infty$. Hence, we complete the result of Theorem 4.3. □

Theorem 4.4 *Let $g \in C_{\nu}^{\tilde{k}}(0,1)$ and $\zeta > 0$. Then,*

$$\lim_{n\to\infty} \sup_{u\in(0,1)} \frac{|G_{l+p}^{\alpha,\beta}(g;u) - g(u)|}{(1+u^2)^{1+\zeta}} = 0.$$

Proof: Since $|g(u)| \leq \|g\|_{\nu}(1+u^2)$, for any real fixed number $u_0 > 0$, we get

$$\begin{aligned}
\sup_{u\in(0,1)} \frac{|G_{l+p}^{\alpha,\beta}(g;u)-g(u)|}{(1+u^2)^{1+\zeta}} &\leq \sup_{u\leq u_0} \frac{|G_{l+p}^{\alpha,\beta}(g;u)-g(u)|}{(1+u^2)^{1+\zeta}} + \sup_{u\geq u_0} \frac{|G_{l+p}^{\alpha,\beta}(g;u)-g(u)|}{(1+u^2)^{1+\zeta}} \\
&\leq \|G_{l+p}^{\alpha,\beta}(g;u) - g(u)|\|_{C_{\nu}^{\tilde{k}}(0,1)} \\
&+ \|g\|_{\nu} \sup_{u\geq u_0} \frac{|G_{l+p}^{\alpha,\beta}(1+t^2;u)|}{(1+u^2)^{1+\zeta}} + \sup_{u\geq u_0} \frac{|g(u)|}{(1+u^2)^{1+\zeta}} \\
&= \tilde{W_1} + \tilde{W_2} + \tilde{W_3}, \quad \textit{say}. \qquad (4.3)
\end{aligned}$$

Now,

$$\tilde{W_3} = \sup_{u\geq u_0} \frac{|g(u)|}{(1+u^2)^{1+\zeta}} \leq \sup_{u\geq u_0} \frac{\|g\|_{\nu}(1+u^2)}{(1+u^2)^{1+\zeta}} \leq \frac{\|g\|_{\nu}}{(1+u_0^2)^{\zeta}}.$$

In view of Lemma 1.2, it gives

$$\lim_{l\to\infty} \sup_{u\in[u_0,1)} \frac{G_{l+p}^{\alpha,\beta}(1+t^2;u)}{1+u^2} = 1.$$

Therefore, for any arbitrary $\epsilon > 0$, there corresponds $n_1 \in \mathbb{N}$ with

$$\sup_{u\in[u_0,1)} \frac{G_{l+p}^{\alpha,\beta}(1+t^2;u)}{1+u^2} \leq \frac{(1+u_0^2)^{\zeta}}{\|g\|_{\nu}} \frac{\epsilon}{3} + 1, \text{ for all } l \geq l_1.$$

That is

$$\tilde{W_2} = ||g||_{\nu} \sup_{u\in[u_0,1)} \frac{G_{l+p}^{\alpha,\beta}(1+t^2;u)}{(1+u^2)^{1+\zeta}} \leq \frac{||g||_{\nu}}{(1+u_0^2)^{\zeta}} + \frac{\epsilon}{3}, \text{ for all } l \geq l_1. \qquad (4.4)$$

Hence, we get

$$\tilde{W_2} + \tilde{W_3} < 2\frac{\|g\|_{\nu}}{(1+u^2)^{\zeta}} + \frac{\epsilon}{3}.$$

If we take u_0 to be so large that $\frac{\|g\|_{\nu}}{(1+u^2)^{\zeta}} < \frac{\epsilon}{6}$, then, we have

$$\tilde{W_2} + \tilde{W_3} < \frac{2\epsilon}{3} \text{ for all } l \geq l_1. \qquad (4.5)$$

Now, from Theorem 4.1, there corresponds $l_2 > l$ with

$$\tilde{W_1} = \|G_{l+p}^{\alpha,\beta}(g;\cdot) - g\|_{C[0,u_0]} < \frac{\epsilon}{3} \text{ for all } l_2 \geq l. \qquad (4.6)$$

Let $l_3 = \max(l_1, l_2)$. Then, with the aid of the (4.3), (4.5) and (4.6), we get

$$\sup_{u\in(0,1)} \frac{|G_{l+p}^{\alpha,\beta}(g;u) - g(u)|}{(1+u^2)^{1+\zeta}} < \epsilon,$$

which completes the desired result. □

5. A-Statistical Approximation

Here, we recall a few abbreviations and notation from [32,33,34]. Suppose that $A = (a_{l\nu})$ represents non-negative infinite summability matrix. Then, for a sequence $z := (z_\nu)$ is said to be A-statistically convergent to $\tilde{L}$, that is $st_A - \lim z = \tilde{L}$, if for all $\epsilon > 0$

$$\lim_l \sum_{\nu:|z_\nu \tilde{L}| \geq \epsilon} a_{l\nu} = 0.$$

Consider $q = \{q_l\}$ to be a sequence with the following conditions

$$st_A - \lim_l q_l = 1 \text{ and } st_A - \lim_l q_l^l = a, \; 0 \leq a < 1. \tag{5.1}$$

Theorem 5.1 *Consider $A = (a_{l\nu})$ to be a nonnegative regular summability matrix and $q = \{q_l\}$ sequence with (5.1) and $q_l \in (0,1)$, $l \in \mathbb{N}$. Then, for every $g \in C_\nu^0[0, 1+p]$, $st_A - \lim_l \|G_{l+p}^{\alpha,\beta}(g; .) - g\|_\nu = 0$.*

Proof: From Lemma 1.2, we get

$$st_A - \lim_l \|G_{l+p}^{\alpha,\beta}(e_0; u) - e_0\|_\nu = 0$$

and

$$\begin{aligned}\|G_{l+p}^{\alpha,\beta}(e_1; .) - u\|_\nu &= \sup_{u\in[0,1+p]} \frac{1}{1+u^2} \left| \frac{2(l+p-2)u + 3 + 2\alpha}{l+p+\beta+1} - u \right| \\ &= \frac{1}{1+u^2} \sup_{u\in[0,1+p]} \frac{(l+p-5-\beta)u}{l+p+\beta+1} + \frac{1}{1+u^2} \sup_{u\in[0,1+p]} \frac{3+2\alpha}{l+p+\beta+1}.\end{aligned}$$

Now

$$\begin{aligned}\tilde{I}_1 &:= \left\{ l : \|G_{l+p}^{\alpha,\beta}(e_1; u) - u\| \geq \epsilon \right\}, \\ \tilde{I}_2 &:= \left\{ l : \frac{(l-5-\beta)}{l+p+\beta+1} \geq \frac{\epsilon}{2} \right\}, \\ \tilde{I}_3 &:= \left\{ l : \frac{3+2\alpha}{l+p+\beta+1} \geq \frac{\epsilon}{2} \right\}.\end{aligned}$$

This implies that $\tilde{I}_1 \subseteq \tilde{I}_2 \cup \tilde{I}_3$ and this shows that $\sum_{\nu\in\tilde{I}_1} a_{l\nu} \leq \sum_{\nu\in\tilde{I}_2} a_{l\nu} + \sum_{\nu\in\tilde{I}_3} a_{l\nu}$. Therefore, we get

$$st_A - \lim_l \|G_{l+p}^{\alpha,\beta}(e_1; u) - u\|_\nu = 0. \tag{5.2}$$

Now, in the light of Lemma 1.2, we get

$$\begin{aligned}\|G_{l+p}^{\alpha,\beta}(e_2; u) - u^2\|_{1+u^2} \leq \sup_{u\in[0,1+p]} \frac{1}{\nu(u)} \bigg| \frac{1}{(l+p+\beta+1)^2} \Big\{ &((l+p)^2 - 7(l+p) - 6)u^2 \\ &+ (6(l+p) - 8 + 2l\alpha - 4\alpha)u + (\frac{7}{3} + 3\alpha + \alpha^2) \Big\} - u^2 \bigg|.\end{aligned}$$

For $\varepsilon > 0$, we have the following sets

$$\tilde{G_1} := \left\{ l : \left\| G^{\alpha,\beta}_{l+p}(e_2; u) - u^2 \right\|_{\nu} \geq \epsilon \right\}$$
$$\tilde{G_2} := \left\{ l : \frac{(l+p)^2 - 7(l+p) - 7}{(l+p+\beta+1)^2} \geq \frac{\epsilon}{3} \right\}$$
$$\tilde{G_3} := \left\{ l : \frac{6(l+p) - 8 + 2(l+p)\alpha - 4\alpha}{(l+p+\beta+1)^2} \geq \frac{\epsilon}{3} \right\}$$
$$\tilde{G_4} := \left\{ l : \frac{7 + 9\alpha + 3\alpha^2}{3(l+p+\beta+1)^2} \geq \frac{\epsilon}{3} \right\}.$$

We note that $\tilde{G_1} \subseteq \tilde{G_2} \bigcup \tilde{G_3} \bigcup \tilde{G_4}$. Therefore, we get

$$\sum_{\nu \in \tilde{G_1}} a_{l\nu} \leq \sum_{\nu \in \tilde{G_2}} a_{l\nu} + \sum_{\nu \in \tilde{G_3}} a_{l\nu} + \sum_{\nu \in \tilde{G_4}} a_{l\nu}.$$

As $l \to \infty$, we have

$$st_A - \lim_{l} \|G^{\alpha,\beta}_{l+p}(e_2; .) - e_2\|_{\nu} = 0. \tag{5.3}$$

Hence, we complete the required result. □

Further, we examine the rate of A-Statistical approximation in the account of Peetre's K-functional for $G^{\alpha,\beta}_{l+p}(.; .)$.

The Peetre's K-functional of $g \in C_B[0, 1+p]$ is

$$K(g; \tilde{\eta}) = \inf_{h \in C^2_B[0,1+p]} \left\{ \|g - h\|_{C_B[0,1+p]} + \tilde{\eta} \|h\|_{C^2_B[0,1+p]} \right\},$$

where $\tilde{\eta} > 0$ and

$$C^2_B[0, 1+p] = \{g \in C_B[0, 1+p] : g', g'' \in C_B[0, 1+p]\},$$

endowed with the following relation of norm

$$\|g\|_{C^2_B[0,1+p]} = \|g\|_{C_B[0,1+p]} + \|g'\|_{C_B[0,1+p]} + \|g''\|_{C_B[0,1+p]}.$$

Theorem 5.2 *Suppose that $g \in C^2_B[0, 1+p]$. Then,*

$$st_A - \lim_{l} \|G^{\alpha,\beta}_{l+p}(g; \cdot) - g\|_{C_B[0,1+p]} = 0.$$

Proof: In the light of Taylor's theorem, we have

$$g(t) = g(u) + g'(u)(t-u) + \frac{1}{2} g''(\psi^0_u(t))(t-u)^2,$$

where $t \leq \psi \leq u$. On operating the operators $G^{\alpha,\beta}_{l+p}(.; .)$, both sides in the above equation, one get

$$G^{\alpha,\beta}_{l+p}(g; u) - g(u) = g'(u) G^{\alpha,\beta}_{l+p}(\psi^1_u(t); u) + \frac{1}{2} g''(\psi G^{\alpha,\beta}_{l+p}(\psi^2_u(t); u),$$

which yields that

$$\begin{aligned} \|G^{\alpha,\beta}_{l+p}(g; \cdot) - g\|_{C_B[0,1+p]} &\leq \|g'\|_{C_B[0,1+p]} \|G^{\alpha,\beta}_{l+p}(e_1 -, .)\|_{C_B[0,1+p]} \\ &\quad + \|g''\|_{C_B[0,1+p]} \|G^{\alpha,\beta}_{l+p}(e_1 -, .)^2\|_{C_B[0,1+p]} \\ &= \tilde{V_1} + \tilde{V_2}, \quad \textit{say}. \end{aligned} \tag{5.4}$$

From Eqs. (5.2) and (5.3), we have

$$\begin{aligned}\lim_{l} \sum_{\nu\in\mathbb{N}:\tilde{V}_1\geq\frac{\epsilon}{2}} a_{l\nu} &= 0,\\ \lim_{l} \sum_{\nu\in\mathbb{N}:\tilde{V}_2\geq\frac{\epsilon}{2}} a_{l\nu} &= 0.\end{aligned}$$

From the (5.4), one has

$$\lim_{l} \sum_{\nu\in\mathbb{N}:\|G_{l+p}^{\alpha,\beta}(g;\cdot)-g\|_{C_B[0,1+p]}\geq\epsilon} a_{l\nu} \leq \lim_{l} \sum_{\nu\in\mathbb{N}:\tilde{V}_1\geq\frac{\epsilon}{2}} a_{l\nu} + \lim_{l} \sum_{\nu\in N:\tilde{V}_2\geq\frac{\epsilon}{2}} a_{l\nu}.$$

Therefore $st_A - \lim_{l} \|G_{l+p}^{\alpha,\beta}(h;\cdot) - h\|_{C_B[0,1+p]} \to 0$. as $l \to \infty$.

Which, arrive at the required result. □

Theorem 5.3 *Let $g \in C_B^2[0, 1+p]$. Then,*

$$\|G_{l+p}^{\alpha,\beta}(g;\cdot) - g\|_{C_B[0,1+p]} \leq M\omega_2(g; \sqrt{\tilde{\eta}}),$$

where $\tilde{\eta} = \|G_{l+p}^{\alpha,\beta}(e_1 - \cdot;\cdot)\|_{C_B[0,1+p]} + \|G_{l+p}^{\alpha,\beta}((e_1 - \cdot)^2;\cdot)\|_{C_B[0,1+p]}$, *and* $\|g\|_{C_B^2[0,1+p]} = \|g\|_{C_B[0,1+p]} + \|g'\|_{C_B[0,1+p]} + \|g''\|_{C_B[0,1+p]}$.

Proof: Let $h \in C_B^2[0, 1+p]$. Using (5.4), we obtain that

$$\begin{aligned}\|G_{l+p}^{\alpha,\beta}(h) - h\|_{C_B[0,1+p]} &\leq \|h'\|_{C_B[0,1+p]}\|G_{l+p}^{\alpha,\beta}((e_1-\cdot);\cdot)\|_{C_B[0,1+p]}\\ &\quad+\frac{1}{2}\|h''\|_{C_B[0,1+p]}G_{l+p}^{\alpha,\beta}((e_1-\cdot)^2;\cdot)\|_{C_B[0,1+p]}\\ &\leq \tilde{\eta}\|h\|_{C_B^2[0,1+p]}. \end{aligned}\tag{5.5}$$

Now, for every $g \in C_B[0, 1+p]$ and $h \in C_B^2[0, 1+p]$, from (5.5), one get

$$\begin{aligned}\|G_{l+p}^{\alpha,\beta}(g;\cdot) - g\|_{C_B[0,1+p]} &\leq \|G_{l+p}^{\alpha,\beta}(g;\cdot) - G_{l+p}^{\alpha,\beta}(h;\cdot)\|_{C_B[0,1+p]}\\ &\quad+\|G_{l+p}^{\alpha,\beta}(h;\cdot) - h\|_{C_B[0,1+p]} + \|g-h\|_{C_B[0,1+p]}\\ &\leq 2\|h-g\|_{C_B[0,1+p]} + \|G_{l+p}^{\alpha,\beta}(h;\cdot) - h\|_{C_B[0,1+p]}\\ &\leq 2\|h-g\|_{C_B[0,1+p]} + \tilde{\eta}\|g\|_{C_B^2[0,1+p]}.\end{aligned}$$

In view of Peetre's K-functional, we have

$$\|G_{l+p}^{\alpha,\beta}(g;\cdot) - g\|_{C_B[0,1+p]} \leq 2K_2(g;\tilde{\eta})$$

and

$$\|G_{l+p}^{\alpha,\beta}(g;\cdot) - g\|_{C_B[0,1+p]} \leq \tilde{M}\big\{\omega_2(g;\sqrt{\tilde{\eta}}) + \min(1,\tilde{\eta})\|g\|_{C_B[0,1+p]}\big\}.$$

In view of (5.3), we have

$$st_A - \lim_{l}\tilde{\eta} = 0, \text{ therefore } st_A - \lim_{l}\omega(h;\sqrt{\tilde{\eta}}) = 0,$$

Hence, we arrive the proof. □

6. Conclusion

In this study, we introduce generalized Bernstein type operators with two shifted nodes and estimate some lemmas to support approximation results in subsequent sections in terms of test function and central moments. Next, the convergence results and approximation rate in the sense of Korovkin theorem and classical modulus of continuity are studied. Further, we investigate direct results of approximation via Peetre's K-functional, modulus of continuity of second order, Lipschitz space of functions and Lipschitz type r^{th} order maximal function. In the last section, we discuss global approximation results and convergence results in terms of statistical approximation.

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