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Positive Solutions for Nonlinear Fractional Differential Equations with a Disturbance Parameter on an Infinite Interval

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ABSTRACT: This paper studies a class of integral boundary value problems of fractional differential equations with a disturbance parameter on an infinite interval. By applying the upper-lower solution method and fixed point index theory, the influence of the disturbance parameter on the existence of positive solutions is rigorously analyzed. We have obtained that the boundaryvalue problem admits at least twopositive solutions, one positivesolution or no solution. An example is provided to validate the results. The results of this study provide a significant complement and refinement to some existing findings.

Key Words: Boundary value problem, disturbance parameter, infinite interval, fractional differential equation, positive solution.

Contents

1. Introduction

Fractional differential equations (FDE for short), as an extension of classical integer differential equations, have garnered significant attention in recent years across fields such as mathematics, physics, engineering and biology, see [1,2,3,4,5,6]. Research on boundary value problems (BVP for short) of FDE has achieved many important results, see [7,8,9,10,11,12,13,14,15,16,17,18,19].

FDE with disturbance parameters are a key focus in mathematics and applied sciences. As errors are inevitable in solving many practical problems, these errors often affect the existence of solutions. Therefore, it is meaningful to study BVP of FDE with disturbance parameters, see [13,14,15,16,17,18,19]. In [18], the authors studied the FDE

$$\begin{cases} D_{0+}^{\alpha} x(t) = f(t, x(t)), \quad t \in (0,1), \\ x(0) = x'(0) = 0, \\ x(1) = \displaystyle\int_0^1 g_1(s)x(s)\mathrm{d}s + a, \\ x'(1) = \displaystyle\int_0^1 g_2(s)x(s)\mathrm{d}s - b, \end{cases}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α, $3 < \alpha \leq 4$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $g_1,\ g_2 \in L^1[0,1]$ and $a, b \geq 0$. By using the Guo–Krasnosel'skii fixed point theorem, the impact of the disturbance parameters a, b on the existence and nonexistence of positive solutions is investigated.

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In [13], the authors investigated a class of Riemann–Stieltjes integral BVP of FDE with parameters

$$\begin{cases} D_{0^+}^{\alpha}\left(p(t)D_{0^+}^{\beta}u(t)\right)+\lambda f(t,u(t))=0, \quad t\in(0,1),\\ \lim\limits_{t\to0^+}t^{2-\beta}u(t)=a, \quad u(1)=\displaystyle\int_0^1 u(s)\mathrm{d}A(s),\\ \lim\limits_{t\to0^+}t^{1-\alpha}p(t)D_{0^+}^{\beta}u(t)=0,\end{cases}$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann–Liouville fractional derivatives with $0<\alpha\le1$, $1<\beta\le2$. Here, $\lambda>0$, $a\ge0$, $p\in C([0,1],(0,+\infty))$, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ are given functions, f satisfies L^q-Carathéodory conditions, and $\int_0^1 u(s)\mathrm{d}A(s)$ denotes the Riemann–Stieltjes integral with respect to A. By using fixed point index theory, some new sufficient conditions for the existence of at least one, two and the nonexistence of positive solutions are obtained. In [20], the authors studied the FDE

$$\begin{cases} D^{\alpha}u(t)+\mu g(t)f(u(t))=0, \quad t\in(0,1),\\ u(0)=u'(0)=0, \quad u(1)=\lambda\displaystyle\int_0^1 u(s)\mathrm{d}s,\end{cases}$$

where $2<\alpha<3$, $0<\lambda<\alpha$, $\mu>0$. Here D^{α} is the Riemann–Liouville fractional derivative. The authors obtained the existence of positive solutions if $\lambda\in(0,\alpha)$, as well as the influence of μ on the number of positive solutions to the problem. In [21], the authors studied the FDE

$$\begin{cases} D_{0^+}^{\alpha}u(t)+\lambda a(t)f(t,u(t))=0, \quad t\in(0,+\infty),\\ u(0)=u'(0)=0, \quad D_{0^+}^{\alpha-1}u(+\infty)=\displaystyle\sum_{i=1}^{m-2}c_iu\left(\xi_i\right),\end{cases}$$

where $2<\alpha<3$, $D_{0^+}^{\alpha}$ is the usual Riemann–Liouville fractional derivative, $a:[0,+\infty)\to[0,+\infty)$, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ are continuous, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<+\infty$, $c_i\ge0$, $i=1,2,\cdots,m-2$, and $0<\sum\limits_{i=1}^{m-2}c_i\xi_i^{\alpha-1}<\Gamma(\alpha)$. Under some new conditions, the authors employed a new method from [22] to establish the existence and uniqueness of positive solutions for any fixed $\lambda>0$. In [19], the authors studied the fractional integral BVP

$$\begin{cases} D_{0^+}^{\alpha}x(t)+q(t)f(t,x(t))=0, \quad t\in(0,+\infty),\\ x(0)=0, \quad x'(0)=0,\\ D_{0^+}^{\alpha-1}x(+\infty)=\beta\displaystyle\int_0^{\eta}x(s)\mathrm{d}s+\lambda,\end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivative of order α, $2<\alpha<3$, β, $\eta>0$ and $\lambda\ge0$, $f\in C([0,+\infty)\times[0,+\infty),\ [0,+\infty))$ is continuous. By employing the Guo–Krasnosel'skii fixed point theorem, they divided the range of parameter into intervals with at least two, one and no positive solutions. In [23], the authors investigated the BVP

$$\begin{cases} D_{0^+}^{\alpha}x(t)+f(t,x(t))=0, \quad t\in(0,+\infty),\\ \lim\limits_{t\to0}t^{2-\alpha}x(t)=\lim\limits_{t\to\infty}D_{0^+}^{\alpha-1}x(t)=\displaystyle\int_0^{\infty}g(s)x(s)\mathrm{d}s,\end{cases}$$

where $1<\alpha\le2$, $D_{0^+}^{\alpha}$ is the standard Riemann–Liouville fractional derivative. The nonlinear term f satisfies that for each $r>0$, there exist constants k, $M_r>0$ and $\sigma_1\in(-1,\sigma)$ such that $0\le f\left(t,\frac{1+t^{\sigma+2}}{t^{2-\alpha}}x\right)\le M_rt^{\sigma_1}e^{-kt}$ for all $t\in(0,+\infty)$, $|x|\le r$. By applying the monotone iterative technique, the existence of positive solutions under some conditions was established and successive iterative schemes for approximating solutions were obtained.

In this paper, we are concerned with the BVP

$$\begin{cases} D_{0+}^{\alpha}x(t)+q(t)f(t,x(t))=0, \quad t\in(0,+\infty), \\ \lim\limits_{t\to 0}t^{3-\alpha}x(t)=\lim\limits_{t\to 0}D_{0+}^{\alpha-2}x(t)=\lim\limits_{t\to\infty}D_{0+}^{\alpha-1}x(t)=\int_0^\infty g(s)x(s)\mathrm{d}s+\lambda, \end{cases} \tag{1.1}$$

where $2<\alpha\le 3$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, $q\in L^1[0,\infty)$ is nonnegative and $q\not\equiv\theta$, $\int_0^\infty g(s)\left[\frac{s^{\alpha-1}}{\Gamma(\alpha)}+\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}+s^{\alpha-3}\right]\mathrm{d}s<1$, $f\in C((0,\infty)\times[0,\infty),[0,\infty))$, f may be singular at $t=0$. For any $x\in(0,+\infty)$, $f(t,x)\not\equiv 0$ on any subinterval of $t\in(0,+\infty)$. The following assumptions will be employed in this paper.

(H1) For each $r>0$, there exists $\varphi_r(t)\ge 0$ on $(0,+\infty)$, with $q\varphi_r\in L^1([0,+\infty))$ such that $f(t,\frac{1+t^{\sigma+3}}{t^{3-\alpha}}x)\le\varphi_r(t)$, $t\in(0,+\infty)$, $0\le x\le r$;

(H2) $f(t,x)$ is monotone increasing with respect to $x\in[0,+\infty)$ for each $t\in(0,+\infty)$.

The main features of this paper are as follows.

Firstly, this paper studies α fractional BVP with a disturbance parameter based on [23], where the α is in $(1,2]$. To utilize the method of upper and lower solutions, we need to obtain some properties (Lemma 2.3) of the Green function. These properties do not exist in the case of $1<\alpha\le 2$. So we change the order of α into $(2,3]$. In addition, the control condition for the nonlinear term is more general. In [23], the nonlinear term f satisfies that for each $r>0$ there exist constants k, $M_r>0$ and $\sigma_1\in(-1,\sigma)$ such that $0\le f\left(t,\frac{1+t^{\sigma+2}}{t^{2-\alpha}}x\right)\le M_r t^{\sigma_1}e^{-kt}$ for all $t\in(0,+\infty)$, $|x|\le r$. However, in this paper, we only require that there exists a control function $\varphi_r(t)\ge 0$ on $(0,+\infty)$, satisfying (H1). So, the control function is more general and weaker.

Secondly, compared with [13,18,19,20,21], we obtain more accurate results. In [13, Theorem 3.2], the authors studied that there exists $a_0>0$ such that whenever $0<a<a_0$, the problem possesses at least one positive solution for all $\lambda>0$. There is still an unknown impact of the parameter a on the solution in $[a_0,+\infty)$. [18] studied the effect of parameters a,b on the existence of solutions. They obtained that there exist small enough a_0, b_0 and big enough a_1, b_1 such that the problem has at least one positive solution and none for $(a,b)\in[0,a_0]\times[0,b_0]$ and $(a,b)\in[a_1,+\infty)\times[b_1,+\infty)$, respectively. There is no discussion for $(a,b)\in(a_0,a_1)\times(a_1,b_1)$. [19] studied the impact of a disturbance parameter λ on the number of positive solutions for $\lambda=0$, $\lambda\in(0,\lambda^*)$ and $\lambda\in[\lambda^{**},+\infty)$, respectively. However, there is no discussion for $\lambda\in[\lambda^*,\lambda^{**})$. In [20, Theorem 3.2], conclusions (1)–(3), (5) and (7) indicate that there exists $\mu_0>0$ such that if $\mu>\mu_0$ or $0<\mu<\mu_0$, the results regarding the nonexistence or existence of positive solutions are obtained. The case where $\mu=\mu_0$ has not been discussed. Conclusions (4) and (6) indicate that the problem has a positive solution for every $\mu>0$. The influence of the parameter μ on the number of solutions is not reflected. [21] investigated the existence and uniqueness of positive solutions for any fixed $\lambda>0$. The impact of λ on the existence and multiplicity of solutions is not reflected. We made up for those omissions in [18,21,19,13,20]. We obtain that there exists $\lambda^*>0$ such that the BVP (1.1) has at least one positive solution for $\lambda=0$ or $\lambda=\lambda^*$, two positive solutions for $0<\lambda<\lambda^*$, and no positive solutions for $\lambda>\lambda^*$.

Thirdly, the method in this paper is the upper-lower solution method and fixed point index theory, which are different from the Guo–Krasnosel'skii fixed point theorem in [18,19], and from a new fixed point theorem for a class of generalized concave operators in [21,22].

Fourthly, our study is on an infinite interval, while the problems in [13,18,20] are on a finite interval This will lead to more complex studies in compactness verification.

The paper is organized as follows. In Section 2, we introduce some necessary definitions and lemmas, and we deduce some new properties for Green's function of the BVP (2.1). In Section 3, we introduce the comparison principle. In Section 4, we obtain the properties of positive solutions and some necessary theorems to verify our main results. In Section 5, we establish the existence, nonexistence and multiplicity of positive solutions for the BVP (1.1). In Section 6, we provide an example to illustrate the main results.

2. Properties of Green's Function and other Preliminaries

In this section, we provide some necessary definitions and lemmas.

Definition 2.1 ([24]) *Let $\kappa:(0,+\infty)\to\mathbb{R}$ be a function and $\varrho>0$. The Riemann–Liouville fractional integral of order ϱ of κ is defined by*

$$I_{0+}^{\varrho}\kappa(\zeta)=\frac{1}{\Gamma(\varrho)}\int_0^{\zeta}(\zeta-s)^{\varrho-1}\kappa(s)\mathrm{d}s,\quad \zeta\in(0,+\infty),$$

provided the integral in the right–hand side exists for each $\zeta\in(0,1)$.

Definition 2.2 ([24]) *The Riemann–Liouville fractional derivative of order ϱ for a continuous function $\kappa:(0,+\infty)\to\mathbb{R}$ is defined by*

$$D_{0+}^{\varrho}\kappa(\zeta)=\frac{1}{\Gamma(n-\varrho)}\left(\frac{d}{d\zeta}\right)^n\int_0^{\zeta}(\zeta-s)^{n-\varrho-1}\kappa(s)\mathrm{d}s,\quad \zeta\in(0,+\infty),$$

provided the right–hand side is pointwise defined on $(0,1)$, where n is the smallest integer greater than or equal to ϱ, and $\Gamma(\varrho)$ is the gamma function.

Lemma 2.1 ([24]) *Let $\alpha>0$. If we assume $u\in C(0,+\infty)\cap L[0,+\infty)$, then the FDE*

$$D_{0+}^{\alpha}u=0$$

has a unique solution

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},\quad c_i\in\mathbb{R},\quad i=1,2,\cdots,n,$$

where $n-1<\alpha\le n$, $n\in\mathbb{Z}$.

We consider the Banach space $X=\{C(0,\infty):\ \|x\|<+\infty\}$ endowed with $\|x\|=\sup\limits_{t\in(0,+\infty)}\frac{|x(t)|t^{3-\alpha}}{1+t^{\sigma+3}}$, where $\sigma>-1$ is any chosen constant. We also put

$$P=\{x\in X:\ x(t)\ge 0,\ t\in(0,+\infty)\}.$$

Lemma 2.2 *Suppose that $h\in L[0,+\infty)\cap C(0,+\infty)$ is a given nonnegative function. Consider the linear fractional BVP*

$$\begin{cases} D_{0+}^{\alpha}x(t)+h(t)=0,\quad t\in(0,+\infty),\\ \lim\limits_{t\to 0}t^{3-\alpha}x(t)=\lim\limits_{t\to 0}D_{0+}^{\alpha-2}x(t)=\lim\limits_{t\to\infty}D_{0+}^{\alpha-1}x(t)=\displaystyle\int_0^{\infty}g(s)x(s)\mathrm{d}s+\lambda.\end{cases}\tag{2.1}$$

Then x is a solution of BVP (2.1) if and only if $x\in X$ and

$$x(t)=\int_0^{\infty}G(t,s)h(s)\mathrm{d}s+\frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right],\tag{2.2}$$

where

$$G(t,s)=G_1(t,s)+G_2(t,s)\tag{2.3}$$

and

$$\psi=1-\int_0^{\infty}g(s)\left[\frac{s^{\alpha-1}}{\Gamma(\alpha)}+\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}+s^{\alpha-3}\right]\mathrm{d}s.$$

Here

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le+\infty,\\ t^{\alpha-1}, & 0\le t\le s\le+\infty,\end{cases}\tag{2.4}$$

$$G_2(t,s)=\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right]\frac{1}{\psi}\int_0^{\infty}G_1(t,s)g(t)\mathrm{d}t,\quad 0<t,s<+\infty.\tag{2.5}$$

Proof: On the one hand, according to Lemma 2.1, we obtain that

$$x(t) = -I^{\alpha}h(t) - c_1t^{\alpha-1} - c_2t^{\alpha-2} - c_3t^{\alpha-3}, \tag{2.6}$$

where $c_1,\ c_2,\ c_3 \in \mathbb{R}$. Note that

$$\left|t^{3-\alpha}\int_0^t (t-s)^{\alpha-1}h(s)\mathrm{d}s\right| = t^2\int_0^t \left(1-\frac{s}{t}\right)^{\alpha-1} h(s)\mathrm{d}s \le t^2\int_0^{\infty} h(s)\mathrm{d}s \to 0, \quad t\to 0,$$

and by (2.6), we have

$$t^{3-\alpha}x(t) = -t^{3-\alpha}I^{\alpha}h(t) - c_1t^2 - c_2t - c_3,$$

$$D_{0^+}^{\alpha-2}x(t) = -I^2h(t) - c_1D_{0^+}^{\alpha-2}t^{\alpha-1} - c_2\Gamma(\alpha-1),$$

$$D_{0^+}^{\alpha-1}x(t) = -I^1h(t) - c_1\Gamma(\alpha).$$

With the boundary conditions given in (2.1), we obtain

$$c_3 = -\int_0^{\infty} g(s)x(s)\mathrm{d}s - \lambda,$$

$$c_2 = -\frac{1}{\Gamma(\alpha-1)}\left(\int_0^{\infty} g(s)x(s)\mathrm{d}s + \lambda\right),$$

$$c_1 = -\frac{1}{\Gamma(\alpha)}\left[\int_0^{\infty} g(s)x(s)\mathrm{d}s + \int_0^{\infty} h(s)\mathrm{d}s + \int_0^{+\infty} g(s)x(s)\mathrm{d}s + \lambda\right].$$

Substituting these three values into (2.6), we obtain

$$\begin{aligned} x(t) &= -I^{\alpha}h(t) + \frac{1}{\Gamma(\alpha)}\left[\int_0^{\infty} g(s)x(s)\mathrm{d}s + \int_0^{\infty} h(s)\mathrm{d}s + \lambda\right]t^{\alpha-1} \\ &\quad + \left[\int_0^{\infty} g(s)x(s)\mathrm{d}s + \lambda\right]\frac{t^{\alpha-2}}{\Gamma(\alpha-2)} + \left[\int_0^{\infty} g(s)x(s)\mathrm{d}s + \lambda\right]t^{\alpha-3} \\ &= -I^{\alpha}h(t) + \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]\int_0^{+\infty} g(s)x(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\infty} h(s)\mathrm{d}s \\ &\quad + \lambda\left[\frac{t^{\alpha}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]. \end{aligned} \tag{2.7}$$

So we have

$$\begin{aligned} \int_0^{+\infty} g(s)x(s)\mathrm{d}s &= -\int_0^{+\infty} g(s)I^{\alpha}h(s)\mathrm{d}s + \int_0^{+\infty} g(s)\frac{s^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} h(\tau)\mathrm{d}\tau\mathrm{d}s \\ &\quad + \int_0^{+\infty} g(s)\left[\frac{s^{\alpha-1}}{\Gamma(\alpha)} + \frac{s^{\alpha-2}}{\Gamma(\alpha-2)} + s^{\alpha-3}\right]\int_0^{+\infty} g(\tau)x(\tau)\mathrm{d}\tau\mathrm{d}s \\ &\quad + \int_0^{+\infty} g(s)\left[\frac{s^{\alpha-1}}{\Gamma(\alpha)} + \frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + s^{\alpha-3}\right]\mathrm{d}s \\ &= \frac{1}{\psi}\Bigg[-\int_0^{+\infty} g(s)I^{\alpha}h(s)\mathrm{d}s + \int_0^{+\infty} g(s)\frac{s^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} h(\tau)\mathrm{d}\tau\mathrm{d}s \\ &\quad + \int_0^{\infty} g(s)\lambda\left[\frac{s^{\alpha-1}}{\Gamma(\alpha)} + \frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + s^{\alpha-3}\right]\mathrm{d}s\Bigg]. \end{aligned} \tag{2.8}$$

By incorporating (2.8) into (2.7), we have

$$
\begin{aligned}
x(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty h(s)\mathrm{d}s + \lambda\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right)\\
&\quad + \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]\frac{1}{\psi}\Bigg[-\int_0^\infty g(s)I^\alpha h(\tau)\mathrm{d}\tau\mathrm{d}s\\
&\quad + \int_0^\infty g(s)\frac{s^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty h(\tau)\mathrm{d}\tau + \int_0^\infty g(s)\lambda\left(\frac{s^{\alpha-1}}{\Gamma(\alpha)} + \frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + s^{\alpha-3}\right)\Bigg]\mathrm{d}s\\
&= -\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]\frac{1}{\psi\Gamma(\alpha)}\int_0^\infty g(s)\int_0^s (s-\tau)^{\alpha-1}h(\tau)\mathrm{d}\tau\mathrm{d}s\\
&\quad + \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]\frac{1}{\psi}\int_0^\infty g(s)\frac{s^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty h(\tau)\mathrm{d}\tau\mathrm{d}s\\
&\quad + \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]\frac{\lambda(1-\psi)}{\psi} + \lambda\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]\\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty h(s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\mathrm{d}s\\
&= \int_0^\infty G(t,s)h(s)\mathrm{d}s + \frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right],
\end{aligned}
\tag{2.9}
$$

where $G(t,s)$ is defined by (2.3).

Now, we prove $x \in X$. It is obvious that $\frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]$ is continuous on $(0,+\infty)$. From (2.9), we need to confirm the continuity of $\int_0^{+\infty} G(t,s)h(s)\mathrm{d}s$ on $(0,+\infty)$ next. It is not difficult to show that $G(t,s)$ is continuous and $G(t,s) \geq 0$ for $(t,s) \in (0,+\infty)\times(0,+\infty)$. For $t_0 \in (0,+\infty)$, for any $t_n \in (0,+\infty)$ and $t_n \to t_0$ $(n\to\infty)$, we have

$$
G(t_n,s)h(s) \to G(t_0,s)h(s)\ (n\to\infty), \quad s \in (0,+\infty).
$$

Since $\{t_n\}_0^\infty$ is convergent, it is bounded. Therefore, there exist $M_i > 0$ $(i=1,2,3)$ such that $\left|\frac{t_n^{\alpha-1}}{\Gamma(\alpha)}\right| \leq M_1$, $\left|\frac{t_n^{\alpha-2}}{\Gamma(\alpha-1)}\right| \leq M_2$, $|t_n^{\alpha-3}| \leq M_3$. Due to $\int_0^\infty g(s)\left[\frac{s^{\alpha-1}}{\Gamma(\alpha)} + \frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + s^{\alpha-3}\right]\mathrm{d}s < 1$, we have

$$
\begin{aligned}
|G(t_n,s)h(s)| &= |G_1(t_n,s)h(s) + G_2(t_n,s)h(s)|\\
&\leq \left|\frac{t_n^{\alpha-1}}{\Gamma(\alpha)}h(s)\right| + \left|\left[\frac{t_n^{\alpha-1}}{\Gamma(\alpha)} + \frac{t_n^{\alpha-2}}{\Gamma(\alpha-1)} + t_n^{\alpha-3}\right]\frac{1}{\psi}h(s)\int_0^\infty \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}g(\tau)d\tau\right|\\
&\leq M|h(s)|, \quad s \in (0,+\infty),
\end{aligned}
$$

where $M = (\frac{1}{\psi}+1)M_1 + \frac{1}{\psi}(M_2+M_3)$. Since $h \in L[0,+\infty)\cap C(0,+\infty)$, by the Lebesgue dominated convergence theorem, we have

$$
\int_0^{+\infty} G(t_n,s)h(s)\mathrm{d}s \to \int_0^{+\infty} G(t_0,s)h(s)\mathrm{d}s, \quad n\to\infty.
$$

Therefore, we know that $\int_0^{+\infty} G(t,s)h(s)\mathrm{d}s$ is continuous on $(0,+\infty)$. Thus $x \in C(0,+\infty)$. Observe that

$$
\frac{t^{3-\alpha}}{1+t^{\sigma+3}}G_1(t,s) \leq \frac{t^2}{\Gamma(\alpha)\,(1+t^{\sigma+3})} < \frac{1}{\Gamma(\alpha)}, \tag{2.10}
$$

$$\begin{aligned}\frac{t^{3-\alpha}}{1+t^{\sigma+3}}G_2(t,s) &= \left[\frac{t^2}{\Gamma(\alpha)\,(1+t^{\sigma+3})}+\frac{t}{\Gamma(\alpha-1)(1+t^{\sigma+3})}+\frac{1}{1+t^{\sigma+3}}\right]\frac{1}{\psi}\int_0^\infty G_1(t,s)g(t)\mathrm{d}t\\ &\le \left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+1\right)\frac{1}{\psi}\int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)}g(t)\mathrm{d}t\\ &= \left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+1\right)\left(L-\frac{1}{\psi\Gamma(\alpha)}\right),\end{aligned}$$

where $L=\frac{1}{\psi\Gamma(\alpha)}\left(1+\int_0^\infty t^{\alpha-1}g(t)\mathrm{d}t\right)$. Consequently,

$$0\le \frac{t^{3-\alpha}}{1+t^{\sigma+3}}G(t,s)\le L_0, \tag{2.11}$$

where $L_0=\frac{1}{\Gamma(\alpha)}+\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+1\right)\left(L-\frac{1}{\psi\Gamma(\alpha)}\right)$. So, we obtain

$$\begin{aligned}&\frac{t^{3-\alpha}}{1+t^{\sigma+3}}|x(t)|\\ &=\left|\int_0^\infty \frac{t^{3-\alpha}}{1+t^{\sigma+3}}G(t,s)h(s)\mathrm{d}s+\frac{\lambda}{\psi}\left[\frac{t^2}{\Gamma(\alpha)(1+t^{\sigma+3})}+\frac{t}{\Gamma(\alpha)(1+t^{\sigma+3})}+\frac{1}{1+t^{\sigma+3}}\right]\right|\\ &\le L_0\left|\int_0^\infty h(s)\mathrm{d}s\right|+\frac{\lambda}{\psi}\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+1\right]\\ &<\infty.\end{aligned}$$

Hence, $x\in X$.

On the other hand, if $x\in X$ and x satisfies (2.2) and $D^\alpha_{0+}x\in L^1[0,+\infty)$, it is not difficult to prove that x satisfies (2.1). The proof is completed. □

Therefore, if $x\in X$, $D^\alpha_{0+}x\in L^1[0,+\infty)$ hold, in virtue of Lemma 2.2, x is the solution of BVP (1.1) if and only if $x\in X$ and

$$x(t)=\int_0^\infty G(t,s)q(s)f(s,x(s))\mathrm{d}s+\frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right], \tag{2.12}$$

where G is defined by (2.3).

Remark 2.1 *Compared to [23], there is a lack of proof for $x\in C(0,+\infty)$ in [23, Lemma 2.8], and we have completed this part of the proof.*

Lemma 2.3 *$G(t,s)$, $G_1(t,s)$ defined by (2.3) and (2.4) possess the following properties:*

(i) $G(t,s)$ and $G_1(t,s)$ are nonnegative continuous functions;

(ii) $G_1(t,s)$ is strictly increasing in the first variable;

(iii) $0<\frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}<L_0$ for $0<s,t<+\infty$, where $L_0=\frac{1}{\Gamma(\alpha)}+\left(\frac{1}{\Gamma(\alpha)}+1\right)\left(L-\frac{1}{\Gamma(\alpha)}\right)$;

(iv) $G_1(t,s)$ is concave in the first variable for $0<s<t<+\infty$;

(v) For any constant $k>1$,

$$\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}\ge \frac{1}{4k^4(1+k^{\sigma+3})}\sup_{t>0}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}, \tag{2.13}$$

$$\min_{\frac{1}{k}\le t\le k}\frac{G_2(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}\ge \frac{1}{1+k^{\sigma+3}}\sup_{t>0}\frac{G_2(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}. \tag{2.14}$$

Proof: Conclusion (i) is obvious. (iii) can be obtained from (2.11).

(ii) For any fixed $s \in (0, +\infty)$, let

$$l_1(t) = \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-1} - (t-s)^{\alpha-1}\right), \quad s \le t, \quad t \in (0, +\infty),$$
$$l_2(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, \quad t \le s, \quad t \in (0, +\infty).$$

Then

$$l_1'(t) = \frac{\alpha-1}{\Gamma(\alpha)} \left[t^{\alpha-2} - (t-s)^{\alpha-2}\right] > 0, \quad l_2'(t) = \frac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-2} > 0.$$

Thus $l_1(t)$ is strictly increasing on $[s, +\infty)$ and $l_2(t)$ is strictly increasing on $(0, s]$. For any $t_1, t_2 \in (0, +\infty)$ and $t_1 < t_2$, if $s \notin (t_1, t_2)$, then either $l_1(t_1) < l_1(t_2)$ or $l_2(t_1) < l_2(t_2)$. This implies $G_1(t_1, s) < G_1(t_2, s)$. If $s \in (t_1, t_2)$, from the definition and monotonicity of l_1, l_2, we have $l_2(t_1) \le l_2(s) = l_1(s) \le l_1(t_2)$. We claim that $l_2(t_1) < l_1(t_2)$. In fact, if $l_2(t_1) = l_1(t_2)$, then $l_2(t_1) = l_2(s) = l_1(s) = l_1(t_2)$. As l_1 and l_2 are strictly increasing, we have $t_1 = s = t_2$, which contradicts with $t_1 < t_2$. This fact implies that $G_1(t_1, s) < G_1(t_2, s)$. Thus $G_1(t, s)$ is increasing on $t \in (0, +\infty)$.

(iv) For $0 < s < t < +\infty$, we get

$$\frac{\partial^2 G_1(t,s)}{\partial t^2} = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \left(t^{\alpha-3} - (t-s)^{\alpha-3}\right) < 0.$$

Therefore, $G_1(t, s)$ is concave in the first variable for $0 < s < t < +\infty$.

(v) Firstly, we prove (2.13). For any $s \in (0, +\infty)$, we know from (2.10) that the function $\frac{G_1(t,s)t^{3-\alpha}}{1+t^{3+\sigma}}$ achieves its supremum in $(0, +\infty)$ at one point $\xi \in (0, +\infty)$. Let

$$\eta = \inf \left\{ \xi \in (0, +\infty) : \sup_{t \in (0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{3+\sigma}} = \frac{G_1(\xi,s)\xi^{3-\alpha}}{1+\xi^{3+\sigma}} \right\}.$$

There are three cases to consider.

Case 1. $\eta \le \frac{1}{k}$. From part (ii) of Lemma 2.3, we have

$$\begin{aligned}
\min_{\frac{1}{k} \le t \le k} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} &\ge \frac{G_1\left(\frac{1}{k}, s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+k^{\sigma+3}} \\
&\ge \frac{G_1(\eta, s)\eta^{3-\alpha}}{1+k^{\sigma+3}} \\
&\ge \frac{G_1(\eta, s)\eta^{3-\alpha}}{1+k^{\sigma+3}} \frac{1}{1+\eta^{\sigma+3}} \\
&= \frac{1}{1+k^{\sigma+3}} \sup_{t \in (0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&\ge \frac{1}{4k^4(1+k^{\sigma+3})} \sup_{t \in (0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}.
\end{aligned}$$

Case 2. $\frac{1}{k} \le \eta \le k$. For fixed $0 \le s \le \frac{1}{2k}$, from part (iv) of Lemma 2.3, we have

$$\frac{G_1\left(\frac{1}{k}, s\right) - G_1\left(\frac{1}{2k}, s\right)}{\frac{1}{k} - \frac{1}{2k}} \ge \frac{G_1(\eta, s) - G_1\left(\frac{1}{2k}, s\right)}{\eta - \frac{1}{2k}},$$

i.e.,

$$\left(\eta - \frac{1}{2k}\right)\left[G_1\left(\frac{1}{k}, s\right) - G_1\left(\frac{1}{2k}, s\right)\right] \ge \left[G_1(\eta, s) - G_1\left(\frac{1}{2k}, s\right)\right] \frac{1}{2k}.$$

Therefore, we have

$$\begin{aligned}\left(\eta-\frac{1}{2k}\right)G_1\left(\frac{1}{k},s\right)&\geq\left(\eta-\frac{1}{2k}\right)G_1\left(\frac{1}{2k},s\right)+\frac{1}{2k}G_1(\eta,s)-\frac{1}{2k}G_1\left(\frac{1}{2k},s\right)\\&=\left(\eta-\frac{1}{k}\right)G_1\left(\frac{1}{2k},s\right)+\frac{1}{2k}G_1(\eta,s).\end{aligned}$$

Then

$$\begin{aligned}G_1\left(\frac{1}{k},s\right)&\geq\left(\frac{\eta-\frac{1}{k}}{\eta-\frac{1}{2k}}\right)G_1\left(\frac{1}{2k},s\right)+\frac{1}{2k(\eta-\frac{1}{2k})}G_1(\eta,s)\\&\geq\frac{1}{2k\eta-1}G_1(\eta,s)\\&\geq\frac{1}{2k\eta}G_1(\eta,s),\end{aligned}\tag{2.15}$$

$$\begin{aligned}\min_{\frac{1}{k}\leq t\leq k}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}&\geq\frac{G_1\left(\frac{1}{k},s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+k^{\sigma+3}}\\&\geq\frac{G_1(\eta,s)\left(\frac{1}{k}\right)^{3-\alpha}}{2k\eta(1+k^{\sigma+3})}\quad(\text{by }(2.15))\\&=\frac{G_1(\eta,s)\left(\frac{1}{k}\right)^{3-\alpha}}{2k\eta(1+k^{\sigma+3})}\frac{(1+\eta^{\sigma+3})\eta^{3-\alpha}}{(1+\eta^{\sigma+3})\eta^{3-\alpha}}\\&\geq\frac{1+\eta^{\sigma+3}}{2k\eta(1+k^{\sigma+3})}\frac{1}{\eta^{3-\alpha}k^{3-\alpha}}\sup_{t\in(0,+\infty)}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}\\&\geq\frac{1+\eta^{\sigma+3}}{2k^{3-\alpha}\eta^{3-\alpha}(1+k^{\sigma+3})}\sup_{t\in(0,+\infty)}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}\\&=\frac{1}{2k^{8-2\alpha}(1+k^{\sigma+3})}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}\\&\geq\frac{1}{2k^4(1+k^{\sigma+3})}\sup_{t\in(0,+\infty)}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}.\end{aligned}\tag{2.16}$$

For $\frac{1}{2k}\leq s\leq\frac{1}{k}$,

$$\begin{aligned}\min_{\frac{1}{k}\leq t\leq k}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}&\geq\frac{G_1\left(\frac{1}{k},s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+k^{\sigma+3}}\\&\geq\frac{G_1\left(\frac{1}{2k},s\right)}{1+k^{\sigma+3}}\left(\frac{1}{k}\right)^{3-\alpha}\\&=\frac{G_1\left(\frac{1}{2k},s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{G_1(\eta,s)\eta^{3-\alpha}}\frac{1+\eta^{\sigma+3}}{1+\eta^{\sigma+3}}\frac{G_1(\eta,s)\eta^{3-\alpha}}{1+k^{\sigma+3}}\\&\geq\frac{G_1\left(\frac{1}{2k},s\right)}{G_1(\eta,s)}\frac{1+\eta^{\sigma+3}}{(1+k^{\sigma+3})k^{6-2\alpha}}\sup_{t\in(0,+\infty)}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}\\&\geq\frac{1+\eta^{\sigma+3}}{(2k)^{\alpha-1}\eta^{\alpha-1}(1+k^{\sigma+3})k^{6-2\alpha}}\sup_{t\in(0,+\infty)}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}\\&=\frac{1}{2^{\alpha-1}k^4(1+k^{\sigma+3})}\sup_{t\in(0,+\infty)}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}\\&\geq\frac{1}{4k^4(1+k^{\sigma+3})}\sup_{t\in(0,+\infty)}\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}.\end{aligned}\tag{2.17}$$

For $s \geq \frac{1}{k}$,

$$\begin{aligned}
\min_{\frac{1}{k} \leq t \leq k} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} &\geq \frac{G_1\left(\frac{1}{k},s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+k^{\sigma+3}} \\
&= \frac{G_1\left(\frac{1}{k},s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{G_1(\eta,s)\eta^{3-\alpha}} \frac{1+\eta^{\sigma+3}}{1+k^{\sigma+3}} \frac{G_1(\eta,s)(\eta)^{3-\alpha}}{1+\eta^{\sigma+3}} \\
&\geq \frac{G_1\left(\frac{1}{k},s\right)}{G_1(\eta,s)} \left(\frac{1}{k}\right)^{6-2\alpha} \frac{1+\eta^{\sigma+3}}{1+k^{\sigma+3}} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&\geq \frac{1+\eta^{\sigma+3}}{k^{\alpha-1}\eta^{\alpha-1}k^{6-2\alpha}(1+k^{\sigma+3})} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&= \frac{1}{k^{\alpha-1}\eta^{\alpha-1}k^{6-2\alpha}(1+k^{\sigma+3})} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&\geq \frac{1}{k^4(1+k^{\sigma+3})} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&\geq \frac{1}{4k^4(1+k^{\sigma+3})} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}. \qquad (2.18)
\end{aligned}$$

Therefore, from (2.16), (2.17) and (2.18), (2.13) holds for $\frac{1}{k} \leq \eta \leq k$.

Case 3. $\eta \geq k$. For $0 \leq s \leq \frac{1}{2k}$, we have

$$\begin{aligned}
\min_{\frac{1}{k} \leq t \leq k} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} &\geq \frac{G_1\left(\frac{1}{k},s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+k^{\sigma+3}} \\
&\geq \frac{G_1(\eta,s)\left(\frac{1}{k}\right)^{3-\alpha}}{2k\eta(1+k^{\sigma+3})} \quad \text{(by (2.15))} \\
&= \frac{\left(\frac{1}{k}\right)^{3-\alpha}}{(2k\eta)\eta^{3-\alpha}} \frac{1+\eta^{\sigma+3}}{1+k^{\sigma+3}} \frac{G_1(\eta,s)\eta^{3-\alpha}}{1+\eta^{\sigma+3}} \\
&= \frac{1}{2k^{4-\alpha}(1+k^{\sigma+3})} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&\geq \frac{1}{4k^4(1+k^{\sigma+3})} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}. \qquad (2.19)
\end{aligned}$$

For $\frac{1}{2k} \leq s \leq \frac{1}{k}$,

$$\min_{\frac{1}{k} \leq t \leq k} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \geq \frac{1}{4k^4(1+k^{\sigma+3})} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}. \qquad (2.20)$$

For $s \geq \frac{1}{k}$,

$$\min_{\frac{1}{k} \leq t \leq k} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \geq \frac{1}{4k^4(1+k^{\sigma+3})} \sup_{t\in(0,+\infty)} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}. \qquad (2.21)$$

Therefore, from (2.19) (2.20) and (2.21), (2.13) holds for $\eta \geq k$.

Next, we prove (2.14). For fixed $s \in (0, +\infty)$,

$$\begin{aligned}
&\min_{\frac{1}{k} \le t \le k} \frac{G_2(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&= \min_{\frac{1}{k} \le t \le k} \left[\frac{t^2}{(1+t^{\sigma+3})\Gamma(\alpha)} + \frac{t}{(1+t^{\sigma+3})\Gamma(\alpha-1)} + \frac{1}{1+t^{\sigma+3}} \right] \frac{1}{\psi} \int_0^\infty G_1(t,s)g(t)\mathrm{d}t \\
&\ge \frac{1}{\psi} \int_0^\infty G_1(t,s)g(t)\mathrm{d}t \left[\min_{\frac{1}{k} \le t \le k} \frac{t^2}{(1+t^{\sigma+3})\Gamma(\alpha)} + \min_{\frac{1}{k} \le t \le k} \frac{t}{(1+t^{\sigma+3})\Gamma(\alpha-1)} \right. \\
&\quad \left. + \min_{\frac{1}{k} \le t \le k} \frac{1}{1+t^{\sigma+3}} \right] \\
&\ge \frac{1}{\psi} \int_0^\infty G_1(t,s)g(t)\mathrm{d}t \left[\frac{k^2}{(1+k^{\sigma+3})\Gamma(\alpha)} + \frac{k}{(1+k^{\sigma+3})\Gamma(\alpha-1)} + \frac{1}{1+k^{\sigma+3}} \right] \\
&= \frac{1}{1+k^{\sigma+3}} \frac{1}{\psi} \int_0^\infty G_1(t,s)g(t)\mathrm{d}t \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + 1 \right] \\
&> \frac{1}{1+k^{\sigma+3}} \frac{1}{\psi} \int_0^\infty G_1(t,s)g(t)\mathrm{d}t \sup_{t \in (0,+\infty)} \left[\frac{t^2}{(1+t^{\sigma+3})\Gamma(\alpha)} + \frac{t}{(1+t^{\sigma+3})\Gamma(\alpha-1)} + \frac{1}{1+t^{\sigma+3}} \right] \\
&= \frac{1}{1+k^{\sigma+3}} \sup_{t \in (0,+\infty)} \frac{G_2(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}.
\end{aligned}$$

The proof is now complete. □

Define an operator $T : X \to X$ by

$$(Tx)(t) = \int_0^\infty G(t,s)q(s)f(s,x(s))\mathrm{d}s + \frac{\lambda}{\psi} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3} \right], \quad t \in (0, +\infty). \tag{2.22}$$

Since $G(t,s) > 0$, $0 < t, s < +\infty$, q is nonnegative and $q \ne \theta$, and for any $x \in (0, +\infty)$, $f(t,x) \not\equiv 0$ on any subinterval of $t \in (0, +\infty)$, it follows that if x is a nonnegative fixed point of the operator T, then x is a positive solution of BVP (1.1).

Lemma 2.4 *If (H1) and (H2) hold, then $T : P \to P$ is well defined and completely continuous.*

Proof: The proof follows with a similar approach as in [23, Lemma 2.8]. □

Lemma 2.5 *If x is a positive solution for BVP (1.1), then we have*

$$\min_{\frac{1}{k} \le t \le k} \frac{x(t)t^{3-\alpha}}{1+t^{\sigma+3}} \ge \frac{1}{4k^4(1+k^{\sigma+3})} \|x\|. \tag{2.23}$$

Proof: From (2.13) and (2.14), we get

$$\begin{aligned}
\min_{\frac{1}{k} \le t \le k} \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} &\ge \min_{\frac{1}{k} \le t \le k} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} + \min_{\frac{1}{k} \le t \le k} \frac{G_2(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&\ge \frac{1}{4k^4(1+k^{\sigma+3})} \sup_{t>0} \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} + \frac{1}{(1+k^{\sigma+3})} \sup_{t>0} \frac{G_2(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \\
&\ge \frac{1}{4k^4(1+k^{\sigma+3})} \sup_{t>0} \left[\frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} + \frac{G_1(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} \right] \\
&= \frac{1}{4k^4(1+k^{\sigma+3})} \sup_{t>0} \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}.
\end{aligned} \tag{2.24}$$

Therefore, we have

$$\begin{aligned}
\min_{\frac{1}{k}\le t\le k}\frac{x(t)t^{3-\alpha}}{1+t^{\sigma+3}} &\ge \int_0^\infty \min_{\frac{1}{k}\le t\le k}\frac{G(t,s)t^{3-\alpha}q(s)f(s,x(s))}{1+t^{\sigma+3}}\,\mathrm{d}s\\
&\quad+\frac{\lambda}{\psi}\min_{\frac{1}{k}\le t\le k}\left[\frac{t^2}{\Gamma(\alpha)(1+t^{\sigma+3})}+\frac{t}{\Gamma(\alpha-1)(1+t^{\sigma+3})}+\frac{t}{1+t^{\sigma+3}}\right]\\
&\ge \frac{1}{4k^4(1+k^{\sigma+3})}\int_0^\infty \sup_{t\in(0,+\infty)}\frac{G(t,s)t^{3-\alpha}q(s)f(s,x(s))}{1+t^{\sigma+3}}\,\mathrm{d}s\\
&\quad+\frac{\lambda}{\psi}\left[\frac{k^2}{\Gamma(\alpha)(1+t^{\sigma+3})}+\frac{k}{\Gamma(\alpha-1)(1+k^{\sigma+3})}+\frac{k}{1+k^{\sigma+3}}\right]\\
&\ge \frac{1}{4k^4(1+k^{\sigma+3})}\sup_{t\in(0,+\infty)}\int_0^\infty \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}q(s)f(s,x(s))\,\mathrm{d}s\\
&\quad+\frac{\lambda}{\psi}\left[\frac{1}{\Gamma(\alpha))}+\frac{1}{\Gamma(\alpha-1)}+1\right]\\
&\ge \frac{1}{4k^4(1+k^{\sigma+3})}\left\{\sup_{t\in(0,+\infty)}\int_0^\infty \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}q(s)f(s,x(s))\,\mathrm{d}s\right.\\
&\quad\left.+\frac{\lambda}{\psi}\sup_{t\in(0,+\infty)}\left[\frac{t^2}{\Gamma(\alpha)(1+t^{\sigma+3})}+\frac{t}{\Gamma(\alpha-1)(1+t^{\sigma+3})}+\frac{t}{1+t^{\sigma+3}}\right]\right\}\\
&\ge \frac{1}{4k^4(1+k^{\sigma+3})}\sup_{t\in(0,+\infty)}\left\{\int_0^\infty \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}q(s)f(s,x(s))\,\mathrm{d}s\right.\\
&\quad\left.+\frac{\lambda}{\psi}\left[\frac{t^2}{\Gamma(\alpha)(1+t^{\sigma+3})}+\frac{t}{\Gamma(\alpha-1)(1+t^{\sigma+3})}+\frac{t}{1+t^{\sigma+3}}\right]\right\}\\
&= \frac{1}{4k^4(1+k^{\sigma+3})}\|x\|.
\end{aligned}$$

This completes the proof. □

Lemma 2.6 ([25, p. 88]) *Let P be a cone of the Banach space X, $\Omega\subset X$ be a bounded open set and zero element $\theta\in\Omega$. Suppose $T: P\cap\overline{\Omega}\to P$ is a completely continuous operator. If $x\ne\mu Tx$ for any $x\in P\cap\partial\Omega$ and $\mu\in[0,1]$, then $i(T,P\cap\Omega,P)=1$.*

3. Comparison Principle

Let $\delta\in X$, $D^{\alpha}_{0+}\delta\in L^1[0,+\infty)$, and we call δ a lower solution to the BVP (1.1) provided that δ meets

$$\begin{cases} -D^{\alpha}_{0+}\delta(t)\le q(t)f(t,\delta(t)), \quad t\in(0,+\infty),\\ \lim\limits_{t\to0}t^{3-\alpha}\delta(t)=\lim\limits_{t\to0}D^{\alpha-2}_{0+}\delta(t)=\lim\limits_{t\to\infty}D^{\alpha-1}_{0+}\delta(t)\le\int_0^\infty g(s)\delta(s)\mathrm{d}s+\lambda.\end{cases}$$

Let $\omega\in X$, $D^{\alpha}_{0+}\omega\in L^1[0,+\infty)$, and we call ω an upper solution to the BVP (1.1) provided that ω meets

$$\begin{cases} -D^{\alpha}_{0+}\omega(t)\ge q(t)f(t,\omega(t)), \quad t\in(0,+\infty),\\ \lim\limits_{t\to0}t^{3-\alpha}\omega(t)=\lim\limits_{t\to0}D^{\alpha-2}_{0+}\omega(t)=\lim\limits_{t\to\infty}D^{\alpha-1}_{0+}\omega(t)\ge\int_0^\infty g(s)\omega(s)\mathrm{d}s+\lambda.\end{cases}$$

Lemma 3.1 *If $x\in X$ satisfies $D^{\alpha}_{0+}x(t)\in L^1[0,+\infty)$ and*

$$\begin{cases} D^{\alpha}_{0+}x(t)\le 0, \quad t\in(0,+\infty),\\ \lim\limits_{t\to0}t^{3-\alpha}x(t)=\lim\limits_{t\to0}D^{\alpha-2}_{0+}x(t)=\lim\limits_{t\to\infty}D^{\alpha-1}_{0+}x(t)\ge\int_0^\infty g(s)x(s)\mathrm{d}s,\end{cases}$$

then $x(t) \geq 0$ for $t \in (0, +\infty)$.
Proof: Let

$$y(t) = -D_{0+}^{\alpha} x(t) \geq 0 \quad \text{for a.e.} \quad t \in (0, +\infty).$$

Note that

$$\begin{aligned}\lambda &= \lim_{t\to 0} t^{3-\alpha} x(t) - \int_0^{\infty} g(s)x(s)\mathrm{d}s \\ &= \lim_{t\to 0} D_{0+}^{\alpha-2} x(t) - \int_0^{\infty} g(s)x(s)\mathrm{d}s \\ &= \lim_{t\to \infty} D_{0+}^{\alpha-1} x(t) - \int_0^{\infty} g(s)x(s)\mathrm{d}s \\ &\geq 0.\end{aligned}$$

We know from Lemma 2.2 that the BVP

$$\begin{cases} -D_{0+}^{\alpha} x(t) = y(t), \quad t \in (0, +\infty), \\ \lim\limits_{t\to 0} t^{3-\alpha} x(t) = \lim\limits_{t\to 0} D_{0+}^{\alpha-2} x(t) = \lim\limits_{t\to \infty} D_{0+}^{\alpha-1} x(t) = \int_0^{\infty} g(s)x(s)\mathrm{d}s + \lambda \end{cases}$$

has a unique solution

$$x(t) = \int_0^{\infty} G(t,s)y(s)\mathrm{d}s + \frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right].$$

We can deduce that $x(t) \geq 0$ for $t \in (0, +\infty)$. □

Remark 3.1 *If all conditions of Lemma 3.1 are satisfied and $\lambda > 0$, then we can further conclude that $x(t) > 0$ for $t \in (0, +\infty)$.*

Theorem 3.1 *Assume (H1) and (H2) hold. Given that the BVP (1.1) possesses a nonnegative lower solution δ and an upper solution ω, with $\delta(t) \leq \omega(t)$ for $t \in (0, +\infty)$, then the BVP (1.1) guarantees at least one positive solution x for which $\delta(t) \leq x(t) \leq \omega(t)$ holds for $t \in (0, +\infty)$.*

Proof:

$$F(t,x) = \begin{cases} f(t,\omega(t)), & x > \omega(t), \\ f(t,x), & \delta(t) \leq x \leq \omega(t), \\ f(t,\delta(t)), & x < \delta(t), \end{cases}$$

where $t \in (0, +\infty)$, $x \in X$. Since $f \in C((0, +\infty) \times [0, +\infty))$, we get $F \in C((0, +\infty) \times [0, +\infty))$, too. Based on (2.12), it can be deduced that the BVP

$$\begin{cases} D_{0+}^{\alpha} x(t) + q(t)F(t, x(t)) = 0, \quad t \in (0, +\infty), \\ \lim\limits_{t\to 0} t^{3-\alpha} x(t) = \lim\limits_{t\to 0} D_{0+}^{\alpha-2} x(t) = \lim\limits_{t\to \infty} D_{0+}^{\alpha-1} x(t) = \int_0^{\infty} g(s)x(s)\mathrm{d}s + \lambda \end{cases} \tag{3.1}$$

is equivalent to the integral equation

$$x(t) = \int_0^{\infty} G(t,s)q(s)F(s,x(s))\mathrm{d}s + \frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right].$$

Define an operator $Q : X \to X$ by

$$(Qx)(t) = \int_0^{\infty} G(t,s)q(s)F(s,x(s))\mathrm{d}s + \frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right], \quad t \in (0, +\infty), \quad x \in X.$$

We can easily show that $Q : P \to P$ is completely continuous since its proof is similar to that of Lemma 2.4.

Let $\Omega_1 = \{x \in P : \|x\| \le R_0\}$, where the constant

$$R_0 = L_0 \int_0^{+\infty} q(s)\varphi_{\|\omega\|}(s)\mathrm{d}s + \frac{\lambda}{\psi}\left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + 1\right].$$

Obviously, Ω_1 is a closed and convex set. From (H1), (H2) and the definition of F, we obtain that

$$0 \le F(t, x(t)) \le f(t, \omega(t)) = f\left(t, \frac{1+t^{\sigma+3}}{t^{3-\alpha}}\frac{t^{3-\alpha}\omega(t)}{1+t^{\sigma+3}}\right) \le \varphi_{\|\omega\|}(t), \quad t \in (0, +\infty), \quad x \in \Omega_1. \tag{3.2}$$

Then

$$\begin{aligned}\frac{t^{3-\alpha}|\left(Qx\right)(t)|}{1+t^{\sigma+3}} &\le L_0 \int_0^{+\infty} q(s)\varphi_{\|\omega\|}(s)\mathrm{d}s + \frac{\lambda}{\psi}\left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + 1\right]\\ &= R_0, \quad t \in (0, +\infty), \quad x \in \Omega_1.\end{aligned} \tag{3.3}$$

Thus $\|Qx\| \le R_0$, which implies $Q : \Omega_1 \to \Omega_1$. Based on the Schauder fixed point theorem, we know that Q has at least one fixed point x. Then the BVP (3.1) possesses a positive solution x.

Next, we prove $\delta(t) \le x(t) \le \omega(t)$ for $t \in (0, +\infty)$. Let $v(t) = x(t) - \delta(t)$ for $t \in (0, +\infty)$. According to (H2), we get

$$D^{\alpha}_{0^+}v(t) = -q(t)F(t, x(t)) - D^{\alpha}_{0^+}\delta(t) \le -q(t)F(t, x(t)) + q(t)f(t, \delta(t)) \le 0,$$

$$\lim_{t\to 0} t^{3-\alpha}v(t) \ge \int_0^{\infty} g(s)x(s)\mathrm{d}s + \lambda - \int_0^{\infty} g(s)\delta(s)\mathrm{d}s - \lambda = \int_0^{\infty} g(s)v(s)\mathrm{d}s,$$

$$\lim_{t\to 0} D^{\alpha-2}_{0^+}v(t) \ge \int_0^{\infty} g(s)x(s)\mathrm{d}s + \lambda - \int_0^{\infty} g(s)\delta(s)\mathrm{d}s - \lambda = \int_0^{\infty} g(s)v(s)\mathrm{d}s,$$

$$\lim_{t\to \infty} D^{\alpha-1}_{0^+}v(t) \ge \int_0^{\infty} g(s)x(s)\mathrm{d}s + \lambda - \int_0^{\infty} g(s)\delta(s)\mathrm{d}s - \lambda = \int_0^{\infty} g(s)v(s)\mathrm{d}s.$$

According to Lemma 3.1, we get $v(t) \ge 0$ for $t \in (0, +\infty)$, which indicates that $x(t) \ge \delta(t)$ for $t \in (0, +\infty)$. Similarly, it can be proven that $x(t) \le \omega(t)$ for $t \in (0, +\infty)$. Therefore, each solution x of BVP (3.1) satisfies $\delta(t) \le x(t) \le \omega(t)$ for $t \in (0, +\infty)$. That is, $F(t, x(t)) = f(t, x(t))$ and x is a positive solution of BVP (1.1). □

4. Properties of Positive Solutions

Theorem 4.1 *Suppose (H1) and (H2) hold.*

(i) If there exists a constant $\overline{\lambda} \ge 0$ such that BVP $(1.1)_{\overline{\lambda}}$ has a positive solution $x_{\overline{\lambda}}$, then for each $0 \le \lambda \le \overline{\lambda}$, BVP $(1.1)_{\lambda}$ has a positive solution x_{λ} and

$$\frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right] \le x_{\lambda}(t) \le x_{\overline{\lambda}}(t), \quad t \in (0, +\infty);$$

(ii) If there exists a constant $\underline{\lambda} \ge 0$ such that BVP $(1.1)_{\underline{\lambda}}$ does not have positive solutions, then for each $\lambda \ge \underline{\lambda}$, BVP $(1.1)_{\lambda}$ does not have positive solutions.

Proof: (i) If $x_{\overline{\lambda}}$ is a positive solution of BVP $(1.1)_{\overline{\lambda}}$, then by (2.12), we have

$$x_{\overline{\lambda}}(t)=\int_0^{\infty}G(t,s)q(s)f(s,x_{\overline{\lambda}}(s))\mathrm{d}s+\frac{\overline{\lambda}}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right],\quad t\in(0,+\infty).$$

Therefore, we can obtain that for any $0\le\lambda\le\overline{\lambda}$,

$$x_{\overline{\lambda}}(t)\ge\frac{\overline{\lambda}}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right]\ge\frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right],\quad t\in(0,+\infty).$$

We can easily show that $x_{\overline{\lambda}}$ is an upper solution of BVP $(1.1)_{\lambda}$. Next we show that $\frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right]$ is a lower solution of BVP $(1.1)_{\lambda}$. We have

$$-D_{0^+}^{\alpha}\frac{\lambda}{\psi}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right)=0\le q(t)f(t,\delta(t)),$$

$$\lim_{t\to0}\frac{\lambda t^{3-\alpha}}{\psi}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right)=\lim_{t\to0}\frac{\lambda}{\psi}\left(\frac{t^{2}}{\Gamma(\alpha)}+\frac{t}{\Gamma(\alpha-1)}+1\right)=\frac{\lambda}{\psi},$$

$$D_{0^+}^{\alpha-1}\frac{\lambda}{\psi}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right)=\frac{\lambda}{\psi},\quad \lim_{t\to\infty}D_{0^+}^{\alpha-1}\frac{\lambda}{\psi}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right)=\frac{\lambda}{\psi},$$

$$\lim_{t\to0}D_{0^+}^{\alpha-2}\frac{\lambda}{\psi}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right)=\lim_{t\to0}\frac{\lambda}{\psi}\left(\frac{t}{\alpha}+1\right)=\frac{\lambda}{\psi},$$

$$\int_0^{\infty}g(s)\frac{\lambda}{\psi}\left(\frac{s^{\alpha-1}}{\Gamma(\alpha)}+\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}+s^{\alpha-3}\right)\mathrm{d}s+\lambda=\frac{\lambda}{\psi}.$$

So $\frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right]$ is a lower solution of BVP $(1.1)_{\lambda}$. Thus according to Theorem 3.1, we obtain that for any $0\le\lambda\le\overline{\lambda}$, BVP $(1.1)_{\lambda}$ has a positive solution x_{λ} and

$$\frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+t^{\alpha-3}\right]\le x_{\lambda}(t)\le x_{\overline{\lambda}}(t),\quad t\in(0,+\infty).$$

(ii) Assume that there exists a constant $\lambda_0\ge\underline{\lambda}$, such that BVP $(1.1)_{\lambda_0}$ has a positive solution. By the conclusion of (i), we can infer that BVP $(1.1)_{\underline{\lambda}}$ has a positive solution since $\underline{\lambda}\le\lambda_0$, which contradicts the assumption. □

Denote

$$\begin{aligned}f^0&=\limsup_{x\to0^+}\sup_{t\in(0,+\infty)}\frac{f\left(t,\frac{1+t^{\sigma+3}}{t^{3-\alpha}}x\right)}{x},\quad f_{\infty}=\liminf_{x\to+\infty}\inf_{t\in\left[\frac{1}{k},k\right]}\frac{f\left(t,\frac{1+t^{\sigma+3}}{t^{3-\alpha}}x\right)}{x},\\ \rho_1&=\frac{1}{L_0\int_0^{+\infty}q(s)\mathrm{d}s+\frac{1}{\psi}[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+1]},\quad \rho_2=\frac{4\Gamma(\alpha)k^4[1+k^{\sigma+3}]^2}{k^{\sigma+1}\int_{\frac{1}{k}}^{k}q(s)\mathrm{d}s}.\end{aligned}\tag{4.1}$$

Theorem 4.2 *If (H1) and $f^0<\rho_1$ hold, then there exists a constant $\lambda_*\ge0$ such that BVP $(1.1)_{\lambda_*}$ has at least one positive solution.*

Proof: Since $f^0 < \rho_1$, there exists a constant $r_1 > 0$ such that $f\left(t, \frac{1+t^{\sigma+3}}{t^{3-\alpha}}u\right) < \rho_1 u \leq \rho_1 r_1$ for any $t \in (0, +\infty)$ and $u \in (0, r_1]$. Set $B_{r_1} = \{x \in P : \|x\| \leq r_1\}$. Choose

$$\lambda_* \in [0, \rho_1 r_1]. \tag{4.2}$$

Then, for any $x \in B_{r_1}$,

$$\begin{aligned} f(t, x(t)) &= f\left(t, \left(\frac{1+t^{\sigma+3}}{t^{3-\alpha}}\right)\frac{t^{3-\alpha}}{1+t^{\sigma+3}}x\right) \\ &< \rho_1 \frac{t^{3-\alpha}}{1+t^{\sigma+3}}x(t) \\ &\leq \rho_1 r_1, \quad t \in (0, +\infty). \end{aligned} \tag{4.3}$$

By part (iii) of Lemma 2.3 and (4.3), we have

$$\begin{aligned} \frac{t^{3-\alpha}|Tx(t)|}{1+t^{\sigma+3}} &= \int_0^{+\infty} \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}q(s)f(s,x(s))\mathrm{d}s + \frac{\lambda_*}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]\frac{t^{3-\alpha}}{1+t^{\sigma+3}} \\ &< L_0\rho_1 r_1 \int_0^{+\infty} q(s)\mathrm{d}s + \frac{\lambda_*}{\psi}\left[\frac{t^2}{\Gamma(\alpha)(1+t^{\sigma+3})} + \frac{t}{\Gamma(\alpha-1)(1+t^{\sigma+3})} + \frac{1}{1+t^{\sigma+3}}\right] \\ &\leq L_0\rho_1 r_1 \int_0^{+\infty} q(s)\mathrm{d}s + \frac{\rho_1 r_1}{\psi}\left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + 1\right] = r_1, \quad t \in (0, +\infty). \end{aligned}$$

So $T(B_{r_1}) \subset B_{r_1}$. According to the Schauder fixed point theorem, we obtain that T has at least one fixed point in B_{r_1}. Thus BVP $(1.1)_{\lambda_*}$ has at least one positive solution. □

Theorem 4.3 *If (H1) and $f_\infty > \rho_2$ hold, then there exists a positive constant $\hat{\lambda}$ such that BVP $(1.1)_{\hat{\lambda}}$ has no positive solution.*

Proof: Assume that BVP $(1.1)_\lambda$ has a positive solution x_λ for any $\lambda > 0$. By $f_\infty > \rho_2$, there exists a constant $r_2 > 0$ such that

$$f\left(t, \frac{1+t^{\sigma+3}}{t^{3-\alpha}}u\right) > \rho_2 u, \quad t \in \left[\frac{1}{k}, k\right], \quad u \geq r_2. \tag{4.4}$$

Take the special constant

$$\widetilde{\lambda} = 2\psi \frac{\Gamma(\alpha)\Gamma(\alpha-1)k^2(1+k^{\sigma+3})}{\Gamma(\alpha-1)+\Gamma(\alpha)k+\Gamma(\alpha)\Gamma(\alpha-1)k^2}r_2. \tag{4.5}$$

We know that BVP $(1.1)_{\widetilde{\lambda}}$ has a positive solution $x_{\widetilde{\lambda}}$. By (2.12), we have

$$x_{\widetilde{\lambda}}(t) = \int_0^{+\infty} G(t,s)q(s)f(s, x_{\widetilde{\lambda}}(s))\mathrm{d}s + \frac{\widetilde{\lambda}}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right], \quad t \in (0, +\infty).$$

Therefore

$$\begin{aligned} \frac{s^{3-\alpha}x_{\widetilde{\lambda}}(s)}{1+s^{\sigma+3}} &\geq \frac{\widetilde{\lambda}}{\psi}\left[\frac{s^2}{\Gamma(\alpha)(1+s^{\sigma+3})} + \frac{s}{\Gamma(\alpha-1)(1+s^{\sigma+3})} + \frac{1}{1+s^{\sigma+3}}\right] \\ &\geq \frac{\widetilde{\lambda}}{\psi}\left[\frac{(\frac{1}{k})^2}{\Gamma(\alpha)(1+k^{\sigma+3})} + \frac{\frac{1}{k}}{\Gamma(\alpha-1)(1+k^{\sigma+3})} + \frac{1}{1+k^{\sigma+3}}\right] \\ &= \frac{\widetilde{\lambda}}{\psi}\frac{\Gamma(\alpha-1)+\Gamma(\alpha)k+\Gamma(\alpha)\Gamma(\alpha-1)k^2}{\Gamma(\alpha)\Gamma(\alpha-1)k^2(1+k^{\sigma+3})} \\ &= 2r_2 > r_2, \quad s \in \left[\frac{1}{k}, k\right]. \end{aligned} \tag{4.6}$$

By (4.4) and (4.6), we get

$$f(s, x_{\widetilde{\lambda}}(s)) = f\left(s, \frac{1+s^{\sigma+3}}{s^{3-\alpha}} \frac{s^{3-\alpha}}{1+s^{\sigma+3}} x_{\widetilde{\lambda}}(s)\right) > \rho_2 \frac{s^{3-\alpha}}{1+s^{\sigma+3}} x_{\widetilde{\lambda}}(s), \quad s \in \left[\frac{1}{k}, k\right]. \tag{4.7}$$

By (2.23), we have

$$\min_{\frac{1}{k} \le t \le k} \frac{x_{\widetilde{\lambda}}(t) t^{3-\alpha}}{1+t^{\sigma+3}} \ge \frac{1}{4k^4\left(1+k^{\sigma+3}\right)} \|x_{\widetilde{\lambda}}\|. \tag{4.8}$$

Based on (4.5), (4.6), (4.7) and (4.8), we have

$$\begin{aligned}
\|x_{\widetilde{\lambda}}\| &\ge \frac{\left(\frac{1}{k}\right)^{3-\alpha} x_{\widetilde{\lambda}}\left(\frac{1}{k}\right)}{1+\left(\frac{1}{k}\right)^{\sigma+3}} \\
&= \int_0^{+\infty} \frac{G\left(\frac{1}{k}, s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+\left(\frac{1}{k}\right)^{\sigma+3}} q(s) f(s, x_{\widetilde{\lambda}}(s)) \mathrm{d}s \\
&\quad + \frac{\widetilde{\lambda}}{\psi}\left[\frac{\left(\frac{1}{k}\right)^{3-\alpha}\left(\frac{1}{k}\right)^{\alpha-1}}{\Gamma(\alpha)\left(1+\left(\frac{1}{k}\right)^{\sigma+3}\right)} + \frac{\left(\frac{1}{k}\right)^{3-\alpha}\left(\frac{1}{k}\right)^{\alpha-2}}{\Gamma(\alpha-1)\left(1+\left(\frac{1}{k}\right)^{\sigma+3}\right)} + \frac{\left(\frac{1}{k}\right)^{3-\alpha}\left(\frac{1}{k}\right)^{\alpha-3}}{1+\left(\frac{1}{k}\right)^{\sigma+3}}\right] \\
&\ge \int_{\frac{1}{k}}^{k} \frac{G_1\left(\frac{1}{k}, s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+\left(\frac{1}{k}\right)^{\sigma+3}} q(s) f(s, x_{\widetilde{\lambda}}(s)) \mathrm{d}s \\
&\quad + \frac{\widetilde{\lambda}}{\psi}\left[\frac{\frac{1}{k^2}}{\Gamma(\alpha)\left(1+\left(\frac{1}{k}\right)^{\sigma+3}\right)} + \frac{\frac{1}{k}}{\Gamma(\alpha-1)\left(1+\left(\frac{1}{k}\right)^{\sigma+3}\right)} + \frac{1}{1+\frac{1}{k^{\sigma+3}}}\right] \\
&> \frac{\rho_2 \frac{1}{k^2}}{\Gamma(\alpha)\left(1+\frac{1}{k^{\sigma+3}}\right)} \int_{\frac{1}{k}}^{k} q(s) \frac{s^{3-\alpha}}{1+s^{\sigma+3}} x_{\widetilde{\lambda}}(s) \mathrm{d}s + r_2 \text{ (by (4.5), (4.6) and (4.7))} \\
&\ge \frac{\rho_2 k^{\sigma+1}}{\Gamma(\alpha)\left(1+k^{\sigma+3}\right)} \|x_{\widetilde{\lambda}}\| \frac{1}{4k^4\left(1+k^{\sigma+3}\right)} \int_{\frac{1}{k}}^{k} q(s) \mathrm{d}s + r_2 \quad \text{(by (4.8))} \\
&= \|x_{\widetilde{\lambda}}\| + r_2,
\end{aligned}$$

which is a contradiction. Thus, there exists a big enough constant $\hat{\lambda} > 0$ such that BVP $(1.1)_{\hat{\lambda}}$ has no positive solution. □

Theorem 4.4 *Let $I \subset [0, +\infty)$ be a bounded set. If (H1) and $f_\infty > \rho_2$ hold, then there exists $\tau > 0$ such that $\|x\| < \tau$, where x is the positive solution of BVP $(1.1)_{\lambda \in I}$.*

Proof: Since $f_\infty > \rho_2$, there exists a constant $r > 0$ such that $f\left(t, \frac{1+t^{\sigma+3}}{t^{3-\alpha}} u\right) > \rho_2 u$ for any $t \in \left[\frac{1}{k}, k\right]$ and $u \ge r$. Let $\tau = 4k^4\left(1+k^{\sigma+3}\right) r$. Suppose that BVP $(1.1)_{\lambda \in I}$ has a solution $\breve{x}$ satisfying $\|\breve{x}\| \ge \tau$. By (2.23),

$$\min_{\frac{1}{k} \le t \le k} \frac{t^{3-\alpha} \breve{x}(t)}{1+t^{\sigma+3}} \ge \frac{1}{4k^4\left(1+k^{\sigma+3}\right)} \|\breve{x}\| \ge \frac{1}{4k^4\left(1+k^{\sigma+3}\right)} \tau = r.$$

By part (ii) of Lemma 2.3 and (2.23), we have

$$\begin{aligned}
\|\breve{x}\| &\geq \frac{\left(\frac{1}{k}\right)^{3-\alpha}\breve{x}\left(\frac{1}{k}\right)}{1+\left(\frac{1}{k}\right)^{\sigma+3}} \\
&\geq \int_0^{+\infty}\frac{G\left(\frac{1}{k},s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+\left(\frac{1}{k}\right)^{\sigma+3}}q(s)f(s,\breve{x}(s))\mathrm{d}s \\
&\geq \int_{\frac{1}{k}}^{k}\frac{G_1\left(\frac{1}{k},s\right)\left(\frac{1}{k}\right)^{3-\alpha}}{1+\left(\frac{1}{k}\right)^{\sigma+3}}q(s)f(s,\breve{x}(s))\mathrm{d}s \\
&> \frac{\rho_2}{\Gamma(\alpha)}\int_{\frac{1}{k}}^{k}\frac{\left(\frac{1}{k}\right)^{\alpha-1}\left(\frac{1}{k}\right)^{3-\alpha}}{1+\left(\frac{1}{k}\right)^{\sigma+3}}q(s)\frac{s^{3-\alpha}\breve{x}(s)}{1+s^{\sigma+3}}\mathrm{d}s \\
&\geq \rho_2\frac{\frac{1}{k^2}\|\breve{x}\|}{\Gamma(\alpha)\left(1+\frac{1}{k^{\sigma+3}}\right)4k^4(1+k^{\sigma+3})}\int_{\frac{1}{k}}^{k}q(s)\mathrm{d}s \\
&= \|\breve{x}\|,
\end{aligned}$$

which is a contraction. Therefore, the conclusion is proved. □

5. Existence, Nonexistence and Multiplicity of Positive Solutions

Theorem 5.1 *If (H1), (H2), $f_0 < \rho_1$ and $f_\infty > \rho_2$ hold, then there exists a constant $\lambda^* \in (0,+\infty)$ such that*

(i) BVP (1.1) has at least one positive solution for $\lambda = 0$ and $\lambda = \lambda^*$;

(ii) BVP (1.1) has at least two positive solutions for each $\lambda \in (0,\lambda^*)$;

(iii) BVP (1.1) does not have any positive solution for each $\lambda \in (\lambda^*,+\infty)$.

Proof: Let

$$\Lambda = \{\lambda \in [0,+\infty):\ \text{BVP } (1.1)_\lambda \text{ has at least one positive solution}\}.$$

Then by Theorem 4.2, we know that $\Lambda \neq \emptyset$. According to Theorem 4.3, we can show that Λ is a bounded set. Therefore, Λ has a supremum, which we indicate as $\lambda^* = \sup\Lambda$.

Firstly, we prove that BVP $(1.1)_{\lambda^*}$ has at least one positive solution. Since $\lambda^* = \sup\Lambda$, there exists a sequence $\{\lambda_m\} \subset \Lambda$ that satisfies $\lambda_m < \lambda^*$, $\lambda_m \to \lambda^*$ as $m \to +\infty$. Let x_{λ_m} be the solution of BVP $(1.1)_{\lambda_m}$. In view of (2.12), we know that BVP $(1.1)_{\lambda_m}$ is equivalent to

$$x_{\lambda_m}(t) = \int_0^{+\infty}G(t,s)q(s)f\left(s,x_{\lambda_m}(s)\right)\mathrm{d}s + \frac{\lambda_m}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right], \quad t \in (0,+\infty), \tag{5.1}$$

where $m \in \mathbb{N}^+$. According to Theorem 4.4, there exists a constant $\tau > 0$ such that $\|x_{\lambda_m}\| < \tau$. Based on Lemma 2.4, we can easily show that $\{x_{\lambda_m}(t)\}$ is relatively compact. Then we know that $\{x_{\lambda_m}\}$ has a convergent subsequence, we assume that $\{x_{\lambda_m}\}$ itself converges to x_0 and $x_0 \in P$. Therefore, $\frac{t^{3-\alpha}}{1+t^{\sigma+3}}\left|x_{\lambda_m}(t) - x_0(t)\right| \to 0$ as $m \to \infty$, $t \in (0,+\infty)$. So $|x_{\lambda_m}(t) - x_0(t)| \to 0$ as $m \to \infty$, $t \in (0,+\infty)$. In addition, since $f \in C((0,+\infty)\times[0,+\infty),[0,+\infty))$, we have

$$q(s)\left|f(s,x_{\lambda_m}(s)) - f(s,x_0(s))\right| \to 0 \quad \text{as} \quad m \to \infty, \quad s \in (0,+\infty).$$

By (H2) and part (iii) of Lemma 2.3, we get

$$\frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}}q(s)\left|f(s,x_{\lambda_m}(s)) - f(s,x_0(s))\right| \leq 2L_0q(s)\varphi_\tau(s), \quad t,s \in (0,+\infty), \quad m \in \mathbb{N}^+.$$

Then, by the Lebesgue dominated convergence theorem, we have

$$\int_0^{+\infty} \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} q(s) \left| f(s, x_{\lambda_m}(s)) - f(s, x_0(s)) \right| \mathrm{d}s \to 0 \quad \text{as} \quad m \to \infty, \quad t \in (0, +\infty).$$

Therefore

$$\begin{aligned} &\left| \int_0^{+\infty} \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} q(s) f(s, x_{\lambda_m}(s)) - \int_0^{+\infty} \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} q(s) f(s, x_0(s)) \mathrm{d}s \right| \\ &\le \int_0^{+\infty} \frac{G(t,s)t^{3-\alpha}}{1+t^{\sigma+3}} q(s) \left| f(s, x_{\lambda_m}(s)) - f(s, x_0(s)) \right| \mathrm{d}s \\ &\to 0 \quad \text{as} \quad m \to \infty, \quad t \in (0, +\infty). \end{aligned} \tag{5.2}$$

In addition, due to the continuity of $\frac{\lambda_m}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right]$ with respect to m, we have

$$\frac{\lambda_m}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right] \to \frac{\lambda^*}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right] \quad \text{as} \quad m \to \infty, \quad t \in (0, +\infty). \tag{5.3}$$

From (5.1), (5.2) and (5.3), we have

$$x_0(t) = \int_0^{+\infty} G(t,s) q(s) f(s, x_0(s)) \mathrm{d}s + \frac{\lambda^*}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right], \quad t \in (0, +\infty).$$

Hence, x_0 is a positive solution of BVP $(1.1)_{\lambda^*}$. By the definition of λ^*, we know that BVP $(1.1)_\lambda$ does not have positive solutions for each $\lambda \in (\lambda^*, +\infty)$.

Finally, we prove that BVP $(1.1)_\lambda$ has at least two positive solutions for $\lambda \in (0, \lambda^*)$. For any given $\lambda \in (0, \lambda^*)$, there exist $\underline{\lambda}$, $\overline{\lambda} \in \overline{\Lambda}$ such that $0 < \underline{\lambda} < \lambda < \overline{\lambda}$. Let $\overline{x}$, $\underline{x}$ be the solutions of BVP $(1.1)_{\overline{\lambda}}$ and BVP $(1.1)_{\underline{\lambda}}$, respectively. We can easily verify that $\underline{x}$ is a lower solution and $\overline{x}$ is an upper solution of BVP $(1.1)_\lambda$ and $\underline{x}(t) < \overline{x}(t)$ for $t \in (0, +\infty)$. Then, according to Theorem 4.1 and Remark 3.1, BVP $(1.1)_\lambda$ has a positive solution x_1 for the given λ, and $\underline{x}(t) < x_1(t) < \overline{x}(t)$, $t \in (0, +\infty)$. Choose $\hat{\lambda} > \lambda^*$. Then $\lambda < \lambda^* < \hat{\lambda}$. We define $K_\gamma : P \to X$ by

$$(K_\gamma x)(t) = \int_0^{+\infty} G(t,s) q(s) f(s, x(s)) \mathrm{d}s + \frac{\gamma}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right], \quad t \in (0, +\infty),$$

where $\gamma \in [\lambda, \hat{\lambda}]$. Let

$$F(t,x) = \begin{cases} f(t, \overline{x}(t)), & x > \overline{x}(t), \\ f(t,x), & \underline{x}(t) \le x \le \overline{x}(t), \\ f(t, \underline{x}(t)), & x < \underline{x}(t), \end{cases}$$

where $t \in (0, +\infty)$, $x \in P$. Define an integral operator $\widehat{K}_\gamma : P \to X$ by

$$\left(\widehat{K}_\gamma x\right)(t) = \int_0^{+\infty} G(t,s) q(s) F(s, u(s)) \mathrm{d}s + \frac{\gamma}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right], \quad t \in (0, +\infty),$$

where $\gamma \in [\lambda, \hat{\lambda}]$. We can easily prove that K_γ and $\widehat{K}_\gamma$ are completely continuous for each $\gamma \in [\lambda, \hat{\lambda}]$ according to Lemma 2.4. Next we demonstrate that BVP $(1.1)_\lambda$ has another positive solution $x_2 \neq x_1$ with the given λ, which is equivalent to proving that K_λ has a fixed point $x_2 \neq x_1$. That means we need to prove (5.12) below, which requires verifying (5.9) and (5.11) below first.

To prove (5.9), we first confirm (5.6) and (5.8) below. By Theorem 4.4, there exists $\tau > 0$ such that any fixed point x_γ of K_γ satisfies

$$\|x_\gamma\| < \tau \quad \text{for each} \quad \gamma \in [\lambda, \hat{\lambda}]. \tag{5.4}$$

Let

$$\Omega = \{x \in P : \ \|x\| < \tau, \ \underline{x}(t) < x(t) < \overline{x}(t), \ t \in (0, +\infty)\}.$$

Obviously, $x_1 \in \Omega$. Then $\Omega \subset P$ is a nonempty open bounded subset. Due to (H1), (H2) and the definition of F, similar to (3.2), we can prove that $F(t, x(t)) \leq \varphi_{\|\overline{x}\|}(t)$. Similar to (3.3), there exists a constant $R_1 > \tau > 0$ such that $\left\|\widehat{K}_\gamma x\right\| < R_1$ for $(\gamma, x) \in [\lambda, \hat{\lambda}] \times P$. Let $B_{R_1} = \{x \in X : \|x\| < R_1\}$. Then $\Omega \subset P \cap B_{R_1}$. Actually,

$$x \neq \mu \widehat{K}_\gamma x, \quad x \in P \cap \partial B_{R_1}, \quad \mu \in [0, 1]. \tag{5.5}$$

Otherwise, suppose there exist $x_0 \in P \cap \partial B_{R_1}$ and $\mu_0 \in [0, 1]$ such that $x_0 = \mu_0 \widehat{K}_\gamma x_0$. If $\mu_0 \in (0, 1]$, then $R_1 = \|x_0\| = \mu_0 \left\|\widehat{K}_\gamma x_0\right\| < \mu_0 R_1 \leq R_1$, which is a contradiction. If $\mu_0 = 0$, then we have $x_0 = \theta$ and then $R_1 = \|x_0\| = 0$, which contradicts $R_1 > 0$. By (5.5),

$$i\left(\widehat{K}_\lambda, P \cap B_{R_1}, P\right) = 1. \tag{5.6}$$

To prove (5.8), we confirm that $\widehat{K}_\lambda$ does not have fixed points on $P \cap (\overline{B_{R_1}} \backslash \Omega)$, next. If x_0 is a fixed point of $\widehat{K}_\lambda$ on $P \cap (\overline{B_{R_1}} \backslash \Omega)$, then

$$\begin{aligned}
x_0(t) &= \int_0^\infty G(t,s)q(s)F(s, x_0(s))\mathrm{d}s + \frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right] \\
&\leq \int_0^{+\infty} G(t,s)q(s)f(s, \overline{x}(s))\mathrm{d}s + \frac{\lambda}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right] \\
&< \int_0^{+\infty} G(t,s)q(s)f(s, \overline{x}(s))\mathrm{d}s + \frac{\overline{\lambda}}{\psi}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + t^{\alpha-3}\right] \\
&= \overline{x}(t), \quad t \in (0, +\infty).
\end{aligned}$$

By the same token, we get that $x_0(t) > \underline{x}(t)$ for $t \in (0, +\infty)$. Then

$$\underline{x}(t) < x_0(t) < \overline{x}(t), \quad t \in (0, +\infty). \tag{5.7}$$

From the definition of F, it follows that $F(s, x_0(s)) = f(s, x_0(s))$. Consequently, x_0 is also a fixed point of K_λ. Based on (5.4), we know that $\|x_0\| < \tau$. Thus $x_0 \in \Omega$, which contradicts $x_0 \in P \cap (\overline{B_{R_1}} \backslash \Omega)$. Hence, $\widehat{K}_\lambda$ does not have any fixed points on $P \cap (\overline{B_{R_1}} \backslash \Omega)$. Therefore, from the solvability of the fixed point index,

$$i\left(\widehat{K}_\lambda, P \cap (B_{R_1} \backslash \overline{\Omega}), P\right) = 0. \tag{5.8}$$

Since $\widehat{K}\Big|_\Omega = K$, by the excision property of the fixed point index, we obtain

$$\begin{aligned}
i\left(K_\lambda, P \cap \Omega, P\right) &= i\left(\widehat{K}_\lambda, P \cap \Omega, P\right) \\
&= i\left(\widehat{K}_\lambda, P \cap B_{R_1}, P\right) - i\left(\widehat{K}_\lambda, P \cap (B_{R_1} \backslash \overline{\Omega}), P\right) \\
&= 1.
\end{aligned} \tag{5.9}$$

Next we prove (5.11). We know from the definition of λ^* and the fact of $\hat{\lambda} > \lambda^*$ that BVP $(1.1)_{\hat{\lambda}}$ does not have positive solutions. That is, $K_{\hat{\lambda}}$ has no fixed point in P. Therefore, from the solvability of the fixed point index,

$$i\left(K_{\hat{\lambda}}, P \cap B_{R_1}, P\right) = 0. \tag{5.10}$$

Define H_μ by

$$H_\mu(x) = K_{(1-\mu)\lambda + \mu\hat{\lambda}}(x), \quad \mu \in [0, 1], \quad x \in P \cap \overline{B_{R_1}}.$$

Obviously, H_μ is completely continuous. We have $H_\mu(x) \neq x$ for $(\mu, x) \in [0,1] \times P \cap \partial B_{R_1}$. Otherwise, if there exists $(\mu_0, x_0) \in [0,1] \times P \cap \partial B_{R_1}$ such that $H_{\mu_0}(x_0) = x_0$, then

$$K_{(1-\mu_0)\lambda+\mu_0\hat{\lambda}}(x_0) = x_0, \quad x_0 \in P, \quad \|x_0\| = R_1.$$

Therefore, x_0 is a solution of BVP $(1.1)_{(1-\mu_0)\lambda+\mu_0\hat{\lambda}}$. By (5.4), we get $\|x_0\| < \tau$, which contradicts $\|x_0\| = R_1 > \tau$. By (5.10) and the homotopy invariance of the fixed point index, we have

$$i\left(K_\lambda, P \cap B_{R_1}, P\right) = i\left(H_0, P \cap B_{R_1}, P\right) = i\left(H_1, P \cap B_{R_1}, P\right) = i\left(K_{\hat{\lambda}}, P \cap B_{R_1}, P\right) = 0. \tag{5.11}$$

According to (5.9), (5.11) and by using the additivity property of the fixed point index, we have

$$i\left(K_\lambda, P \cap B_{R_1} \backslash \overline{\Omega}, P\right) = -1. \tag{5.12}$$

So K has a fixed point $x_2 \in P \cap B_{R_1} \backslash \overline{\Omega}$. Therefore, BVP $(1.1)_\lambda$ has a solution $x_2 \in P \cap B_{R_1} \backslash \overline{\Omega}$. Since $x_1 \in \Omega$, it follows that $x_2 \neq x_1$. Hence, BVP $(1.1)_\lambda$ has at least two positive solutions for $\lambda \in (0, \lambda^*)$. The proof is completed. □

6. An Example

In this section, we demonstrate our results through the following example. Consider the BVP

$$\begin{cases} D_{0^+}^{\frac{11}{4}} x(t) + e^{-t} \dfrac{t^{-\frac{1}{8}} e^{-t} \sin t}{(1+t^3)^2} x^2 = 0, \quad t \in (0, +\infty), \\ \lim\limits_{t\to 0} t^{\frac{1}{4}} x(t) = \lim\limits_{t\to 0} D_{0^+}^{\frac{3}{4}} x(t) = \lim\limits_{t\to\infty} D_{0^+}^{\frac{7}{4}} x(t) = \dfrac{1}{5}\displaystyle\int_0^\infty e^{-2s} x(s) \mathrm{d}s + \lambda. \end{cases} \tag{6.1}$$

Here, BVP (6.1) is a special case of BVP (1.1) if

$$\alpha = \frac{11}{4}, \quad \sigma = 0, \quad g(s) = \frac{1}{5} e^{-2s}, \quad q(t) = e^{-t}, \quad f(t,x) = \frac{t^{-\frac{1}{8}} e^{-t} \sin t}{(1+t^3)^2} x^2.$$

Clearly,

$$\int_0^\infty g(s) \left[\frac{s^{\alpha-1}}{\Gamma(\alpha)} + \frac{s^{\alpha-2}}{\Gamma(\alpha-1)} + s^{\alpha-3}\right] \mathrm{d}s \approx 0.2349 < 1$$

and $f \in C((0,+\infty) \times [0,+\infty))$. If $\varphi_r(t) = \frac{e^{-t}\sin t}{t^{\frac{5}{8}}} r^2$, then we get

$$\int_0^{+\infty} q(s) \varphi_r(s) \mathrm{d}s \approx 0.3032 r^2 < +\infty$$

and $f\left(t, \frac{1+t^3}{t^{\frac{1}{4}}} x\right) \leq \varphi_r(t)$ for $t \in (0,+\infty)$, $0 \leq x \leq r$. Thus (H1) holds. (H2) obviously holds.

Let $k = 2$. From (4.1), we have $\rho_1 \approx 0.2769$, $\rho_2 \approx 3346.7165$,

$$f^0 = \limsup_{x\to 0^+} \sup_{t\in(0,+\infty)} \frac{f\left(t, \left(\frac{1+t^3}{t^{\frac{1}{4}}}\right) x\right)}{x} = \limsup_{x\to 0^+} \sup_{t\in(0,+\infty)} \frac{e^{-t}\sin t}{t^{\frac{5}{8}}} x = 0 < \rho_1, \tag{6.2}$$

$$f_\infty = \liminf_{x\to+\infty} \inf_{t\in[\frac{1}{2},2]} \frac{f\left(t, \left(\frac{1+t^3}{t^{\frac{1}{4}}}\right) x\right)}{x} = \liminf_{x\to+\infty} \inf_{t\in[\frac{1}{2},2]} \frac{e^{-t}}{t^{\frac{5}{8}}} x = \infty > \rho_2. \tag{6.3}$$

Therefore, all the conditions of Theorem 5.1 have been verified.

According to Theorem 5.1, we know that there exists a constant λ^* such that BVP $(6.1)_\lambda$ has at least one positive solution for $\lambda = 0$ and $\lambda = \lambda^*$, two positive solutions for any $\lambda \in (0, \lambda^*)$, and no positive solution for $\lambda \in (\lambda^*, +\infty)$. Below we give a more accurate range for λ^*. Based on (6.2), there exists a sufficiently small $r_1 > 0$ such that $f\left(t, \left(\frac{1+t^3}{t^{\frac{1}{4}}}\right) x\right) < \rho_1 r_1$ for $x \in (0, r_1]$, $t \in (0, +\infty)$. Here, we might as well assume $r_1 = 0.2$. By (4.2), let $\lambda_* = \rho_1 r_1 \approx 0.05538$. By Theorem 4.1 and Theorem 4.2, BVP $(6.1)_\lambda$ has a positive solution for $\lambda \in [0, 0.05538]$. Based on (6.3), there exists a sufficiently large $r_2 > 0$ such that $f\left(t, \left(\frac{1+t^3}{t^{\frac{1}{4}}}\right) x\right) > \rho_2 r_2$ for $x \in [r_2, +\infty)$ and $t \in \left[\frac{1}{2}, 2\right]$. Here, we might as well assume $r_2 = 1000$. By (4.5), let $\hat{\lambda} = 2\psi \frac{\Gamma(\alpha)\Gamma(\alpha-1)k^2(1+k^{\sigma+3})}{\Gamma(\alpha-1)+\Gamma(\alpha)k+\Gamma(\alpha)\Gamma(\alpha-1)k^2} r_2 \approx 8102.92$. By Theorem 4.1 and Theorem 4.3, BVP $(6.1)_\lambda$ does not have positive solutions for $\lambda \in (8102.92, +\infty)$. Thus, we know that $\lambda^* \in [0.05538, 8102.92)$.

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