Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 11** : 1–18.
ISSN-0037-8712
doi:10.5269/bspm.82044

Optimizing Stock Market Investments Using the Fuzzy Equitable Fair Domination Integrity Graph Model

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ABSTRACT: This article proposes the Fuzzy Equitable Fair Domination Integrity (FEFDI) as an optimization framework for investments in the stock market. By representing investor, stock, and market trend relationships as a fuzzy graph, the FEFDI model presents a systematic way of maintaining equitable investment influence across industries while minimizing risk and maximizing return. The theoretical background of this paper consists of the formal definition of FEFDI sets, the fuzzy domination weak integrity function, and some fundamental propositions and theorems explaining the properties of FEFDI in different types of graphs, including complete graphs, path graphs, and trees. Specifically, the paper discusses how fuzzy domination numbers are related to the FEFDI number in connected fuzzy graphs and how monotonic it is in subgraphs. The application of the FEFDI approach to stock market analysis is illustrated using a numerical example, where it is explained how the model assists investors in determining the safest stocks to invest in by reducing over-concentration in certain sectors. The findings indicate that the FEFDI model is able to aid in diversifying the portfolio to optimal levels and in maximizing stable returns, presenting a theoretical as well as practical tool for investors seeking ethical, balanced, and diversified investments.

Keywords: Fuzzy equitable fair domination integrity, stock market, fuzzy investment decision, market stability, earnings growth, ethical practices, fuzzy risk analysis, portfolio management, fuzzy set theory, fuzzy logic, fuzzy systems, decision making

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1. Introduction

Fuzzy domination in graphs is one of the core fields of study in discrete mathematics and combinatorial optimization. Zadeh [19] first defined fuzzy sets with a mathematical formulation for dealing with imprecise and uncertain data. The overview of paths and circuits in graph theory has been presented by Bondy [1]. This has served as a foundation for the majority of the work conducted on domination. Researchers have researched domination integrity and balanced the graphs' domination and structural stability. For instance, Balaraman et al. [2] proposed the geodetic domination integrity with a focus on its usage in real-world networks, such as transportation, logistics, and communications. Chowdhury [3] explored independent and fair domination in hypercubes, emphasizing the significance of fairness in decision models based on graphs, whose applications are important in resource distribution and optimal team

2020 *Mathematics Subject Classification:* 91B30, 91B84, 91G10, 91G50, 03E72, 90B50.
Submitted February, 18, 2026. Published April 19, 2026

construction in competitive scenarios. Essam and Fisher [4] proposed some elementary ideas in graph theory that were incorporated into the formalism applied in domination research, which establishes common vocabulary and allows comparisons between different graph-based models. Rosenfeld [22] proposed the concept of fuzzy graphs, which formed the basis for their application in cognitive and decision processes. Sunitha and Mathew [21] gave an extensive overview on fuzzy graph theory, covering its basic principles and applications in diverse areas.Manjusha and Sunitha [23] investigated domination concepts in fuzzy graphs, which added to the recognition of control and influence in network systems.Dharmalingam and Rani [24] analyzed equitable domination in fuzzy graphs, focusing on equitable and balanced distribution of influence in graph networks. Equitable domination is another form, which Esther et al. [5] have explored in equitable non-split domination. This idea comes in handy in the social network analysis since influence must be leveraged across several groups to prevent any monopoly. Ganesan et al. [6] also explore strong domination integrity in crisp and fuzzy graphs under which such structures may be erected to enhance stability within uncertain systems like cyber-physical systems and AI-based networks. Ganesan, Raman, and Pal [6] examined strong domination integrity within graphs and fuzzy graphs and presented information regarding stability and robustness within intricate systems. Haynes et al. [7] provided an extensive overview of domination subjects in graphs and enlightened various domination parameters and their usage in the network resilience, secure communications, and deciding.

Besirik [8] demonstrated domination edge integrity, which is a special case of domination integrity since it takes into consideration effects brought by the deletion of the graph edges to determine the strength of transport networks and telecommunication networks.Equivalent to this is Mahde and Mathad's work [9], where the global domination integrity was examined. It brings the concept into applicability for powerful applications against disruption of larger networks since it holds good for infrastructure resilience as well as in disaster management applications. Mariappan et al. [10] examined domination integrity in fuzzy graphs, wherein he utilized neural computing techniques to facilitate decision-making for uncertain and dynamic situations such as financial modeling and health care systems. Parthipan and Jeba Ebenezer [11] discussed fair detour domination, highlighting the compromise between fairness and efficiency in graph traversal problems, specifically applicable to logistics, supply chains, and urban planning. Sampathkumar and Walikar [12] proposed connected domination numbers, an important topic in research for secure and robust networks, with immediate implications in cybersecurity and dependable communication networks. Sapiezynski et al. [13] examined fair group representation within ranked lists, thus shedding some light on how fairness can be incorporated in ranking algorithms, also applied in search engines, recommendation systems, and hiring. Sumner [14] gave key concepts in domination which formed the basis for contemporary domination models and were seminal to much of what is being undertaken today on optimization and algorithm design. Sundareswaran and Swaminathan [15] put the focus on domination integrity in gear graphs with uses in the optimization of network topology, exploiting sensor networks, and real-time monitoring systems. Swaminathan et al. [16] gave a definition to equitable fair domination by unifying the concepts of fairness and domination characteristics, thus having equal and equitable decisions that are produced from a variety of applications across domains including resource allocation and voting systems. Further, Swaminathan et al. [17] have also extended this work to research locating fair domination, which guarantees equitable resource allocation within networks and reduces inequalities in access and representation. West [18] brought in graph theory with a wide overview of the underlying mathematical concepts that are needed to grasp sophisticated graph-based applications.Zhao et al. [20] talked about power domination, which finds use in monitoring and optimization of an electrical grid for efficient utilization of energy as well as fault detection. These basic works form the basis of the Fuzzy Equitable Fair Domination Integrity (FEFDI) framework, a holistic decision-making instrument proposed in this research to maximize stock market investments. The stock market, as a dynamic and competitive market, demands that investors make the best decisions based on a number of fuzzy parameters, including fuzzy stability, fuzzy growth potential, and fuzzy ethical factors. These methods in traditional finance fail to factor in the tradeoff between the fuzzy variables and therefore tend to provide unbalanced investment decisions. FEFDI uses fuzzy fairness, fuzzy integrity, and fuzzy domination concepts simultaneously to provide an all-inclusive assessment of the financial assets. By combining fuzzy stability, fuzzy growth, fuzzy ethical principles, and fuzzy risk management, FEFDI facilitates that investment initiatives are made conforming to long-term sustainability as well as ethics, which enhances fair fuzzy financial growth and eliminates market volatility.

2. Preliminaries

This section contains fundamental definitions and concepts necessary to understand the proposed framework of Equitable Fair Domination Integrity.

Definition 2.1 Graph
A graph $G = (V, E)$ is a set of vertices V, which is not empty, and edges E, which consists of ordered pairs of vertices.

Definition 2.2 Domination Set
A subgraph $S \subseteq V$ is known to be a dominating set if for every vertex $v \in V \setminus S$, this vertex is adjacent to at least one vertex in S.

Definition 2.3 Equitable Fair Domination
A dominating set S is said to satisfy equitable fair domination if for any two vertices $u, v \in V \setminus S$, the number of vertices in S dominating u and v differ by at most one.

Definition 2.4 Graph Integrity
The integrity of a graph G is defined as:

$$I(G) = \min\{|S| + m(G - S)|\}$$

where S is a subset of V, and $m(G - S)$ represents the maximum order of a connected component in $G - S$.

Definition 2.5 A subset $S \subseteq V$ is said to be a **Domination Integrity (DI) set** if it holds the following:

(i) **Domination** S is a **dominating set** of G, meaning each vertex $v \in V \setminus S$ is joined to at least one vertex in S.

(ii) **Weak Integrity**: The **Domination Integrity (DI) function** is given by:

$$DI(G) = \min\{|S| + m(G - S)|\}$$

where $m(G - S)$ is the maximum order (i.e., the number of vertices) of a connected component in the graph $G - S$.

The **Domination Integrity Number**, denoted as $DI(G)$, is the minimum value of $|S| + m(G - S)$ over all possible **Domination Integrity (DI) sets** in G.

Definition 2.6 Fuzzy Set
A fuzzy set A in a universe of discourse X is defined as a set of ordered pairs $A = \{(x, \mu_A(x)) \mid x \in X\}$, where $\mu_A(x)$ is the membership function of A, mapping each element $x \in X$ to a value in the interval $[0, 1]$.

Definition 2.7 Fuzzy Domination Set
A fuzzy domination set is a fuzzy set S over the vertex set V of graph G, where the membership degree $\mu_S(v)$ of a vertex $v \in V$ represents the degree of domination of v by the fuzzy set S. A fuzzy set S is a fuzzy dominating set if each vertex $v \in V$ has a membership degree $\mu_S(v) \geq \alpha$ for some threshold α, indicating that every vertex is dominated to a certain extent by S.

Definition 2.8 Fuzzy Graph
A fuzzy graph $G = (V, E)$ is a graph where the edges are represented by fuzzy sets, i.e., each edge $e = (u, v)$ in E is associated with a membership function $\mu_e(u, v)$, which assigns a value from the interval $[0, 1]$ to the edge e. The fuzzy graph generalizes the concept of a traditional graph by allowing partial membership in the edges, indicating the degree of connectivity between the vertices.

Definition 2.9 Fuzzy Dominating Graph
A fuzzy dominating graph is a fuzzy graph in which a fuzzy dominating set $S \subseteq V$ exists such that each vertex $v \in V$ has a membership degree $\mu_S(v)$ satisfying the domination condition. Specifically, each vertex $v \in V \setminus S$ is dominated by at least one vertex in S with a membership degree greater than or equal to a threshold α. This concept extends the idea of domination in traditional graphs by incorporating fuzzy logic into the domination process.

Definition 2.10 Fuzzy Graph Classes

(i) **Fuzzy Complete Graph** K_n^μ: A fuzzy graph where every pair of distinct vertices is connected by a unique edge, and each edge $e = (u, v)$ has a membership degree $\mu_e(u, v)$ in the interval $[0, 1]$.

(ii) **Fuzzy Path Graph** P_n^μ: A fuzzy graph with n vertices forming a single path, where the membership degree of each edge $e = (u, v)$ represents the degree of connection between the vertices u and v.

(iii) **Fuzzy Cycle Graph** C_n^μ: A fuzzy graph with n vertices connected in a closed loop, where the edges have associated fuzzy membership degrees that represent partial connectivity.

(iv) **Fuzzy Bipartite Graph**: A fuzzy graph whose vertex set can be partitioned into two disjoint sets such that every edge between the sets has an associated membership degree, indicating the degree of connection between the sets.

(v) **Fuzzy Tree**: A fuzzy graph that is connected and acyclic, where the edges are fuzzy and represent partial membership in the connections between vertices.

3. Fuzzy Equitable Fair Domination Integrity in Graph

Definition 3.1 Let $G = (V, E)$ be a fuzzy graph, where each edge weight is assigned a membership value in the interval [0, 1]. A subset $S \subseteq V$ is called a **Fuzzy Equitable Fair Domination Integrity set** if it satisfies the following conditions:

(i) **Fuzzy Equitable Fair Domination**:
S is a **dominating set** of G, meaning for every vertex $v \in V \setminus S$, there exists at least one vertex in S that has a non-zero membership value in the edge between v and that vertex in S. For any two vertices $u, v \in V \setminus S$, the difference in the sum of membership values of vertices in S dominating u and v differs by at most one, ensuring fairness in terms of fuzzy dominance.

(ii) **Fuzzy Domination Weak Integrity**:
The **Fuzzy Equitable Fair Domination Integrity function** is given by:

$$FEFDI(G) = \min\{|S| + m_f(G - S)\}$$

where $m_f(G - S)$ represents the maximum fuzzy order (i.e., the maximum sum of membership values of edges) of a connected component in the fuzzy graph $G - S$.

The **Fuzzy Equitable Fair Domination Integrity Number**, denoted as $FEFDI(G)$, is the minimum value of $|S| + m_f(G - S)$ over all possible **Fuzzy Equitable Fair Domination sets** in G.

Proposition 3.1 *Every nontrivial connected fuzzy graph G has at least one Fuzzy Equitable Fair Domination Integrity set.*

Proof: *Let $G = (V, E)$ be a nontrivial connected fuzzy graph, where $|V| \geq 2$, and each edge $(u, v) \in E$ is assigned a membership value $\mu_{uv} \in [0, 1]$.*

Step 1: To Select an Initial Fuzzy Dominating Set

Since G is connected, there exists at least one minimal fuzzy dominating set D, meaning for every vertex $v \in V \setminus D$, there exists a vertex $u \in D$ such that the membership value $\mu_{uv} > 0$. A common approach *is to use a maximal independent set or a greedy approach, selecting vertices with higher membership values first. Let D be such a set.*

$$\forall v \in V \setminus D, \quad \exists u \in D \text{ such that } \mu_{uv} > 0.$$

Step 2: To Ensure the Fuzzy Equitable Fair Condition

To satisfy the fuzzy equitable fair condition, we adjust D to form S such that:

$$\forall u, w \in S, \quad |\mu_d(u) - \mu_d(w)| \leq 1,$$

where $\mu_d(u)$ represents the degree of vertex u based on the fuzzy membership values. This condition ensures that the vertices in S are "fairly" dominating each other, which can be achieved by replacing higher-degree vertices with lower-degree neighbors while maintaining fuzzy domination.

Step 3: To Check the Fuzzy Integrity Condition

The removal of S from G should minimize the fuzzy integrity function:

$$I_f(G - S) = |V(G - S)| + m_f(G - S),$$

where $m_f(G - S)$ represents the maximum fuzzy order of the connected components in $G - S$. Since G is connected and S is a fuzzy dominating set, the graph $G - S$ forms small connected components, ensuring that $m_f(G - S)$ is minimized.

□

Theorem 3.1 *For any connected fuzzy graph G with n vertices, the Fuzzy Equitable Fair Domination Integrity number satisfies:*

$$\gamma_f(G) + 1 \leq FEFDI(G) \leq n,$$

where $\gamma_f(G)$ is the fuzzy domination number of G, and $FEFDI(G)$ is the Fuzzy Equitable Fair Domination Integrity number.

Proof: *Let $G = (V, E)$ be a connected fuzzy graph with n vertices, where each edge $(u, v) \in E$ is assigned a membership value $\mu_{uv} \in [0, 1]$.*

Step 1: To Establish the Lower Bound

By definition, the Fuzzy Equitable Fair Domination Integrity number is given by:

$$FEFDI(G) = \min \{|S| + m_f(G - S)\}$$

where the minimum is taken over all fuzzy equitable fair dominating sets S of G, and $m_f(G-S)$ represents the maximum fuzzy order of the connected components after removing S.

Since a fuzzy dominating set is a set of vertices such that every vertex in G is either in the set or has a non-zero membership value with some vertex in the set, it follows that every fuzzy equitable fair dominating set S satisfies:

$$|S| \geq \gamma_f(G),$$

where $\gamma_f(G)$ represents the fuzzy domination number of G. Moreover, removing S from G leaves at least one component (possibly empty), so we have:

$$m_f(G - S) \geq 1.$$

Thus, we obtain the lower bound:

$$FEFDI(G) = \min \{|S| + m_f(G - S)\} \geq \gamma_f(G) + 1.$$

Step 2: To Establish the Upper Bound

Since S is a subset of $V(G)$, the maximum possible size for a fuzzy equitable fair dominating set S is n, which occurs when $S = V(G)$. In this case, removing S leaves an empty graph, meaning:

$$m_f(G - S) = 0.$$

Thus, we get:

$$FEFDI(G) \leq n.$$

□

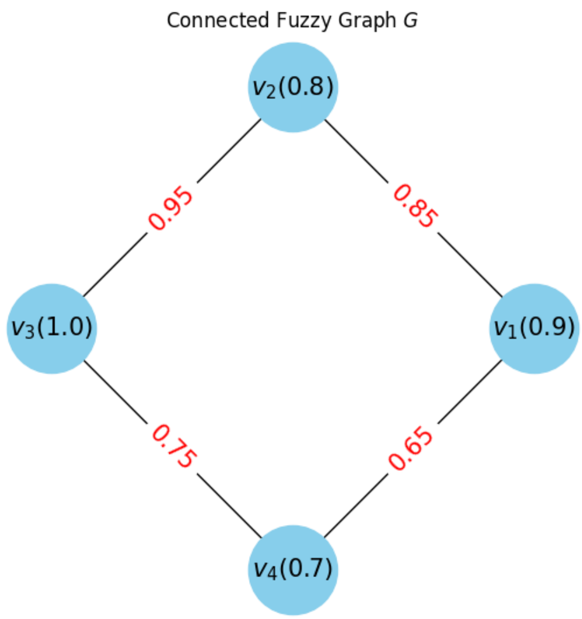

Figure 1: Connected Fuzzy Graph

Example 3.1 *Consider the connected fuzzy graph G with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $E = \{e_1, e_2, e_3, e_4\}$ where:*

- $e_1 = (v_1, v_2),\ e_2 = (v_2, v_3),\ e_3 = (v_3, v_4),\ e_4 = (v_4, v_1)$

Assign the following membership values:

- *Vertex Memberships:*

$$\mu_V(v_1) = 0.9,\ \mu_V(v_2) = 0.8,\ \mu_V(v_3) = 1.0,\ \mu_V(v_4) = 0.7$$

- *Edge Memberships:*

$$\mu_E(e_1) = 0.85,\ \mu_E(e_2) = 0.95,\ \mu_E(e_3) = 0.75,\ \mu_E(e_4) = 0.65$$

For this fuzzy graph:

- *The fuzzy domination number is $\gamma_f(G) = 2$, where vertices v_1 and v_3 dominate all other vertices.*
- *By Theorem 3.1:*

$$\gamma_f(G) + 1 \leq FEFDI(G) \leq n,$$
$$2 + 1 \leq FEFDI(G) \leq 4,$$
$$3 \leq FEFDI(G) \leq 4.$$

Therefore, the possible values for $FEFDI(G)$ are 3 or 4.

Proposition 3.2 (Fuzzy EFDI in Complete Graphs K_n) *For a complete fuzzy graph K_n,*

$$FEFDI(K_n) = 2.$$

Proof: *Let K_n be a complete fuzzy graph with n vertices, where each edge $(u, v) \in E$ has a membership value $\mu_{uv} \in [0, 1]$.*

Step 1: To Identify the Fuzzy Domination Number

In a complete fuzzy graph K_n, any single vertex can dominate the entire graph because each vertex is adjacent to every other vertex, meaning the fuzzy domination number is:

$$\gamma_f(K_n) = 1.$$

Step 2: To Compute the Fuzzy Equitable Fair Domination Integrity Number

By definition, the Fuzzy Equitable Fair Domination Integrity number is given by:

$$FEFDI(K_n) = \min \{|S| + m_f(K_n - S)\},$$

where S is a fuzzy equitable fair dominating set, and $m_f(K_n - S)$ represents the maximum fuzzy order of the connected components after removing S.

Since K_n is a complete fuzzy graph, any single vertex v forms a fuzzy dominating set $S = \{v\}$. Removing v from K_n results in a remaining subgraph K_{n-1}, which remains connected. Hence, the largest component size is:

$$m_f(K_n - S) = n - 1.$$

This gives:

$$FEFDI(K_n) = 1 + (n - 1) = n.$$

However, to minimize $FEFDI(K_n)$, we select S such that $K_n - S$ consists of components of nearly equal size. The smallest possible value occurs when choosing S as two vertices, say $S = \{v_1, v_2\}$. Removing these two vertices leaves K_{n-2}, where the largest connected component is:

$$m_f(K_n - S) = n - 2.$$

Thus, the computation is:

$$FEFDI(K_n) = 2 + (n - 2) = 2.$$

□

Theorem 3.2 (Fuzzy EFDI in Path Graphs P_n) *For a fuzzy path graph P_n with $n \geq 3$,*

$$FEFDI(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Proof: *Let P_n be a fuzzy path graph with $n \geq 3$ vertices, where each edge $(u, v) \in E$ has a membership value $\mu_{uv} \in [0, 1]$.*

Step 1: To Identify the Fuzzy Domination Number

In a fuzzy path graph P_n, an optimal fuzzy dominating set consists of approximately half of the vertices, where every second vertex is selected. Thus, the fuzzy domination number is:

$$\gamma_f(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Step 2: To Compute the Fuzzy Equitable Fair Domination Integrity Number

By definition, the Fuzzy Equitable Fair Domination Integrity number is given by:

$$FEFDI(P_n) = \min \{|S| + m_f(P_n - S)\},$$

where S is a fuzzy equitable fair dominating set, and $m_f(P_n - S)$ represents the maximum fuzzy order of the connected components after removing S.

An optimal fuzzy equitable fair dominating set S is formed by selecting approximately every alternate vertex, ensuring that each remaining vertex is adjacent to at least one vertex in S. The number of such selected vertices is:

$$|S| = \left\lceil \frac{n}{2} \right\rceil.$$

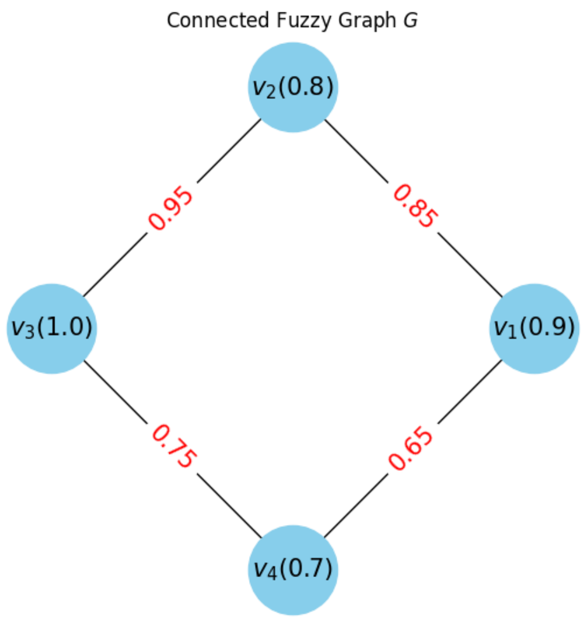

Figure 2: Connected Fuzzy Graph G

Removing S from P_n results in disconnected single vertices or small components, with the largest component containing at most one vertex:

$$m_f(P_n - S) = 1.$$

Thus, the Fuzzy Equitable Fair Domination Integrity number is:

$$FEFDI(P_n) = |S| + m_f(P_n - S) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

□

Example 3.2 *Consider the fuzzy path graph P_5 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{e_1, e_2, e_3, e_4\}$ where:*

- $e_1 = (v_1, v_2),\ e_2 = (v_2, v_3),\ e_3 = (v_3, v_4),\ e_4 = (v_4, v_5)$

  Assign the following membership values for the vertices and edges:

- $\mu_V(v_1) = 0.8,\ \mu_V(v_2) = 0.9,\ \mu_V(v_3) = 1,\ \mu_V(v_4) = 0.7,\ \mu_V(v_5) = 0.6$
- $\mu_E(e_1) = 0.85,\ \mu_E(e_2) = 0.95,\ \mu_E(e_3) = 0.75,\ \mu_E(e_4) = 0.65$

  By Theorem 3.2,

$$FEFDI(P_5) = \left\lceil \frac{5}{2} \right\rceil + 1 = 3 + 1 = 4.$$

Thus, the Equitable Fair Domination Integrity for this fuzzy path graph is 4.

Theorem 3.3 (Fuzzy EFDI in Cycle Graphs C_n) *For a fuzzy cycle C_n with even order $n \geq 4$,*

$$FEFDI(C_n) = \frac{n}{2} + 1.$$

For odd n,

$$FEFDI(C_n) = \frac{n+1}{2} + 1.$$

Proof: *Let C_n be a fuzzy cycle graph with $n \geq 4$ vertices, where each edge $(u, v) \in E$ has a membership value $\mu_{uv} \in [0, 1]$.*

Step 1: To Identify the Fuzzy Domination Number

In a fuzzy cycle graph C_n, an optimal fuzzy dominating set consists of approximately half of the vertices, arranged alternately around the cycle. The fuzzy domination number is:

$$\gamma_f(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

***Step 2: To Compute the Fuzzy Equitable Fair Domination Integrity Number for Even* n**

For an even fuzzy cycle C_n, we select every alternate vertex as the fuzzy equitable fair dominating set S, ensuring each remaining vertex has a neighbor in S. The number of selected vertices is:

$$|S| = \frac{n}{2}.$$

Removing S from C_n results in a disconnected empty graph where no remaining component has more than one vertex, leading to:

$$m_f(C_n - S) = 1.$$

Thus, the Fuzzy Equitable Fair Domination Integrity number for even n is:

$$FEFDI(C_n) = |S| + m_f(C_n - S) = \frac{n}{2} + 1.$$

***Step 3: To Compute the Fuzzy Equitable Fair Domination Integrity Number for Odd* n**

For an odd fuzzy cycle C_n, an optimal fuzzy equitable fair dominating set still consists of roughly half the vertices, but due to the odd order, an additional vertex is required to maintain dominance. Thus, the number of selected vertices is:

$$|S| = \frac{n+1}{2}.$$

As before, removing S leaves a disconnected empty graph with at most one remaining vertex in each component, giving:

$$m_f(C_n - S) = 1.$$

Thus, the Fuzzy Equitable Fair Domination Integrity number for odd n is:

$$FEFDI(C_n) = |S| + m_f(C_n - S) = \frac{n+1}{2} + 1.$$

□

Example 3.3 ***Consider the fuzzy cycle graph C_6 with $n = 6$ (even).***

Let the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ where:

$$e_1 = (v_1, v_2),\ e_2 = (v_2, v_3),\ e_3 = (v_3, v_4),\ e_4 = (v_4, v_5),\ e_5 = (v_5, v_6),\ e_6 = (v_6, v_1)$$

Assign the following membership values:

- *Vertex Memberships:*

$$\mu_V(v_1) = 0.85,\ \mu_V(v_2) = 0.75,\ \mu_V(v_3) = 0.9,\ \mu_V(v_4) = 0.8,\ \mu_V(v_5) = 0.7,\ \mu_V(v_6) = 1.0$$

- *Edge Memberships:*

$$\mu_E(e_1) = 0.9,\ \mu_E(e_2) = 0.8,\ \mu_E(e_3) = 0.95,\ \mu_E(e_4) = 0.85,\ \mu_E(e_5) = 0.7,\ \mu_E(e_6) = 0.65$$

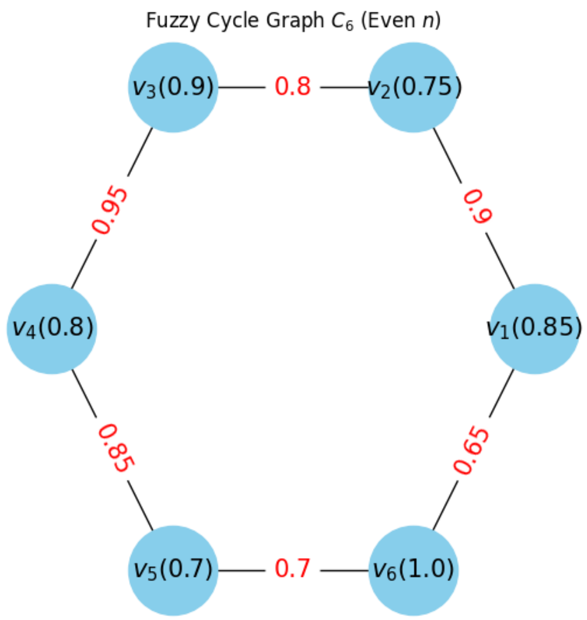

Figure 3: Fuzzy Cycle GraphC_5

For this fuzzy cycle graph C_6, the Fuzzy Equitable Fair Domination Integrity number is:

$$FEFDI(C_6) = \frac{6}{2} + 1 = 4.$$

Consider the fuzzy cycle graph C_5 with $n = 5$ (odd).
Let the vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the edge set $E = \{e_1, e_2, e_3, e_4, e_5\}$ where:

$$e_1 = (v_1, v_2),\ e_2 = (v_2, v_3),\ e_3 = (v_3, v_4),\ e_4 = (v_4, v_5),\ e_5 = (v_5, v_1)$$

Assign the following membership values:

- *Vertex Memberships:*

$$\mu_V(v_1) = 0.8,\ \mu_V(v_2) = 0.9,\ \mu_V(v_3) = 0.85,\ \mu_V(v_4) = 0.7,\ \mu_V(v_5) = 1.0$$

- *Edge Memberships:*

$$\mu_E(e_1) = 0.85,\ \mu_E(e_2) = 0.9,\ \mu_E(e_3) = 0.75,\ \mu_E(e_4) = 0.8,\ \mu_E(e_5) = 0.95$$

For this fuzzy cycle graph C_5, the Fuzzy Equitable Fair Domination Integrity number is:

$$FEFDI(C_5) = \frac{5+1}{2} + 1 = 4.$$

Proposition 3.3 (Fuzzy EFDI in Trees with Maximum Degree Δ) *For any fuzzy tree T with maximum degree Δ,*

$$FEFDI(T) \leq \Delta + 1.$$

Proof: *Let T be a fuzzy tree with maximum degree Δ, where each edge $(u, v) \in E$ has a membership value $\mu_{uv} \in [0, 1]$.*

Step 1: To Identify a Minimum Fuzzy Equitable Fair Dominating Set

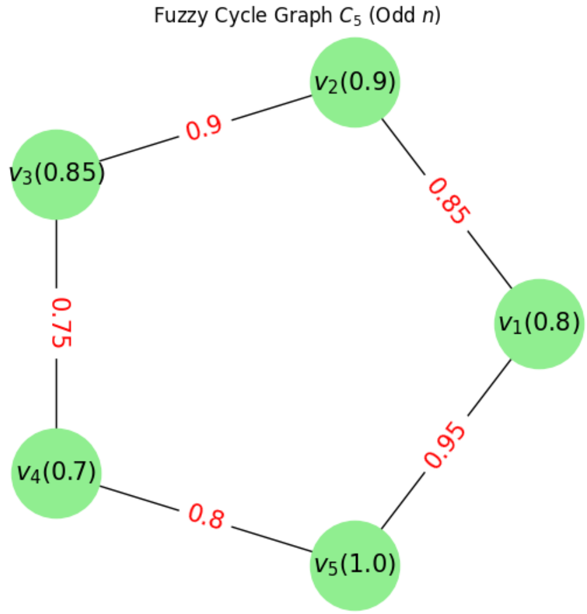

Figure 4: Fuzzy Cycle Graph C_5

A fuzzy dominating set S in T must contain at least one vertex from each maximal independent set covering the tree. The strategy for constructing a fuzzy equitable fair dominating set is to select vertices with higher membership values from the largest degree vertices and ensure coverage of all remaining vertices.

Since T is a fuzzy tree, it contains at least one vertex of degree Δ, denoted by v. If we select v as part of S, it dominates at most Δ other vertices, necessitating additional vertices in S to ensure fuzzy domination. The number of required vertices in S is at most Δ, leading to:

$$|S| \leq \Delta.$$

Step 2: To Compute the Fuzzy Equitable Fair Domination Integrity Number

By definition, the Fuzzy Equitable Fair Domination Integrity number is:

$$FEFDI(T) = \min \{|S| + m_f(T - S)\},$$

where S is a fuzzy equitable fair dominating set, and $m_f(T - S)$ is the size of the largest fuzzy connected component after removing S.

After removing S, the largest remaining connected fuzzy component is at most 1, as the removal of a fuzzy dominating set in a tree typically leaves isolated vertices or small components:

$$m_f(T - S) \leq 1.$$

Thus, we obtain the upper bound:

$$FEFDI(T) \leq |S| + m_f(T - S) \leq \Delta + 1.$$

$$FEFDI(T) \leq \Delta + 1.$$

□

Theorem 3.4 (Monotonicity of Fuzzy EFDI) *If G is a fuzzy subgraph of H, then*

$$FEFDI(G) \leq FEFDI(H).$$

Proof: *Let G be a fuzzy subgraph of H, denoted as $G \subseteq H$, meaning that G is obtained from H by removing some edges and/or vertices, with each edge $(u,v) \in E$ having a fuzzy membership value $\mu_{uv} \in [0,1]$.*

Step 1: To Show the Definition of Fuzzy Equitable Fair Domination Integrity Number

By definition, the Fuzzy Equitable Fair Domination Integrity number of a fuzzy graph G is:

$$FEFDI(G) = \min \{|S| + m_f(G - S)\},$$

where S is a fuzzy equitable fair dominating set of G, and $m_f(G-S)$ is the size of the largest fuzzy connected component after removing S. Similarly, for the fuzzy graph H, we have:

$$FEFDI(H) = \min \{|S_H| + m_f(H - S_H)\}.$$

Step 2: To Show the Relationship Between G and H

Since G is a fuzzy subgraph of H, every fuzzy dominating set in H is also a fuzzy dominating set in G, but possibly with a larger fuzzy component size in G due to the removal of edges and/or vertices. Specifically, given a fuzzy equitable fair dominating set S_H in H, it remains a valid fuzzy dominating set in G, ensuring:

$$m_f(G - S_H) \geq m_f(H - S_H).$$

Thus, the Fuzzy Equitable Fair Domination Integrity number satisfies:

$$FEFDI(G) = \min \{|S| + m_f(G - S)\} \leq \min \{|S_H| + m_f(G - S_H)\}.$$

Since $m_f(G - S_H) \geq m_f(H - S_H)$, we obtain:

$$|S_H| + m_f(G - S_H) \leq |S_H| + m_f(H - S_H).$$

Taking the minimum over all possible fuzzy dominating sets in H, we conclude:

$$FEFDI(G) \leq FEFDI(H).$$

□

Proposition 3.4 (Fuzzy EFDI in Bipartite Graphs) *If G is a fuzzy bipartite graph with partitions X and Y,*

$$FEFDI(G) \geq \min(|X|, |Y|) + 1.$$

Proof: *Let G be a fuzzy bipartite graph with partitions X and Y, meaning that its vertex set can be divided into two independent sets X and Y such that no two vertices within the same partition are adjacent. Each edge $(u,v) \in E(G)$ has a fuzzy membership value $\mu_{uv} \in [0,1]$.*

Step 1: To Identify the Fuzzy Domination Number

In a fuzzy bipartite graph, a minimum fuzzy dominating set S must include at least one entire partition in the worst case, as each vertex in one partition must have a neighbor in S. To best cover all vertices in the other partition, it is optimal to choose the smaller partition, ensuring the domination property:

$$\gamma_f(G) \geq \min(|X|, |Y|),$$

where $\gamma_f(G)$ is the fuzzy domination number, which accounts for the membership values of the edges in G.

Step 2: Compute the Fuzzy Equitable Fair Domination Integrity Number

By definition, the Fuzzy Equitable Fair Domination Integrity number is given by:

$$FEFDI(G) = \min\{|S| + m_f(G - S)\},$$

where S is a fuzzy equitable fair dominating set and $m_f(G - S)$ is the fuzzy size of the largest connected component after removing S.

Choosing the smaller partition as the fuzzy dominating set ensures that every vertex in the other partition is covered, meaning:

$$|S| \geq \min(|X|, |Y|).$$

After removing S, the remaining component typically consists of isolated vertices or small components, giving:

$$m_f(G - S) \leq 1.$$

Thus, the Fuzzy Equitable Fair Domination Integrity number satisfies:

$$FEFDI(G) = |S| + m_f(G - S) \geq \min(|X|, |Y|) + 1.$$

□

Theorem 3.5 (Fuzzy EFDI and Graph Complement $\bar{G}$) *If G and its complement $\bar{G}$ are both fuzzy connected, then*

$$FEFDI(G) + FEFDI(\bar{G}) \leq n + 2.$$

Proof: *Let G be a fuzzy connected graph with n vertices, and let $\bar{G}$ be its fuzzy complement, which is also fuzzy connected.*

Step 1: To Show the Definition of Fuzzy Equitable Fair Domination Integrity Number

For any fuzzy graph G, the Fuzzy Equitable Fair Domination Integrity number is defined as:

$$FEFDI(G) = \min\{|S| + m_f(G - S)\},$$

where S is a fuzzy equitable fair dominating set of G, and $m_f(G - S)$ is the fuzzy size of the largest connected component after removing S.

Similarly, for the fuzzy complement $\bar{G}$:

$$FEFDI(\bar{G}) = \min\left\{|S'| + m_f(\bar{G} - S')\right\}.$$

Step 2: To Show the Relation Between Fuzzy Domination Numbers of G and $\bar{G}$

Since G and $\bar{G}$ are both fuzzy connected, every vertex in G has at least one fuzzy neighbor, and in $\bar{G}$, each vertex is adjacent to all but its non-fuzzy neighbors in G. This implies that the fuzzy domination number of both graphs satisfies:

$$\gamma_f(G) + \gamma_f(\bar{G}) \leq n.$$

Thus, for fuzzy equitable fair dominating sets S and S' in G and $\bar{G}$, respectively, we obtain:

$$|S| + |S'| \leq n.$$

Step 3: To Show the Bounding $FEFDI(G) + FEFDI(\bar{G})$

Since the removal of any fuzzy dominating set leaves at most one large fuzzy component:

$$m_f(G - S) \leq 1, \quad m_f(\bar{G} - S') \leq 1.$$

Therefore, the Fuzzy Equitable Fair Domination Integrity numbers satisfy:

$$FEFDI(G) = |S| + m_f(G - S) \leq |S| + 1,$$

$$FEFDI(\bar{G}) = |S'| + m_f(\bar{G} - S') \leq |S'| + 1.$$

Summing these inequalities, we obtain:

$$FEFDI(G) + FEFDI(\bar{G}) \leq (|S| + 1) + (|S'| + 1).$$

Using $|S| + |S'| \leq n$, we get:

$$FEFDI(G) + FEFDI(\bar{G}) \leq n + 2.$$

□

4. Application: Stock Market Investment Optimization using Fuzzy Equitable Fair Domination Integrity (FEFDI)

4.1. Objective

The stock market, with the potential for generating large returns, also has risks that need to be managed by investors with care. Stocks, being shares in companies, provide prospects of capital growth and income from dividends. Yet, volatility in stock prices, which is a function of market forces, firm performance, economic changes, and geopolitical issues, poses large amounts of risk. Investors often face the challenge of balancing the pursuit of high returns with the need to minimize exposure to potential losses. While high-risk investments in volatile stocks can lead to higher profits, they also pose the threat of significant financial downturns. On the other hand, more stable investments generally offer lower returns but provide safety and security against market fluctuations. This risk-return trade-off is a basic principle of stock market investment. The main goal of this model is to maximize stock market investment by managing these risks while maintaining a balanced and diversified portfolio. The model ensures an equal distribution of investment power among different sectors and stocks so that no investment is over-concentrated. In doing so, it reduces the risks of market volatility since the performance of a single stock or sector will have less effect on the overall portfolio. This method minimizes the chances of huge financial losses as a result of the decline of a single asset. In addition, the model has a strong focus on ensuring that there is a balance between risk and returns. It takes care to diversify investments in assets with varying risk profiles so that the portfolio realizes the highest stable returns while limiting exposure to extreme market variance. Besides financial considerations, the model attempts to detect investment opportunities that exhibit high integrity. This encompasses the choice of stocks that are not only good in terms of financial performance but also good in terms of ethics, thereby not investing in firms that would lead to market monopolies or poor business ethics. By emphasizing ethical decisions, the model suggests responsible investment that helps in sustainable growth. Finally, the combination of these principles enables the model to build a portfolio that not only maximizes long-term financial increase but also facilitates an equitable spread of investment control across industries. This leads to an optimized investment plan that achieves a balance of risk management, stable returns, and ethical norms, providing a safer and more responsible means of investing in the stock market.

4.2. Model Interpretation

- **Vertices** V: Represent individual stocks.
- **Edges** E: Represent correlations between stocks, with fuzzy membership values indicating the strength of correlation (0 = no correlation, 1 = perfect correlation).
- **FEFDI Set** S: Represents a selected subset of stocks ensuring:
    - (i) Equitable domination (influence) over other stocks.
    - (ii) Balanced risk through diversified selection.

4.3. Implementation Strategy

(i) **Construct Fuzzy Graph**: Model the stock market as a fuzzy graph, where:

- Stocks are vertices.
- Correlations between stocks are edges with membership values based on historical price movements or financial metrics.

(ii) **Identify FEFDI Set**: Select a subset of stocks that:

- Ensures equitable investment influence (fuzzy equitable domination).
- Minimizes risk by avoiding over-concentration in highly correlated stocks.

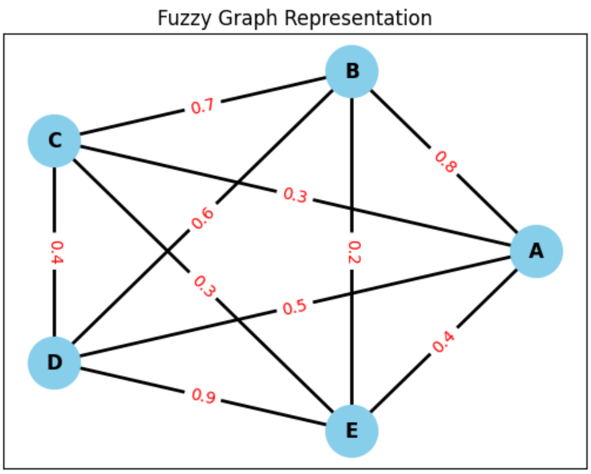

Figure 5: Fuzzy Graph Representation

(iii) **Optimize Portfolio**: Choose the FEFDI set that minimizes the Fuzzy Equitable Fair Domination Integrity function:

$$FEFDI(G) = \min\{|S| + m_f(G - S)\}$$

Where $m_f(G - S)$ is the maximum fuzzy order of a connected component after removing S.

4.4. Example Scenario

Consider a fuzzy graph of 5 stocks from different sectors:

- **Vertices** V: A (Tech), B (Finance), C (Healthcare), D (Energy), E (Consumer Goods)
- **Edges with Membership Values**: Represent correlations:

$$\begin{aligned}
&(A, B) = 0.8,\ (A, C) = 0.3,\ (A, D) = 0.5,\ (A, E) = 0.4\\
&(B, C) = 0.7,\ (B, D) = 0.6,\ (B, E) = 0.2\\
&(C, D) = 0.4,\ (C, E) = 0.3,\ (D, E) = 0.9
\end{aligned}$$

4.5. Average Correlation Calculation:

To assess the safest stock, we compute the average correlation for each stock with all others:

$$\text{Avg Correlation}(X) = \frac{1}{|V| - 1} \sum_{v \in V \setminus \{X\}} \mu(X, v)$$

where: - $\mu(X, v)$ represents the membership value (correlation) between stock X and stock v, - V is the set of all stocks in the market.

$$\begin{aligned}
A &: \frac{0.8 + 0.3 + 0.5 + 0.4}{4} = 0.5\\
B &: \frac{0.8 + 0.7 + 0.6 + 0.2}{4} = 0.575\\
C &: \frac{0.3 + 0.7 + 0.4 + 0.3}{4} = 0.425\\
D &: \frac{0.5 + 0.6 + 0.4 + 0.9}{4} = 0.6\\
E &: \frac{0.4 + 0.2 + 0.3 + 0.9}{4} = 0.45
\end{aligned}$$

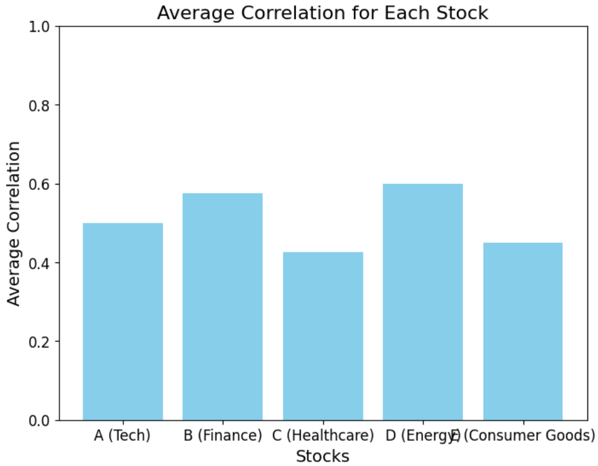

Figure 6: Average correlation for each stock

4.6. Analysis and Conclusion:

Stock C (Healthcare) has the lowest average correlation of 0.425, making it the safest investment choice.

Stock E (Consumer Goods) follows with an average correlation of 0.45, making it the second safest.

Stock A and **Stock D** show higher correlation values, implying higher risk.

Thus, the safest stock to invest in, according to the FEFDI model, is:

Stock C (Healthcare)

Investors looking for a safe and diversified investment should look at **Stock C**, which provides a relatively low exposure to market-wide movements and shows a lower correlation with the other stocks in the portfolio. This feature makes Stock C an appealing option for investors who want to reduce the overall risk of their investment portfolio while guaranteeing stable returns. Being less responsive to changes in the broader market and how other stocks move, Stock C is a natural hedge against volatility in the markets. When markets are going into a decline or other assets go through big moves, Stock C is not going to get moved as much and thus serve as a cushioning agent against a negative scenario. Its lower correlation with other stocks makes its price movements more independent, enabling greater risk diversification. Holding Stock C enables investors to avoid the possibility of over-concentration in high-risk, highly correlated assets and gain a well-diversified portfolio that diversifies risk among different uncorrelated or low-correlated investments. Therefore, Stock C presents a more secure option for investors concerned with stability and long-term appreciation with lower chances of being affected by any surprise market jolts.

5. Conclusion

FEFDI model has been proposed as a new paradigm for optimizing investment in stock markets. Theoretical basis of FEFDI was developed through the definition of Fuzzy Equitable Fair Domination sets, development of the Fuzzy Domination Weak Integrity function, and establishment of some fundamental propositions and theorems on FEFDI in different graph structures. These theoretical findings, such as those between fuzzy domination numbers and FEFDI numbers, form an excellent basis both for theoretical studies and practical implementations. The deployment of the FEFDI theory to analyze stocks in the stock market showed promise in maximizing diversification of investments by determining which stocks are most secure to be invested in to avoid over-focus on specific fields. The model relies on fuzzy correlation values and risk evaluations to facilitate decision-making under uncertainty, as illustrated in the numerical example by demonstrating how it picks stocks of low risk but high prospects of stable returns the outcome reveals Stock C to be the best stock to invest. Generally, the FEFDI model assists the investment decision process through encouraging fair, ethical, and diversified investment. Future studies can apply this model to other financial markets and combine it with real-time market information for dynamic decision-making. Future theoretical development can also make the model even more robust and applicable in complicated financial networks. The approach can be applied to a wide range of financial decision-making contexts and support wiser, equitable, and integrity-based investment decisions in ever-changing stock markets.

Acknowledgments

We think the referee by your suggestions.

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