



## $\mathbb{P}_{\text{CIF}}$ - $Z$ -Open Sets in Cubic Intuitionistic Fuzzy Topology with Application to IoT-Enabled Smart Greenhouse Monitoring

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**ABSTRACT:** This paper introduces and investigates the concept of  $Z$ -open sets in cubic intuitionistic fuzzy topological spaces under  $\mathbb{P}_{\text{CIF}}$ -order. We establish fundamental properties of  $\mathbb{P}_{\text{CIF}}$ - $Z$ -open sets and their relationships with other generalized open sets including  $\delta$ -open, preopen, and  $\delta$ -semiopen sets. Several characterizations of  $\mathbb{P}_{\text{CIF}}$ - $Z$ -open sets are provided through interior and closure operators. We prove that the family of  $\mathbb{P}_{\text{CIF}}$ - $Z$ -open sets forms a topology and investigate properties of  $\mathbb{P}_{\text{CIF}}$ - $Z$ -interior and  $\mathbb{P}_{\text{CIF}}$ - $Z$ -closure operators. The results demonstrate that  $\mathbb{P}_{\text{CIF}}$ - $Z$ -open sets provide a unifying framework that generalizes several existing notions of openness in cubic intuitionistic fuzzy topological spaces, with significant implications for modeling hierarchical uncertainty in decision-making systems.

**Keywords:** Cubic intuitionistic fuzzy set,  $\mathbb{P}_{\text{CIF}}$ -order,  $Z$ -open set,  $\delta$ -open set, preopen set, cubic intuitionistic fuzzy topology.

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### 1. Introduction

Uncertainty, indeterminacy, and vagueness are inherent characteristics of real-world phenomena that classical mathematics often fails to capture adequately. Zadeh’s fuzzy set theory [10] initiated a paradigm shift by allowing partial membership, while Atanassov’s intuitionistic fuzzy sets [1] further enriched this framework by incorporating non-membership and hesitation degrees. The evolution continued with interval-valued intuitionistic fuzzy sets [2] and cubic intuitionistic fuzzy sets (CIFS) [6,7], which integrate both interval-valued and crisp representations of uncertainty within a unified structure.

The extension of these concepts into topology has yielded significant theoretical developments. Coker [3] pioneered intuitionistic fuzzy topological spaces, while Jun et al. [5] introduced cubic sets, leading to cubic intuitionistic fuzzy topological spaces (CIFTS). A crucial advancement was the incorporation of order structures into CIFTS by Riaz et al. [9], who developed  $\mathbb{P}_{\text{CIF}}$ -cubic and  $\mathbb{R}$ -cubic intuitionistic fuzzy topologies to model hierarchical and relational uncertainties.

Within generalized topology,  $Z$ -open sets were originally introduced by El-Magharabi and Mubarki [4] in classical topological spaces as a unifying concept that bridges preopen, semiopen, and  $e$ -open sets. Moogambigai et al. [8] later extended this concept to neutrosophic topological spaces. However, the study of  $Z$ -open sets within the dual-order framework of cubic intuitionistic fuzzy topology remains largely unexplored.

This paper addresses this gap by introducing  $Z$ -open sets under  $\mathbb{P}_{\text{CIF}}$ -order in CIFTS. The  $\mathbb{P}_{\text{CIF}}$ -order provides a natural framework for modeling hierarchical decision-making where membership degrees increase and non-membership degrees decrease along the preference hierarchy. Our contributions include:

- Formal definition of  $\mathbb{P}_{\text{CIF}}$ - $Z$ -open sets in CIFTS and establishment of their fundamental properties

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- Characterization theorems relating  $\mathbb{P}_{\text{CIF}}$ - $Z$ -open sets to other generalized open sets
- Development of  $\mathbb{P}_{\text{CIF}}$ - $Z$ -interior and  $\mathbb{P}_{\text{CIF}}$ - $Z$ -closure operators with their algebraic properties
- Proof that the family of  $\mathbb{P}_{\text{CIF}}$ - $Z$ -open sets forms a topology finer than the original cubic intuitionistic fuzzy topology

These results provide a robust theoretical foundation for applications in multi-criteria decision analysis, pattern recognition, and intelligent systems where hierarchical uncertainty modeling is essential.

## 2. Preliminaries

Let  $X$  be a non-empty universe of discourse. We recall essential definitions for our study.

**Definition 2.1 ([1])** An intuitionistic fuzzy set (IFS)  $A$  in  $X$  is defined as

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where  $\mu_A, \nu_A : X \rightarrow [0, 1]$  satisfy  $0 \leq \mu_A(x) + \nu_A(x) \leq 1$  for all  $x \in X$ . Here  $\mu_A(x)$  and  $\nu_A(x)$  represent membership and non-membership degrees of  $x$  to  $A$ , respectively.

**Definition 2.2 ([2])** An interval-valued intuitionistic fuzzy set (IVIFS)  $A$  in  $X$  is given by

$$A = \{(x, [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)]) \mid x \in X\},$$

where  $0 \leq \mu_A^-(x) \leq \mu_A^+(x) \leq 1$ ,  $0 \leq \nu_A^-(x) \leq \nu_A^+(x) \leq 1$ , and  $\mu_A^+(x) + \nu_A^+(x) \leq 1$  for all  $x \in X$ .

**Definition 2.3 ([6])** A cubic intuitionistic fuzzy set (CIFS)  $\mathcal{A}$  in  $X$  is an ordered pair  $\mathcal{A} = (\tilde{A}, \lambda)$  where:

- $\tilde{A} = \{([\mu_{\tilde{A}}^-(x), \mu_{\tilde{A}}^+(x)], [\nu_{\tilde{A}}^-(x), \nu_{\tilde{A}}^+(x)]) \mid x \in X\}$  is an IVIFS
- $\lambda = \{(\mu_\lambda(x), \nu_\lambda(x)) \mid x \in X\}$  is an IFS

satisfying  $0 \leq \mu_{\tilde{A}}^-(x) \leq \mu_{\tilde{A}}^+(x) \leq 1$ ,  $0 \leq \nu_{\tilde{A}}^-(x) \leq \nu_{\tilde{A}}^+(x) \leq 1$ ,  $\mu_{\tilde{A}}^+(x) + \nu_{\tilde{A}}^+(x) \leq 1$ , and  $\mu_\lambda(x) + \nu_\lambda(x) \leq 1$  for all  $x \in X$ .

**Definition 2.4 ([9])** For two CIFSs  $\mathcal{A} = (\tilde{A}, \lambda)$  and  $\mathcal{B} = (\tilde{B}, \mu)$  in  $X$ , the  $\mathbb{P}_{\text{CIF}}$ -order is defined as:

$$\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{B} \iff \begin{cases} [\mu_{\tilde{A}}^-(x), \mu_{\tilde{A}}^+(x)] \subseteq [\mu_{\tilde{B}}^-(x), \mu_{\tilde{B}}^+(x)], \\ [\nu_{\tilde{A}}^-(x), \nu_{\tilde{A}}^+(x)] \supseteq [\nu_{\tilde{B}}^-(x), \nu_{\tilde{B}}^+(x)], \\ \mu_\lambda(x) \leq \mu_\mu(x), \\ \nu_\lambda(x) \geq \nu_\mu(x) \end{cases} \quad \forall x \in X.$$

$$\mathcal{A} \bigcup_{\mathbb{P}_{\text{CIF}}} \mathcal{B} \iff \begin{cases} [\mu_{\tilde{A}}(x)] \subseteq [\mu_{\tilde{B}}(x)], [\nu_{\tilde{A}}(x)] \supseteq [\nu_{\tilde{B}}(x)], \\ \mu_\lambda(x) \leq \mu_\mu(x), \\ \nu_\lambda(x) \geq \nu_\mu(x) \end{cases}$$

$$\mathcal{A} \bigcap_{\mathbb{P}_{\text{CIF}}} \mathcal{B} \iff \begin{cases} [\mu_{\tilde{A}}(x)] \supseteq [\mu_{\tilde{B}}(x)], [\nu_{\tilde{A}}(x)] \subseteq [\nu_{\tilde{B}}(x)], \\ \mu_\lambda(x) \geq \mu_\mu(x), \\ \nu_\lambda(x) \leq \nu_\mu(x) \end{cases}$$

**Definition 2.5 ([9])** A  $\mathbb{P}_{\text{CIF}}$ -cubic intuitionistic fuzzy topology ( $\mathbb{P}_{\text{CIF}}$ -CIFT) on  $X$  is a family  $\mathcal{F}_{\mathbb{P}_{\text{CIF}}}$  of CIFSs in  $X$  satisfying:

- $\tilde{0}, \tilde{1} \in \mathcal{F}_{\mathbb{P}_{\text{CIF}}}$ , where  $\tilde{0} = (([0, 0], [1, 1]), (0, 1))$  and  $\tilde{1} = (([1, 1], [0, 0]), (1, 0))$
- If  $\mathcal{A}_i \in \mathcal{F}_{\mathbb{P}_{\text{CIF}}}$  for  $i \in \Lambda$ , then  $\bigcup_{\mathbb{P}_{\text{CIF}}, i \in \Lambda} \mathcal{A}_i \in \mathcal{F}_{\mathbb{P}_{\text{CIF}}}$

(iii) If  $\mathcal{A}, \mathcal{B} \in \mathcal{F}_{\mathbb{P}_{\text{CIF}}}$ , then  $\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{B} \in \mathcal{F}_{\mathbb{P}_{\text{CIF}}}$

The pair  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is called a  $\mathbb{P}_{\text{CIF}}$ -cubic intuitionistic fuzzy topological space ( $\mathbb{P}_{\text{CIF}}$ -CIFTS). Members of  $\mathcal{F}_{\mathbb{P}_{\text{CIF}}}$  are called  $\mathbb{P}_{\text{CIF}}$ -open sets.

For a CIFS  $\mathcal{A}$  in a  $\mathbb{P}_{\text{CIF}}$ -CIFTS  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ , the  $\mathbb{P}_{\text{CIF}}$ -interior and  $\mathbb{P}_{\text{CIF}}$ -closure are defined as:

$$\begin{aligned} \text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) &= \bigcup_{\mathbb{P}_{\text{CIF}}} \{\mathcal{G} \mid \mathcal{G} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A} \text{ and } \mathcal{G} \text{ is } \mathbb{P}_{\text{CIF}}\text{-open}\}, \\ \text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) &= \bigcap_{\mathbb{P}_{\text{CIF}}} \{\mathcal{F} \mid \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{F} \text{ and } \mathcal{F} \text{ is } \mathbb{P}_{\text{CIF}}\text{-closed}\}. \end{aligned}$$

### 3. Structural Analysis of $\mathbb{P}_{\text{CIF}}$ -Z-Open Sets

We now introduce the central concept of our study.

**Definition 3.1** A CIFS  $\mathcal{A}$  in a  $\mathbb{P}_{\text{CIF}}$ -CIFTS  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is called a

(i)  $\mathbb{P}_{\text{CIF}}$ -regular open set ( $\mathbb{P}_{\text{CIF}}$ -ros) if

$$\mathcal{A} = \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})),$$

(ii)  $\mathbb{P}_{\text{CIF}}$ -regular closed set ( $\mathbb{P}_{\text{CIF}}$ -rcs) if

$$\mathcal{A} = \text{cl}_{\mathbb{P}_{\text{CIF}}}(\text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})),$$

**Definition 3.2** For a CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ :

(i) The  $\mathbb{P}_{\text{CIF}}$ - $\delta$ -interior is defined as:

$$\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \bigcup_{\mathbb{P}_{\text{CIF}}} \{\mathcal{F} \mid \mathcal{A} \supseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{F} \text{ and } \mathcal{F} \text{ is } \mathbb{P}_{\text{CIF}}\text{-ros}\},$$

(ii) The  $\mathbb{P}_{\text{CIF}}$ - $\delta$ -closure is defined as:

$$\delta\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \bigcap_{\mathbb{P}_{\text{CIF}}} \{\mathcal{F} \mid \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{F} \text{ and } \mathcal{F} \text{ is } \mathbb{P}_{\text{CIF}}\text{-rcs}\},$$

**Definition 3.3** A CIFS  $\mathcal{A}$  in a  $\mathbb{P}_{\text{CIF}}$ -CIFTS  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is called a

(i)  $\mathbb{P}_{\text{CIF}}$ - $\delta$ -semiopen set ( $\mathbb{P}_{\text{CIF}}$ - $\delta$ Sos) if

$$\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})),$$

(ii)  $\mathbb{P}_{\text{CIF}}$ - $\delta$ -semiclosed set ( $\mathbb{P}_{\text{CIF}}$ - $\delta$ Scs) if

$$\mathcal{A} \supseteq_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})),$$

(iii)  $\mathbb{P}_{\text{CIF}}$ -preopen set ( $\mathbb{P}_{\text{CIF}}$ -Pos) if

$$\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$$

(iv)  $\mathbb{P}_{\text{CIF}}$ -preclosed set ( $\mathbb{P}_{\text{CIF}}$ -Pcs) if

$$\mathcal{A} \supseteq_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$$

(v)  $\mathbb{P}_{\text{CIF}}$ -Z-open set ( $\mathbb{P}_{\text{CIF}}$ -Zos) if

$$\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})),$$

(vi)  $\mathbb{P}_{\text{CIF}}\text{-Z-closed set } (\mathbb{P}_{\text{CIF}}\text{-Zcs})$  if

$$\mathcal{A} \supseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cap_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(\delta-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})),$$

**Definition 3.4** For a CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ , the  $\mathbb{P}_{\text{CIF}}\text{-Z-interior}$  and  $\mathbb{P}_{\text{CIF}}\text{-Z-closure}$  are defined as:

$$\begin{aligned} Z-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) &= \bigcup_{\mathbb{P}_{\text{CIF}}} \{\mathcal{G} \mid \mathcal{G} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A} \text{ and } \mathcal{G} \text{ is a } \mathbb{P}_{\text{CIF}}\text{-Zos}\}, \\ Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) &= \bigcap_{\mathbb{P}_{\text{CIF}}} \{\mathcal{F} \mid \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{F} \text{ and } \mathcal{F} \text{ is a } \mathbb{P}_{\text{CIF}}\text{-Zcs}\}. \end{aligned}$$

**Proposition 3.1** The following inclusion relations hold for any CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ :

$$int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} Z-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}).$$

**Proof:** Follows directly from the definitions of the respective operators and the fact that every  $\mathbb{P}_{\text{CIF}}\text{-open}$  set is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$ .  $\square$

**Theorem 3.1** In a  $\mathbb{P}_{\text{CIF}}\text{-CIFS}$   $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ , the following implications hold:

- (i) Every  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-open}$  set is a  $\mathbb{P}_{\text{CIF}}\text{-open}$  set
- (ii) Every  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-open}$  set is a  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-semiopen}$  set
- (iii) Every  $\mathbb{P}_{\text{CIF}}\text{-open}$  set is a  $\mathbb{P}_{\text{CIF}}\text{-preopen}$  set
- (iv) Every  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-semiopen}$  set is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$
- (v) Every  $\mathbb{P}_{\text{CIF}}\text{-preopen}$  set is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$

**Proof:** (i) Let  $\mathcal{A}$  be  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-open}$ . Then  $\mathcal{A} = \delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \bigcup_{\mathbb{P}_{\text{CIF}}} \{\mathcal{G} \mid \mathcal{G} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A}, \mathcal{G} \text{ is } \mathbb{P}_{\text{CIF}}\text{-ro}\}$ . Since every  $\mathbb{P}_{\text{CIF}}\text{-ros}$  is  $\mathbb{P}_{\text{CIF}}\text{-open}$ , we have  $\delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$ . Hence  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$ , implying  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-open}$ .

(ii) Let  $\mathcal{A}$  be  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-open}$ . Then  $\mathcal{A} = \delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$ . Since the closure operator is extensive ( $\mathcal{B} \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{B})$  for any CIFS  $\mathcal{B}$ ), we have:

$$\delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})).$$

Substituting  $\mathcal{A} = \delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$ , we obtain  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$ , so  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-semiopen}$ .

(iii) Similar to (ii), using  $int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$ .

(iv) Let  $\mathcal{A}$  be  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-semiopen}$ . Then  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$ , so  $\mathcal{A}$  is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$ .

(v) Let  $\mathcal{A}$  be  $\mathbb{P}_{\text{CIF}}\text{-preopen}$ . Then  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta-int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$ , so  $\mathcal{A}$  is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$ .  $\square$

**Remark 3.1** The converse of the above theorem 3.1 need not be true, as verified by the following counterexample 3.1:

**Example 3.1** Let  $X = \{x\}$  be a singleton set and let  $\mathcal{F}_{\mathbb{P}_{\text{CIF}}}$  be the  $\mathbb{P}_{\text{CIF}}\text{-CIFS}$  on  $X$  defined by

$$\mathcal{F}_{\mathbb{P}_{\text{CIF}}} = \{\ddot{0}, \ddot{1}, A\}$$

where

$$\begin{aligned} \ddot{0} &= ([0, 0], [1, 1], (0, 1)), \\ \ddot{1} &= ([1, 1], [0, 0], (1, 0)), \end{aligned}$$

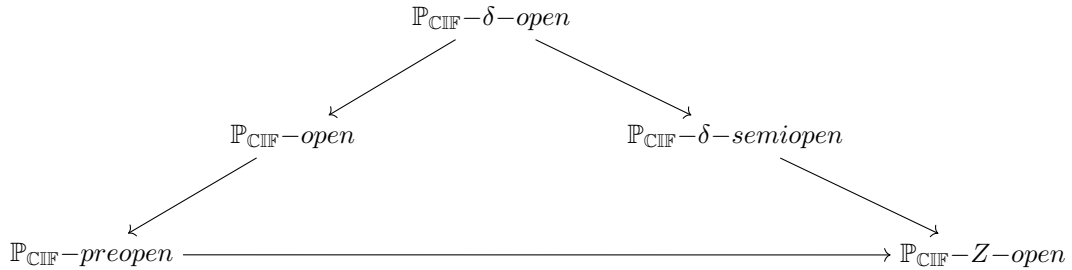
and the following cubic intuitionistic fuzzy sets are defined on  $X$ :

$$\begin{aligned} A &= ([0.30, 0.60], [0.20, 0.40], (0.50, 0.30)), \\ B &= ([0.25, 0.55], [0.35, 0.55], (0.40, 0.50)), \\ C &= ([0.20, 0.75], [0.05, 0.45], (0.65, 0.15)), \\ D &= ([0.35, 0.75], [0.05, 0.35], (0.65, 0.20)), \\ E &= ([0.20, 0.30], [0.50, 0.60], (0.25, 0.55)). \end{aligned}$$

Now we classify the given sets:

- (i)  $A$  is  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ -open but not  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ - $\delta$ -open
- (ii)  $B$  is  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ - $\delta$ -semiopen but not  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ -open
- (iii)  $C$  is  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ -preopen but not  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ -open
- (iv)  $D$  is  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ -Z-open but not  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ - $\delta$ -semiopen
- (v)  $E$  is  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ -Z-open but not  $\text{CIF}_{\mathbb{P}_{\text{CIF}}}$ -preopen

**Remark 3.2** The following inclusion diagram summarizes relationships among generalized open sets in  $\mathbb{P}_{\text{CIF}}\text{-CIFTS}$  (arrows indicate proper inclusion as verified by Counterexample 3.1):



**Theorem 3.2** A CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$  if and only if  $\mathcal{A} = P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} \delta S\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$ , where  $P\text{-int}_{\mathbb{P}_{\text{CIF}}}$  and  $\delta S\text{-int}_{\mathbb{P}_{\text{CIF}}}$  denote pre-interior and  $\delta$ -semi-interior operators respectively.

**Proof:** ( $\Rightarrow$ ) Let  $\mathcal{A}$  be a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$ . Then  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$ . Using properties of pre-interior and  $\delta$ -semi-interior:

$$\begin{aligned} P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} \delta S\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) &= (\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \cup_{\mathbb{P}_{\text{CIF}}} (\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \\ &= \mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} (int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \\ &= \mathcal{A}. \end{aligned}$$

( $\Leftarrow$ ) If  $\mathcal{A} = P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} \delta S\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$ , then:

$$\begin{aligned} \mathcal{A} &= (\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \cup_{\mathbb{P}_{\text{CIF}}} (\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \\ &= \mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} (int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \\ &\subseteq_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})), \end{aligned}$$

so  $\mathcal{A}$  is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$ . □

**Theorem 3.3** The union of any family of  $\mathbb{P}_{\text{CIF}}\text{-Zos}$  is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$ . The intersection of any finite family of  $\mathbb{P}_{\text{CIF}}\text{-Zos}$  is a  $\mathbb{P}_{\text{CIF}}\text{-Zos}$ .

**Proof:** Let  $\{\mathcal{A}_i \mid i \in \Lambda\}$  be a family of  $\mathbb{P}_{\text{CIF}}$ -Zos. For each  $i$ ,  $\mathcal{A}_i \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta - int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}_i)) \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}_i))$ . Then:

$$\begin{aligned} \bigcup_{\mathbb{P}_{\text{CIF}}, i \in \Lambda} \mathcal{A}_i &\subseteq_{\mathbb{P}_{\text{CIF}}} \bigcup_{\mathbb{P}_{\text{CIF}}, i \in \Lambda} (cl_{\mathbb{P}_{\text{CIF}}}(\delta - int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}_i)) \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}_i))) \\ &\subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta - int_{\mathbb{P}_{\text{CIF}}}(\bigcup_{\mathbb{P}_{\text{CIF}}, i \in \Lambda} \mathcal{A}_i)) \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\bigcup_{\mathbb{P}_{\text{CIF}}, i \in \Lambda} \mathcal{A}_i)), \end{aligned}$$

so  $\bigcup_{\mathbb{P}_{\text{CIF}}, i \in \Lambda} \mathcal{A}_i$  is a  $\mathbb{P}_{\text{CIF}}$ -Zos.

For finite intersections, let  $\mathcal{A}$  and  $\mathcal{B}$  be  $\mathbb{P}_{\text{CIF}}$ -Zos. Using distributive properties and monotonicity of closure/interior operators:

$$\begin{aligned} \mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{B} &\subseteq_{\mathbb{P}_{\text{CIF}}} (cl_{\mathbb{P}_{\text{CIF}}}(\delta - int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \cap_{\mathbb{P}_{\text{CIF}}} (cl_{\mathbb{P}_{\text{CIF}}}(\delta - int_{\mathbb{P}_{\text{CIF}}}(\mathcal{B})) \\ &\quad \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{B}))) \subseteq_{\mathbb{P}_{\text{CIF}}} cl_{\mathbb{P}_{\text{CIF}}}(\delta - int_{\mathbb{P}_{\text{CIF}}}(\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{B})) \cup_{\mathbb{P}_{\text{CIF}}} int_{\mathbb{P}_{\text{CIF}}}(cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{B})), \end{aligned}$$

so  $\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{B}$  is a  $\mathbb{P}_{\text{CIF}}$ -Zos. The result extends to finite families by induction.  $\square$

**Theorem 3.4** For any CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ :

- (i)  $Z-cl_{\mathbb{P}_{\text{CIF}}}(\ddot{0}) = \ddot{0}$  and  $Z-cl_{\mathbb{P}_{\text{CIF}}}(\ddot{1}) = \ddot{1}$
- (ii)  $Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$  is a  $\mathbb{P}_{\text{CIF}}$ -Zcs
- (iii)  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{B}$  implies  $Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{B})$
- (iv)  $Z-cl_{\mathbb{P}_{\text{CIF}}}(Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) = Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$
- (v)  $\mathcal{A}$  is a  $\mathbb{P}_{\text{CIF}}$ -Zcs iff  $Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \mathcal{A}$

**Proof:** (i)–(iv) follow directly from the definition of  $Z$ -closure and properties of  $\mathbb{P}_{\text{CIF}}$ -order operations.

(v) If  $\mathcal{A}$  is a  $\mathbb{P}_{\text{CIF}}$ -Zcs, then by definition  $\mathcal{A} \in \{\mathcal{F} \mid \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{F}, \mathcal{F} \text{ is } \mathbb{P}_{\text{CIF}}\text{-Zcs}\}$ , so  $Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \bigcap_{\mathbb{P}_{\text{CIF}}} \{\mathcal{F} \mid \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{F}, \mathcal{F} \text{ is } \mathbb{P}_{\text{CIF}}\text{-Zcs}\} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$ . Since  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$  always holds, we have equality.

Conversely, if  $Z-cl_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \mathcal{A}$ , then  $\mathcal{A}$  is the intersection of  $\mathbb{P}_{\text{CIF}}$ -Zcs sets, hence itself a  $\mathbb{P}_{\text{CIF}}$ -Zcs (by the finite intersection property extended to arbitrary intersections for closed sets).  $\square$

**Theorem 3.5** Let  $\mathcal{A}$  be a  $\mathbb{P}_{\text{CIF}}$ -Zcs in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ . Then  $\mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$  contains no non-empty  $\mathbb{P}_{\text{CIF}}$ -closed subset of  $X$ .

**Proof:** Let  $\mathcal{A}$  be a  $\mathbb{P}_{\text{CIF}}$ -Zcs and let  $\mathcal{B}$  be a  $\mathbb{P}_{\text{CIF}}$ -closed subset of  $\mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$ . Then:

$$\mathcal{B} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} (\ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A}).$$

This implies  $\mathcal{B} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A})$  and  $\mathcal{B} \subseteq_{\mathbb{P}_{\text{CIF}}} \ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$ , so  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{B}$  where  $\ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{B}$  is  $\mathbb{P}_{\text{CIF}}$ -open. Since  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}$ - $Z$ -closed, we have:

$$\mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} \ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{B}.$$

Thus  $\mathcal{B} \subseteq_{\mathbb{P}_{\text{CIF}}} \ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A})$ , which gives:

$$\mathcal{B} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} (\ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A})) = \ddot{0}.$$

Hence  $\mathcal{B} = \ddot{0}$ , proving the result.  $\square$

**Theorem 3.6** A CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is a  $\mathbb{P}_{\text{CIF}}$ -Zcs if and only if  $\mathbb{P}_{\text{CIF}}-Z-cl(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}$ -closed.

**Proof:** ( $\Rightarrow$ ) If  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closed}$ , then  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) = \mathcal{A}$ , so  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A} = \ddot{0}$ , which is  $\mathbb{P}_{\text{CIF}}\text{-closed}$ .

( $\Leftarrow$ ) Suppose  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-closed}$  and  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closed}$ . By the previous theorem,  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$  contains no non-empty  $\mathbb{P}_{\text{CIF}}\text{-closed}$  subset. Since it is itself  $\mathbb{P}_{\text{CIF}}\text{-closed}$ , we must have  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A} = \ddot{0}$ . Therefore  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) = \mathcal{A}$ , so  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closed}$ .  $\square$

**Theorem 3.7** Let  $\mathcal{A}$  be a CIFS in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ . Then  $x_{(\alpha, \beta)} \in \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$  if and only if  $\mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{A} \neq_{\mathbb{P}_{\text{CIF}}} \ddot{0}$  for every  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-os}$   $\mathcal{U}$  containing  $x_{(\alpha, \beta)}$ .

**Proof:** ( $\Rightarrow$ ) Suppose  $x_{(\alpha, \beta)} \in \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$  and let  $\mathcal{U}$  be a  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-os}$  containing  $x_{(\alpha, \beta)}$ . If  $\mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{A} =_{\mathbb{P}_{\text{CIF}}} \ddot{0}$ , then  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{U}$  where  $\ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{U}$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closed}$ . Hence  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} \ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{U}$ , contradicting  $x_{(\alpha, \beta)} \in \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{U}$ .

( $\Leftarrow$ ) Suppose  $x_{(\alpha, \beta)} \notin \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$ . Then  $\ddot{1} \setminus_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$  is a  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-os}$  containing  $x_{(\alpha, \beta)}$  that is disjoint from  $\mathcal{A}$ , contradicting the hypothesis.  $\square$

**Theorem 3.8** Let  $\mathcal{A}$  be a CIFS in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ . Then:

$$(i) \quad \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) = \delta S\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$$

$$(ii) \quad \mathbb{P}_{\text{CIF}}\text{-}Z\text{-int}(\mathcal{A}) = \delta S\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$$

where  $P\text{-cl}_{\mathbb{P}_{\text{CIF}}}$  and  $P\text{-int}_{\mathbb{P}_{\text{CIF}}}$  denote pre-closure and pre-interior operators, and  $\delta S\text{-cl}_{\mathbb{P}_{\text{CIF}}}$  and  $\delta S\text{-int}_{\mathbb{P}_{\text{CIF}}}$  denote  $\delta$ -semi-closure and  $\delta$ -semi-interior operators.

**Proof:** (i) First note that  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} \delta S\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$  since every  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cs}$  is both  $\delta$ -semi-closed and pre-closed.

Conversely, since  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closed}$ :

$$\begin{aligned} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) &\subseteq_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}))) \cap_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\text{int}_{\mathbb{P}_{\text{CIF}}}(\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}))) \\ &\subseteq_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cap_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})). \end{aligned}$$

Therefore:

$$\begin{aligned} \delta S\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) &= (\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \cap_{\mathbb{P}_{\text{CIF}}} (\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \\ &= \mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} (\text{int}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cap_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) = \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}). \end{aligned}$$

(ii) Follows from (i) by taking complements and applying De Morgan's laws under  $\mathbb{P}_{\text{CIF}}\text{-order}$ .  $\square$

**Theorem 3.9** Let  $\mathcal{A}$  be a CIFS in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ . Then the following are equivalent:

(i)  $\mathcal{A}$  is a  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-os}$

$$(ii) \quad \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \delta P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$$

(iii) There exists a  $\mathbb{P}_{\text{CIF}}\text{-preopen}$  set  $\mathcal{U}$  such that  $\mathcal{U} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \delta P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{U})$

$$(iv) \quad \delta P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \delta P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$$

where  $\delta P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{B}) = \mathcal{B} \cup_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{B}))$  denotes the  $\delta P$ -closure operator.

**Proof:** (i)  $\Rightarrow$  (ii): If  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-open}$ , then by the decomposition theorem:

$$\begin{aligned}\mathcal{A} &= \mathbb{P}_{\text{CIF}}\text{-}Z\text{-int}(\mathcal{A}) = \delta S\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \\ &= (\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \cup_{\mathbb{P}_{\text{CIF}}} (\mathcal{A} \cap_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \\ &\quad \subseteq_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})).\end{aligned}$$

Using properties of  $\delta P$ -closure:

$$\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \delta P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})).$$

(ii)  $\Rightarrow$  (iii): Take  $\mathcal{U} = P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$ , which is  $\mathbb{P}_{\text{CIF}}\text{-preopen}$  by definition.

(iii)  $\Rightarrow$  (ii): If  $\mathcal{U} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-preopen}$ , then  $P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \supseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{U}$ , so:

$$\delta P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \supseteq_{\mathbb{P}_{\text{CIF}}} \delta P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{U}) \supseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A}.$$

(ii)  $\Leftrightarrow$  (iv): Direct consequence of the definition of  $\delta P$ -closure operator and monotonicity properties.  $\square$

**Proposition 3.2** Let  $\mathcal{A}$  be a  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-os}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ . Then  $\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$  is a  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-semiopen}$  set.

**Proof:** Since  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-open}$ :

$$\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})).$$

Taking  $\mathbb{P}_{\text{CIF}}\text{-closure}$  on both sides:

$$\begin{aligned}\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) &\subseteq_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) \\ &\subseteq_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))) = \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})).\end{aligned}$$

Hence  $\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$ , so  $\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$  is  $\mathbb{P}_{\text{CIF}}\text{-}\delta\text{-semiopen}$ .  $\square$

**Proposition 3.3** Let  $\mathcal{A}$  be a  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-os}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  such that  $\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \ddot{0}$ . Then  $\mathcal{A}$  is a  $\mathbb{P}_{\text{CIF}}\text{-preopen}$  set.

**Proof:** Since  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-open}$  and  $\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \ddot{0}$ :  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \text{cl}_{\mathbb{P}_{\text{CIF}}}(\delta\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) \cup_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) = \text{cl}_{\mathbb{P}_{\text{CIF}}}(\ddot{0}) \cup_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) = \ddot{0} \cup_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})) = \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$ . Therefore  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \text{int}_{\mathbb{P}_{\text{CIF}}}(\text{cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}))$ , so  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-preopen}$ .  $\square$

**Theorem 3.10** For any CIFSs  $\mathcal{A}, \mathcal{B}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ :

$$(i) \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\ddot{0}) = \ddot{0}$$

$$(ii) \mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$$

$$(iii) \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathcal{B}) = \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{B})$$

$$(iv) \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})) = \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$$

**Proof:** (i) and (ii) follow directly from the definition of  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closure}$ .

(iii) Since  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathcal{B}$  and  $\mathcal{B} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathcal{B}$ , we have:

$$\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathcal{B}) \quad \text{and} \quad \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{B}) \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathcal{B}),$$

so  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{B}) \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathcal{B})$ .

Conversely, since  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$  and  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{B})$  are  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closed}$ , their union is also  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closed}$  (by Theorem ??). Since  $\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathcal{B} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{B})$ , we have:

$$\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} \mathcal{B}) \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{B}).$$

(iv) Since  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$  is  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-closed}$ , by the fixed-point characterization (Theorem ??), we have  $\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})) = \mathbb{P}_{\text{CIF}}\text{-}Z\text{-cl}(\mathcal{A})$ .  $\square$

**Definition 3.5** A CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is called locally  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$  if  $\mathcal{A} = \mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{F}$  where  $\mathcal{U}$  is  $\mathbb{P}_{\text{CIF}}\text{-open}$  and  $\mathcal{F}$  is  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$ .

**Theorem 3.11** A CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is locally  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$  if and only if  $\mathcal{A} = \mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A})$  for some  $\mathbb{P}_{\text{CIF}}\text{-open}$  set  $\mathcal{U}$ .

**Proof:**  $(\Rightarrow)$  If  $\mathcal{A}$  is locally  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$ , then  $\mathcal{A} = \mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{F}$  where  $\mathcal{U}$  is  $\mathbb{P}_{\text{CIF}}\text{-open}$  and  $\mathcal{F}$  is  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$ . Then:

$$\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{F}) = \mathcal{F}.$$

Therefore:

$$\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \subseteq_{\mathbb{P}_{\text{CIF}}} \mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathcal{F} = \mathcal{A},$$

so  $\mathcal{A} = \mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A})$ .

$(\Leftarrow)$  If  $\mathcal{A} = \mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A})$  with  $\mathcal{U}$   $\mathbb{P}_{\text{CIF}}\text{-open}$ , then since  $\mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A})$  is  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$ ,  $\mathcal{A}$  is locally  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$  by definition.  $\square$

**Theorem 3.12** Let  $\mathcal{A}$  be a locally  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$  set in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$ . Then:

- (i)  $\mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$
- (ii)  $\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} (\check{\text{i}} \setminus_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}))$  is  $\mathbb{P}_{\text{CIF}}\text{-Z-open}$
- (iii)  $\mathcal{A} \subseteq_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-int}(\mathcal{A} \cup_{\mathbb{P}_{\text{CIF}}} (\check{\text{i}} \setminus_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A})))$

**Proof:** (i) Since  $\mathcal{A}$  is locally  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$ ,  $\mathcal{A} = \mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A})$  for some  $\mathbb{P}_{\text{CIF}}\text{-open}$  set  $\mathcal{U}$ . Then:

$$\begin{aligned} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{A} &= \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} (\check{\text{i}} \setminus_{\mathbb{P}_{\text{CIF}}} (\mathcal{U} \cap_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}))) \\ &= \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} ((\check{\text{i}} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{U}) \cup_{\mathbb{P}_{\text{CIF}}} (\check{\text{i}} \setminus_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}))) \\ &= (\mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} (\check{\text{i}} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{U})) \cup_{\mathbb{P}_{\text{CIF}}} (\mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} (\check{\text{i}} \setminus_{\mathbb{P}_{\text{CIF}}} \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}))) \\ &= \mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} (\check{\text{i}} \setminus_{\mathbb{P}_{\text{CIF}}} \mathcal{U}), \end{aligned}$$

which is the intersection of a  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$  set and a  $\mathbb{P}_{\text{CIF}}\text{-closed}$  set, hence  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$ .

(ii) and (iii) follow from (i) by taking complements and applying interior operator properties.  $\square$

**Definition 3.6** A CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is called a  $\mathbb{P}_{\text{CIF}}\text{-D}(c, z)$  set if  $\text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \mathbb{P}_{\text{CIF}}\text{-Z-int}(\mathcal{A})$ .

**Theorem 3.13** A CIFS  $\mathcal{A}$  in  $(X, \mathcal{F}_{\mathbb{P}_{\text{CIF}}})$  is  $\mathbb{P}_{\text{CIF}}\text{-open}$  if and only if it is both a  $\mathbb{P}_{\text{CIF}}\text{-Z-os}$  and a  $\mathbb{P}_{\text{CIF}}\text{-D}(c, z)$  set.

**Proof:**  $(\Rightarrow)$  If  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-open}$ , then it is  $\mathbb{P}_{\text{CIF}}\text{-Z-open}$  (by Theorem 3.1(iii)). Also, since  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-open}$ :

$$\text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) = \mathcal{A} = \mathbb{P}_{\text{CIF}}\text{-Z-int}(\mathcal{A}),$$

so  $\mathcal{A}$  is a  $\mathbb{P}_{\text{CIF}}\text{-D}(c, z)$  set.

$(\Leftarrow)$  If  $\mathcal{A}$  is both  $\mathbb{P}_{\text{CIF}}\text{-Z-open}$  and a  $\mathbb{P}_{\text{CIF}}\text{-D}(c, z)$  set, then:

$$\mathcal{A} = \mathbb{P}_{\text{CIF}}\text{-Z-int}(\mathcal{A}) = \text{int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}),$$

so  $\mathcal{A}$  is  $\mathbb{P}_{\text{CIF}}\text{-open}$ .  $\square$

#### 4. Real Life Application: IoT-Based Smart Greenhouse Monitoring

**Example 4.1 (Smart Greenhouse Monitoring Using  $\mathbb{P}_{\text{CIF}}$ -Z-Open Sets)** Modern greenhouses use electronic sensors to monitor important environmental conditions such as temperature, humidity, soil moisture, and light intensity. These sensors help maintain the best environment for plant growth automatically.

However, in real life, sensor readings are never perfectly accurate. For example:

- Temperature sensors may fluctuate slightly due to air circulation,
- Humidity sensors may have delayed response,
- Soil moisture sensors may vary during watering,
- Light sensors may change suddenly due to clouds.

Because of these uncertainties, the greenhouse control system must make decisions even when measurements are not perfectly precise.

##### **Step 1: Mathematical model of the greenhouse system**

We represent the greenhouse system using a simple model with one control point:

$$X = \{x\}$$

Each environmental condition is represented using a cubic intuitionistic fuzzy set (CIFS), which can be understood in simple terms as a way to describe:

- the expected safe range,
- the possible error range,
- and the confidence level of the measurement.

For example:

- *Set A: Temperature state* This represents stable temperature where the system can safely operate.
- *Set B: Humidity state* This represents humidity that fluctuates slightly but still follows a general pattern.
- *Set C: Soil moisture state* This represents soil moisture that changes slowly and predictably.
- *Set D: Light intensity state* This represents light that may change suddenly due to clouds.
- *Set E: Combined system state* This represents situations where multiple environmental factors change at once.

##### **Step 2: Understanding control capability of the system**

The greenhouse control system can handle different types of environmental conditions differently. Using the mathematical theory developed in Example 3.1, we classify the states as follows:

- **Open state (A) — Directly controllable**

This represents stable temperature. The system can immediately adjust heating or cooling.

Example: Maintaining temperature at 25°C.

- **$\delta$ -semiopen state (B) — Noisy but manageable**

*Humidity fluctuates slightly but remains predictable.*

*Example: Humidity varying between 60% and 65%.*

*The system applies small adjustments.*

- **Preopen state (C) — Predictable state**

*Soil moisture changes slowly and can be predicted.*

*Example: Soil drying gradually after watering.*

*The system can schedule irrigation.*

- **Z-open states (D, E) — Complex and uncertain states**

*Light intensity may change suddenly due to clouds, or multiple conditions may change simultaneously.*

*Example:*

*Cloud suddenly blocks sunlight  $\rightarrow$  temperature drops  $\rightarrow$  humidity changes.*

*The system must use intelligent adaptive control.*

**Step 3: Easy interpretation of the diagram**

*The diagram in Remark 3.2 shows levels of stability:*

- *Top level: Very stable conditions (easy to control)*
- *Middle level: Moderately stable conditions (predictable, manageable)*
- *Bottom level: Highly uncertain conditions (require intelligent control)*

**Why  $\mathbb{P}_{\text{CIF-Z-open sets are important}}$**

$\mathbb{P}_{\text{CIF-Z-open sets help model the most realistic situations where:$

- *measurements are uncertain,*
- *conditions change suddenly,*
- *multiple environmental factors interact.*

*This allows greenhouse systems to make better automatic decisions, improving plant growth and reducing manual intervention.*

**Conclusion**

*This mathematical model helps greenhouse systems:*

- *understand stable conditions,*
- *predict changing conditions,*
- *and handle uncertain conditions intelligently.*

Thus,  $\mathbb{P}_{\text{CIF-Z}}$ -open sets provide a powerful and practical tool for smart greenhouse control.

### **Simple numerical example from real greenhouse**

Suppose a smart greenhouse measures temperature using a sensor.  
The ideal temperature for plant growth is:

$$24^{\circ}\text{C to } 28^{\circ}\text{C}$$

But due to sensor uncertainty, the reading is not exact.  
Suppose the system represents temperature using a CIFS

$$A = ([0.60, 0.80], [0.10, 0.20], (0.75, 0.15))$$

This means:

- $[0.60, 0.80] \rightarrow$  Temperature is in the safe range with 60% to 80% confidence.
- $[0.10, 0.20] \rightarrow$  There is 10% to 20% uncertainty.
- $(0.75, 0.15) \rightarrow$  Overall confidence is 75%, hesitation is 15%.

This corresponds to a stable temperature state.

Thus,  $A$  is a  $\mathbb{P}_{\text{CIF}}$ -open set.

This means:

The greenhouse system can safely control temperature automatically.

Example action:

Turn heater ON slightly if temperature drops.

Now consider light intensity measured during cloudy weather:

$$D = ([0.40, 0.70], [0.20, 0.40], (0.60, 0.25))$$

This means:

- Light level varies between 40% and 70%.
- Uncertainty is higher due to cloud movement.
- Confidence is moderate.

This state is not fully stable but still manageable.

Thus,  $D$  is a  $\mathbb{P}_{\text{CIF-Z}}$ -open set.

This means:

The greenhouse system must use intelligent adaptive control.

Example action:

Turn artificial grow lights ON automatically.

Similarly, soil moisture measured as

$$C = ([0.50, 0.75], [0.10, 0.30], (0.70, 0.20))$$

represents predictable moisture level.

Thus,  $C$  is a  $\mathbb{P}_{\text{CIF}}$ -preopen set.

This allows predictive irrigation scheduling.

Example action:

Start watering in 30 minutes.

### **Summary**

<i>Environmental Condition</i>	<i>Mathematical Type</i>	<i>Control Action</i>
<i>Temperature</i>	<i>Open</i>	<i>Direct automatic control</i>
<i>Humidity</i>	<i><math>\delta</math>-semiopen</i>	<i>Small adjustments</i>
<i>Soil moisture</i>	<i>Preopen</i>	<i>Predictive irrigation</i>
<i>Light intensity</i>	<i>Z-open</i>	<i>Intelligent adaptive control</i>
<i>Combined uncertainty</i>	<i>Z-open</i>	<i>Advanced intelligent control</i>

*This example shows how  $\mathbb{P}_{\text{CIF}}\text{-Z-open}$  sets help real greenhouse systems make intelligent decisions even when sensor readings are uncertain.*

### 5. Conclusion

*In this paper, we introduced and investigated  $\mathbb{P}_{\text{CIF}}\text{-Z-open}$  sets in cubic intuitionistic fuzzy topological spaces under the  $\mathbb{P}_{\text{CIF}}$ -order framework. We established that  $\mathbb{P}_{\text{CIF}}\text{-Z-open}$  sets form a topology  $\mathcal{F}_{\mathbb{P}_{\text{CIF}}}^Z$  finer than the original  $\mathbb{P}_{\text{CIF}}$ -cubic intuitionistic fuzzy topology, with fundamental decomposition theorems  $\mathbb{P}_{\text{CIF}}\text{-Z-cl}(\mathcal{A}) = \delta S\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cap_{\mathbb{P}_{\text{CIF}}} P\text{-cl}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$  and  $\mathbb{P}_{\text{CIF}}\text{-Z-int}(\mathcal{A}) = \delta S\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A}) \cup_{\mathbb{P}_{\text{CIF}}} P\text{-int}_{\mathbb{P}_{\text{CIF}}}(\mathcal{A})$  unifying  $\delta$ -semiopen and preopen structures. Through the concepts of locally  $\mathbb{P}_{\text{CIF}}\text{-Z-closed}$  sets and  $\mathbb{P}_{\text{CIF}}\text{-D}(c, z)$  sets, we characterized  $\mathbb{P}_{\text{CIF}}\text{-open}$  sets as those simultaneously being  $\mathbb{P}_{\text{CIF}}\text{-Zos}$  and  $\mathbb{P}_{\text{CIF}}\text{-D}(c, z)$  sets. The practical applicability of this hierarchy was demonstrated in IoT-based smart greenhouse monitoring systems, where the nuanced inclusion relationships among generalized open sets provide an effective mathematical framework for modeling environmental uncertainties with varying degrees of stability and sensor reliability. This work significantly enriches cubic intuitionistic fuzzy topology and opens avenues for applications in intelligent decision-making under hierarchical uncertainty.*

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