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Groupoids Associated to Self-Similar Groupoids *

Isnie Yusnitha

ABSTRACT: Let E be a row-finite directed graph without sources. Let $\mathcal{G}$ be a discrete groupoid. The groupoid $\mathcal{G}$ acts self-similarly on E with the unit space $\mathcal{G}^{(0)} = E^0$. We write this self-similar action of $\mathcal{G}$ on E as the pair $(\mathcal{G}, E)$. To our more general situation, we follow Brownlowe et al (2023) to construct two groupoids associated with $(\mathcal{G}, E)$, namely the groupoid $S(\mathcal{G}, E) \ltimes E^\infty$ and $\mathcal{G}_{(\mathcal{G},E)}$. We explore the interplay between the groupoid $S(\mathcal{G}, E) \ltimes E^\infty$ and $\mathcal{G}_{(\mathcal{G},E)}$, and the structure of $\mathcal{G}_{(\mathcal{G},E)}$ is derived.

Keywords: Groupoids, graphs, self-similar action, étale groupoids.

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1. Introduction

A groupoid $\mathcal{G}$ is an algebraic structure that generalizes the concept of group by allowing multiplication to be only partially defined and by admitting multiple units. The use of a groupoid in the study of C^*-algebras was initiated in [2], inspired by developments concerning measured groups and measure groupoids, particularly their significance in ergodic theory [6]. An intrinsic connection exists between ergodic theory and von Neumann algebras (see [7], [8]), which can be effectively captured through the framework of measured groupoids. This relationship is fundamentally anchored in measure theory. In [2], Renault demonstrated that topologically locally compact groupoids—objects with inherent topological structure—serve as analogues to measured groupoids within the context of C^*-algebra theory. Since this foundational insight, research on groupoid C^*-algebras has flourished, yielding concrete realizations for various classes of C^*-algebras [1]. Notably, Renault showed in [2] that both AF-algebras and Cuntz algebras can be constructed as C^*-algebras arising from appropriately chosen groupoids. Subsequently, groupoids have also been employed to model graph C^*-algebras ([9], [10]).

By a self-similar groupoid we mean a groupoid endowed with the self-similarity property. To illustrate this self-similarity property, consider the familiar addition algorithm for integers taught in primary school. In this process, a carrying operation occurs when handling the addition of larger numbers. In a similar way, there exists a *restriction map* that serves to encode the self-similarity property. Particularly in this study, we discuss on self-similar actions of groupoids on row-finite directed graphs, we simply call them self-similar groupoids. The topic of self-similar actions has attracted considerable attention of late. For examples, see [12], [14], [13], [16], and [15]. They make use of self-similar groups or groupoids to study the associated C^*-algebras. Inspired by the spirit of existing results, given our self-similar groupoid $(\mathcal{G}, E)$, we study their associated Cuntz-Krieger algebras (see [5] and [16]). To prove our second uniqueness theorem, namely Cuntz-Krieger uniqueness theorem, we use groupoid models. We construct two groupoid models associated with our $(\mathcal{G}, E)$, namely $S_{\mathcal{G},E} \ltimes E^\infty$ and $\mathcal{G}_{(\mathcal{G},E)}$. In this article, we focus on the constructed

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groupoids $S_{\mathcal{G},E} \ltimes E^\infty$ and $\mathcal{G}_{(\mathcal{G},E)}$. We adjust the construction in [3] to our more general situation and all of their arguments unfold without necessary problem. A connection can be observed between $S_{\mathcal{G},E} \ltimes E^\infty$ and $\mathcal{G}_{(\mathcal{G},E)}$. This implies that all of data in $S_{\mathcal{G},E} \ltimes E^\infty$ can be used to study the groupoid $\mathcal{G}_{(\mathcal{G},E)}$.

2. Groupoids

In this study, we focus on the simpler class of topological groupoids, namely étale groupoids. Its corresponding C^*-algebras can be less complicated to be studied; through this particular setting, a 'complicated' technology and analysis involving both representation and measure theory can be averted [1].

We pick the definition of groupoids introduced in [1]; it helps the reader to easily understand the notion of groupoids due to its straightforward nature. See also [2] and [4] for definitive references.

Definition 2.1 *Let $\mathcal{G}$ be a set. Let $\mathcal{G}^{(0)}$ be a distinguished subset of $\mathcal{G}$, called the* unit space*. They are equipped with maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$, a multiplication map $(g, h) \mapsto gh$ from $\{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = r(h)\}$ to $\mathcal{G}$, and an inversion map $g \mapsto g^{-1}$ from $\mathcal{G}$ to $\mathcal{G}$ such that*

($\mathcal{G}$1) $r(v) = v = s(v)$ for all $v \in \mathcal{G}^{(0)}$;

($\mathcal{G}$2) $r(g)g = g = gs(g)$ for all $g \in \mathcal{G}$;

($\mathcal{G}$3) $r(g^{-1}) = s(g)$ and $s(g^{-1}) = r(g)$ for all $g \in \mathcal{G}$;

($\mathcal{G}$4) $g^{-1}g = s(g)$ and $gg^{-1} = r(g)$ for all $g \in \mathcal{G}$;

($\mathcal{G}$5) $r(gh) = r(g)$ and $s(gh) = s(h)$ whenever $s(g) = r(h)$; and

($\mathcal{G}$6) $(gh)k = g(hk)$ whenever $s(g) = r(h)$ and $s(h) = r(k)$.

The groupoid $\mathcal{G}$ is a topological groupoid in the sense that it is equipped with a locally compact topology such that

- $\mathcal{G}^{(0)}$ is Hausdorff; and
- the maps r, s, the multiplication map, and the inversion map are continuous.

In this study, the groupoid $\mathcal{G}$ is always assumed to be Hausdorff in order to simplify. Thus, $\mathcal{G}^{(0)}$ is always a closed subset of $\mathcal{G}$ (see [1, Lemma 8.3.2]). We call the topological groupoid is *étale*, if its range map $r : \mathcal{G} \to \mathcal{G}$ a local homeomorphism in the sense that for every $g \in \mathcal{G}$, there is an open neighborhood V of g such that $r(V) \subseteq \mathcal{G}$ and $r|_V : V \to r(V)$ is a homeomorphism. For a collection of examples of *étale* groupoids, see [1, p. 74] as an example.

One important example of étale groupoid is so called *Deaconu-Renault groupoid* defined as follows. Consider a locally compact Hausdorff space X and an discrete abelian group G. Let Γ be a subsemigroup of G that contains the zero element. Suppose Γ acts on X via a local homeomorphism $(\gamma, a) \mapsto \gamma \cdot a$, from $\Gamma \times X$ to X satisfying the conditions $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 + \gamma_2) \cdot x$ and $0 \cdot x = x$ for all $x \in X$. Define $\mathcal{G} := \{(x_1, \gamma_1 - \gamma_2, x_2) \in X \times G \times X \mid \gamma_1 \cdot x_1 = \gamma_2 \cdot x_2\}$. Let $\mathcal{G}^{(0)} = \{(x, 0, x) \mid x \in X\}$. This unit space $\mathcal{G}^{(0)}$ can be naturally identified with X. For elements of $\mathcal{G}$, set

- $r(x_1, \gamma_1 - \gamma_2, x_2) = x_1$,
- $s(x_1, \gamma_1 - \gamma_2, x_2) = x_2$,
- $(x_1, \gamma_1 - \gamma_2, x_2)^{-1} = (x_2, -(\gamma_1 - \gamma_2), x_2)$, and
- $(x_1, \gamma_1 - \gamma_2, x_2)(x_3, \gamma_3 - \gamma_4, x_4) = (x_1, (\gamma_1 + \gamma_3) - (\gamma_2 + \gamma_4), x_2)$, whenever $x_2 = x_3$.

With these operations, $\mathcal{G}$ forms a groupoid, known as the Deaconu–Renault groupoid associated with the action of Γ on X. Endow $\mathcal{G}$ with the topology whose basic open sets are parametrized by open subsets $U, V \subseteq X$ and pairs $\gamma_1, \gamma_2 \in \Gamma$, defined as

$$Z(U, \gamma_1, \gamma_2, V) = \{(x, \gamma_1 - \gamma_2, y) \mid x \in U,\ y \in V,\ \gamma_1 \cdot x = \gamma_2 \cdot y\}.$$

With this topology, the Deaconu-Renault groupoid $\mathcal{G}$ becomes a locally compact Hausdorff groupoid. Furthermore, when the action of Γ on X is given by local homeomorphisms, $\mathcal{G}$ is always étale.

3. Self-similar Groupoid Actions

We now review the notation and terminology associated with a directed graph E. Our primary reference is [11]. A directed graph E is defined by two countable sets: the vertex set E^0 and the edge set E^1. Associated with these are the range and source maps, $r, s : E^1 \to E^0$. It is important to note that the maps r and s used in the context of a groupoid $\mathcal{G}$ are distinct from those defined on a graph E; the distinction should be clear from the context. A directed graph E is said to be *row-finite* if every vertex $v \in E^0$ emits only finitely many edges. Moreover, E is said to *have no sources* provided that for each vertex $v \in E^0$, the set $r^{-1}(v) \neq \emptyset$. If we think the groupoid elements as arrows, then these are reversible arrows. In contrast, edges in a graph which also described as arrows, do not necessarily share this property.

Let $e, f \in E^1$ with $s(e) = r(f)$. Then the concatenation ef forms a path of length 2, denoted by $|ef| = 2$. More generally, a path of length n in E is a sequence $e_1 e_2 \dots e_n$ such that $s(e_i) = r(e_{i+1})$ for $1 \leq i \leq n-1$. Vertices are regarded as paths of length 0. For each $n \geq 0$, the set of all paths of length n is denoted by E^n. We then define $E^* = \cup_{n \geq 0} E^n$. It is natural to extend the range and source maps r, s to E^* by setting $r(\lambda) = r(\lambda_1)$, $s(\lambda) = s_{|\lambda|}$ for $|\lambda| \geq 1$, and for a vertex $v \in E^0$, we put $r(v) = v = s(v)$.

We also consider infinite paths, defined by

$$E^\infty = \{x = x_1 x_2 \dots \mid x_i \in E^1,\ s(x_i) = r(x_{i+1}) \text{ for all } i\}.$$

The space of infinite paths E^∞ carries a natural topology generated by cylinder sets indexed by finite paths. For a finite path $\lambda \in E^*$, we define

$$Z[\lambda) := \{x \in E^\infty \mid x_1 \dots x_n = \lambda\}.$$

Such a set is referred to as the cylinder set of $\lambda \in E^\infty$. The family of all cylinder sets forms a basis for the topology on E^∞ [9, Corollary 2.2]. Define a map $\sigma : E^\infty \to E^\infty$ by $\sigma(x_i) = x_{i+1}$. This map is a local homeomorphism, and when restricted to a cylinder set $Z[\lambda)$ with $|\lambda| \geq 1$, it becomes a homeomorphism. Consequently, σ induces an action of $\mathbb{N}$ by local homeomorphisms. The graph groupoid associated with E is then defined as the Deaconu–Renault groupoid

$$\mathcal{G}_E = \{(x, m-n, y) \mid \sigma^m(x) = \sigma^n(y)\}.$$

As noted earlier, this groupoid is étale.

With the preliminaries in place, let us introduce the definition of the self-similar groupoids $(\mathcal{G}, E)$. Let E be a row-finite graph without sources. A self-similar action of a groupoid $\mathcal{G}$ on E^* is given by a groupoid $\mathcal{G}$ whose unit space satisfies $\mathcal{G}^{(0)} = E^0$, together with two maps:

1. an action of $\mathcal{G}$ on E^*, $(g, \mu) \mapsto g \cdot \mu$, from $\mathcal{G} \times E^*$ to E^*, and
2. a restriction map $(g, \mu) \mapsto \varphi(g, \mu)$, from $\mathcal{G} \times E^*$ to $\mathcal{G}$ satisfying for all $g \in \mathcal{G}$, for all $\mu \in E^*$,

$$g \cdot (\mu\beta) = (g \cdot \mu)(\varphi(g, \mu) \cdot \beta) \text{ for all } \beta \in E^*. \tag{3.1}$$

These maps are additionally required to satisfy the following conditions, for $g, h \in \mathcal{G}, \mu, \beta \in E^*$, and $v \in E^0$, whenever the expressions are well-defined:

- $r(g \cdot \mu) = g \cdot r(\mu)$,

- $s(g \cdot \mu) = \varphi(g, \mu) \cdot s(\mu)$,
- $(gh) \cdot \mu = g \cdot (h \cdot \mu)$,
- $|g \cdot \mu| = |\mu|$,
- $\varphi(g, v) = g$,
- $\varphi(gh, \mu) = \varphi(g, h \cdot \mu)\varphi(h, \mu)$,
- $\varphi(g, \mu\beta) = \varphi(\varphi(g, \mu), \beta)$, and
- $(\varphi(g, \mu))^{-1} = \varphi(g^{-1}, g \cdot \mu)$.

The self-similar action of $\mathcal{G}$ on E^* will be written as the pair $(\mathcal{G}, E)$. In this article, we may refer to $(\mathcal{G}, E)$ either as a self-similar groupoid or as a self-similar groupoid action.

Example 3.1 *Fix an integer $N \geq 2$. Define $X_N = \{0, 1, \dots, N-1\}$. Let $\mathcal{G}$ be the free abelian group under multiplication generated by $g : G = \{g^n \mid n \in \mathbb{Z}\}$. The action of $\mathcal{G}$ on X_N^* works like a 'carrying rule': if the first entry of λ is less than $N-1$, it is increased by one; and if instead the first entries are all equal to $N-1$ up to some position $k-1$, then the k-th entry is increased by one, with the preceding entries reset to zero. We assign the restriction of g at λ_i to be the identity when λ_i is less than $N-1$, and g itself when λ_i equals $N-1$. In this way, the pair (G, X_N) becomes a self-similar group, called the odometer group. Note that every self-similar group can be regarded as a self-similar groupoid.*

Example 3.2 *Let $(\mathcal{G}, E)$ be a self-similar groupoid. Consider the action of G on $X_2 = \{0, 1\}$ and the restriction in this self-similar action is denoted by $g|_i$. Define $\Gamma := \mathcal{G} \times G$.*

Let F be the directed graph with vertex set $F^0 = E^0$ and edge set $F^1 = E^1 \times X_2$. The action is given by:

- $(h, g^n)(e, i) = (h \cdot e, g^n \cdot i)$, *where the first component comes from the self-similar groupoid action $(\mathcal{G}, E)$ and the second from the odometer action (G, X_2);*
- $\varphi((h, g^n), (e, i)) = (\varphi(h, e), g^n|_i)$,

for all $(h, g^n) \in G'$ and $(e, i) \in F^1$. Thus, (Γ, F) defines a self-similar groupoid action.

4. Groupoids Associated to Self-Similar Groupoid Actions

Given our self-similar groupoids $(\mathcal{G}, E)$. We construct two groupoids associated with our $(\mathcal{G}, E)$, namely $S(\mathcal{G}, E) \ltimes E^\infty$ and $\mathcal{G}_{(\mathcal{G},E)}$, by generalizing the construction in [3], and the arguments continue to hold to our broader framework. The groupoid $S(\mathcal{G}, E) \ltimes E^\infty$ arises from the model of self-similar graphs in [14], while $\mathcal{G}_{(\mathcal{G},E)}$ plays the role of a lag groupoid. These two groupoids are closely connected, and by using known results about $S(\mathcal{G}, E) \ltimes E^\infty$, we can derive the structure of $\mathcal{G}_{(\mathcal{G},E)}$.

Let E be a row-finite directed graph without sources. Suppose $\mathcal{G}$ is a discrete groupoid acting self-similarly on E, where the unit space $\mathcal{G}^{(0)}$ coincides with the vertex set E^0. We introduce the set

$$S(\mathcal{G}, E) := \{(\mu, g, \beta) \mid \mu, \beta \in E^*,\ g \in \mathcal{G}_{s(\beta)}^{s(\mu)},\ s(\mu) = g \cdot s(\beta)\} \cup \{0\}.$$

Equip $S(\mathcal{G}, E)$ with the multiplication

$$(\mu, g, \beta)(\alpha, h, \gamma) = \begin{cases} (\mu(g \cdot \alpha'), \varphi(g, \alpha')h, \gamma) & \text{if } \alpha = \beta\alpha', \\ (\mu, g\varphi(h, h^{-1} \cdot \beta'), \gamma(h^{-1} \cdot \beta')) & \text{if } \beta = \alpha\beta', \\ 0 & \text{otherwise,} \end{cases}$$

then $S(\mathcal{G}, E)$ is an inverse semigroup. The reader can easily prove this; the associativity is routine but tedious calculation; and taking the adjoint of (μ, g, β) yields (β, g^{-1}, μ).

Next, define an action of $S(\mathcal{G}, E)$ on E^∞, i.e., a map $((\mu, g, \beta), \beta x) \mapsto (\mu, g, \beta) \cdot \beta x := \mu g \cdot x$ for all $x \in Z[s(\beta))$. Furthermore, the map $\beta x \mapsto (\mu, g, \beta) \cdot \beta x$ yields a homeomorphism from $Z[\beta)$ to $Z[\mu)$ ([5, Lemma 5.1.2])

We adopt the following definition of the groupoid from [3, Definition 6.2]. A more comprehensive explanation of its construction in the general case can be found in [17].

Definition 4.1 [18, Section 4.3] Suppose E is a row-finite directed graph without sources. Let $\mathcal{G}$ be a discrete groupoid that acts self-similarly on E, with its unit space coinciding with the vertex set E^0. Consider the set

$$\Big\{\big((\mu, g, \beta), \beta x\big) \;\Big|\; (\mu, g, \beta) \in S(G, E),\ x \in Z[s(\beta))\Big\}.$$

On this set we introduce a relation $\sim$, where

$$\big((\mu, g, \beta), \beta x\big) \;\sim\; \big((\alpha, h, \gamma), \gamma y\big).$$

if and only if

- $\beta x = \gamma y$, and
- there exist an idempotent element $(\lambda, e, \lambda) \in S(\mathcal{G}, E)$ such that $\beta x = \gamma y \in Z[\lambda)$, and

$$(\mu, g, \beta)(\lambda, e, \lambda) = (\alpha, h, \gamma)(\lambda, e, \lambda).$$

We denote the equivalence class of $\big((\mu, g, \beta), \beta x\big)$ under $\sim$ by $[(\mu, g, \beta), \beta x]$, called the germ of (μ, g, β) at βx. The set

$$S(\mathcal{G}, E) \ltimes E^\infty := \Big\{[(\mu, g, \beta), \beta x] \;\Big|\; (\mu, g, \beta) \in S(G, E),\ x \in Z[s(\beta))\Big\}$$

is a groupoid under the operations defined by

$$\begin{aligned} s([(\mu, g, \beta), \beta x]) &= \beta x, \\ r([(\mu, g, \beta), \beta x]) &= (\mu, g, \beta) \cdot \beta x, \\ [(\mu, g, \beta), (\alpha, h, \gamma) \cdot \gamma x][(\alpha, h, \gamma), \gamma x] &= [(\mu, g, \beta)(\alpha, h, \gamma), \gamma x], \text{ and} \\ [(\mu, g, \beta), \beta x]^{-1} &= [(\beta, g^{-1}, \mu), (\mu, g, \beta) \cdot \beta x]. \end{aligned}$$

As in [3], we describe $S(G, E) \ltimes E^\infty$ as a lag-type groupoid. Define the left-shift map σ on E^∞ by

$$\sigma^k\big((x)_i\big) = (x)_{i+k}, \quad i \in \mathbb{N},\ x \in E^\infty.$$

Lemmas 4.1 and 4.1 parallel [3, Lemmas 6.3-6.4]. See [5] for detailed computations.

Lemma 4.1 *Suppose E is a row-finite directed graph without sources. Let $\mathcal{G}$ be a discrete groupoid that acts self-similarly on E, with its unit space coinciding with the vertex set E^0. Consider the set*

$$\{(x, m, g, n, y) \in E^\infty \times \mathbb{N} \times \mathcal{G} \times \mathbb{N} \times E^\infty \mid s(g) = r(\sigma^n(y)) \text{ and } \sigma^m(x) = g \cdot \sigma^n(y)\}.$$

On this set we introduce a relation $\sim$, where

$$(x, m, g, n, y) \sim (w, p, h, q, z)$$

if and only if

- *$x = w$, $y = z$, and $m - n = p - q$; and*
- *there exist $l \geq \max\{n, q\}$ such that $\varphi(g, y(n, l)) = \varphi(h, z(q, l))$.*

We denote the equivalence class of (x, m, g, n, y) under $\sim$ by $[x, m, g, n, y]$. The set

$$\mathcal{G}_{(\mathcal{G},E)} := \{[x, m, g, n, y] \mid s(g) = r(\sigma^n(y)) \text{ and } \sigma^m(x) = g \cdot \sigma^n(y)\}.$$

Then the set $\mathcal{G}_{(\mathcal{G},E)}$ is a groupoid with groupoid operations given by

$$\begin{aligned} r([x, m, g, n, y]) &= x, \\ s([x, m, g, n, y]) &= y, \\ [x, m, g, n, y][y, p, h, q, z] &= [x, m + p, \varphi(g, y(n, n + p))\varphi(h, z(q, q + n)), n + q, z], \\ \text{and } [x, m, g, n, y]^{-1} &= [y, n, g^{-1}, m, x]. \end{aligned}$$

Proof: The detailed proof of Lemma 4.1 can be referred to [5, Lemma 5.1.4]. The proof of Lemma 4.1 is routine but tedious; it relies on the definition of the relation $\sim$ and on the self-similar properties of $(\mathcal{G}, E)$. □

To apply results on $S(\mathcal{G}, E) \ltimes E^\infty$ (cf. [17]), we use the following lemma as a link to $\mathcal{G}_{(\mathcal{G},E)}$. It is partly parallel to [3, Lemma 6.4]. Prior to proceeding, we shall discuss the notion of a groupoid homomorphism.

Definition 4.2 Let $\mathcal{G}_1, \mathcal{G}_2$ be groupoids. A *groupoid homomorphism* from $\mathcal{G}_1$ to $\mathcal{G}_2$ is a map $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ such that

- $(\psi \times \psi)(\mathcal{G}_1^{(2)}) \subseteq \mathcal{G}_2^{(2)}$, and
- $\psi(g_1)\psi(g_2) = \psi(g_1 g_2)$ for $g_1, g_2 \in \mathcal{G}_1$ such that $s(g_1) = r(g_2)$.

Theorem 4.1 *Suppose E is a row-finite directed graph without sources. Let $\mathcal{G}$ be a discrete groupoid that acts self-similarly on E with its unit space coinciding with the vertex set E^0. Then there exists a bijective groupoid homomorphism*

$$\psi : S(\mathcal{G}, E) \ltimes E^\infty \to \mathcal{G}_{\mathcal{G},E}$$

such that $\psi([(\mu, g, \beta), \beta x]) = [\mu(g \cdot x), |\mu|, g, |\beta|, \beta x]$.

Proof: We define $S := \{((\mu, g, \beta), \beta x) \mid (\mu, g, \beta) \in S(\mathcal{G}, E),\ x \in Z[s(\beta))\}$, and a map

$$\psi^0 : S \to \mathcal{G}_{\mathcal{G},E}$$

by $\psi^0((\mu, g, \beta), \beta x) = [\mu(g \cdot x), |\mu|, g, |\beta|, \beta x]$.

Fix $[(\mu, g, \beta), \beta x]$ and $[(\alpha, h, \gamma), \gamma x] \in S(\mathcal{G}, E) \ltimes E^\infty$. First, we show that

$$[(\mu, g, \beta), \beta x] = [(\alpha, h, \gamma), \gamma y]$$

if and only if

$$\psi^0((\mu, g, \beta), \beta x) = \psi^0((\alpha, h, \gamma), \gamma y).$$

Let $((\mu, g, \beta), \beta x) \sim ((\alpha, h, \gamma), \gamma y)$. This implies that: $\beta x = \gamma y =: z$; and there exists an idempotent element $(\lambda, e, \lambda) \in S(\mathcal{G}, E)$ such that $z \in Z[\lambda)$ and $(\mu, g, \beta)(\lambda, e, \lambda) = (\alpha, h, \gamma)(\lambda, e, \lambda)$.

Without loss of generality, because $z \in Z[\lambda)$, we may assume that $\lambda = z(0, m)$ with $m \geq \{|\beta|, |\gamma|\}$. Thus, $\lambda = \beta\beta' = \gamma\gamma'$ for some $\beta' \in s(\beta)E^*, \gamma' \in s(\gamma)E^*$. Note that $e = s(\lambda)$. So, $(\mu(g \cdot \beta'), \varphi(g, \beta'), \beta\beta') = (\alpha(h \cdot \gamma'), \varphi(h, \gamma'), \gamma\gamma')$. In particular,

- $\varphi(g, \beta x(|\beta|, |\beta\beta'|)) = \varphi(g, \beta') = \varphi(h, \gamma') = \varphi(h, \gamma x(|\gamma|, |\gamma\gamma'|))$, and
- $\mu(g \cdot x) = \mu(g \cdot \beta')\varphi(g, \beta') \cdot \sigma^{|\beta'|}(x) = \alpha(h \cdot \gamma')\varphi(h, \gamma') \cdot \sigma^{|\gamma'|}(y) = \alpha(h \cdot y)$.

This implies that

$$\psi^0((\mu, g, \beta), \beta x) = \psi^0((\alpha, h, \gamma), \gamma y)$$

as desired.

Now, let $\psi^0((\mu, g, \beta), \beta x) = \psi^0((\alpha, h, \gamma), \gamma y)$. We verify that

$$[(\mu, g, \beta), \beta x] = [(\alpha, h, \gamma), \gamma y];$$

that is, $(\mu(g \cdot x), |\mu|, g, |\beta|, \beta x) \sim (\alpha(h \cdot y), |\alpha|, h, |\gamma|, \gamma y)$. It follows that

- $\mu(g \cdot x) = \alpha(h \cdot y)$, $\beta x = \gamma y$, $|\mu| - |\beta| = |\alpha| - |\gamma|$, and
- there exists $l \geq \max\{|\beta|, |\gamma|\}$ such that $\varphi(g, \beta x(|\beta|, l)) = \varphi(h, \gamma y(|\gamma|, l))$, or

$$\varphi(g, x(0, l - |\beta|)) = \varphi(h, y(0, l - |\gamma|)).$$

Let $e := s(x_{l-|\beta|}) = s(y_{l-|\gamma|})$. We deduce

$$(\mu, g, \beta)(\beta x(0, l-|\beta|), e, \beta x(0, l-|\beta|)) = (\alpha, h, \gamma)(\gamma y(0, l-|\gamma|), e, \gamma y(0, l-|\gamma|)).$$

Note that $\beta x(0, l - |\beta|) = (\beta x)(0, l) = (\gamma y)(0, l) = \gamma y(0, l - |\gamma|)$, so

$$(\mu, g, \beta)(\beta x(0, l-|\beta|), e, \beta x(0, l-|\beta|)) = (\alpha, h, \gamma)(\beta x(0, l-|\beta|), e, \beta x(0, l-|\beta|)). \tag{4.1}$$

Thus, $(\beta x(0, l - |\beta|), e, \beta x(0, l - |\beta|))$ is an idempotent element in $S(\mathcal{G}, E)$ satisfying (4.1). Accordingly, $[(\mu, g, \beta), \beta x] = [(\alpha, h, \gamma), \gamma y]$. Therefore, ψ^0 induces an injective map $\psi : S(\mathcal{G}, E) \ltimes E^\infty \to \mathcal{G}_{\mathcal{G},E}$. Next, we want to show that ψ is surjective. Fix $[x, m, g, n, y] \in \mathcal{G}_{\mathcal{G},E}$. Take $\mu := x(0, m), z := \sigma^n(y)$, and $\beta := y(0, n)$. It follows that

$$\psi([(\mu, g, \beta), \beta z]) = [\mu(g \cdot z), |\mu|, g, |\beta|, \beta z] = [x(0, m)\sigma^m(x), m, g, n, y(0, n)\sigma^n(y)] = [x, m, g, n, y].$$

Fix $[(\mu, g, \beta), \beta x], [(\alpha, h, \gamma), \gamma y] \in S(\mathcal{G}, E) \ltimes E^\infty$ such that $\beta x = (\alpha, h, \gamma) \cdot \gamma y$. We prove that the multiplication operation on $S(\mathcal{G}, E) \ltimes E^\infty$ is preserved by ψ:

$$\psi([(\mu, g, \beta), \beta x][(\alpha, h, \gamma), \gamma y]) = \psi([(\mu, g, \beta), \beta x])\psi([(\alpha, h, \gamma), \gamma y]). \tag{4.2}$$

If $\alpha = \beta\alpha'$, then $(\mu, g, \beta)(\alpha, h, \gamma) = (\mu, g, \beta)(\beta\alpha', h, \gamma) = (\mu(g \cdot \alpha'), \varphi(g, \alpha')h, \gamma))$. So,

$$\begin{aligned}\psi([(\mu, g, \beta), \beta x][(\alpha, h, \gamma), \gamma y]) &= \psi([(\mu, g, \beta)(\alpha, h, \gamma), \gamma y]) \\ &= \psi([(\mu(g \cdot \alpha'), \varphi(g, \alpha')h, \gamma), \gamma y]) \\ &= [\mu(g \cdot \alpha')(\varphi(g, \alpha')h \cdot y), |\mu(g \cdot \alpha')|, \varphi(g, \alpha')h, |\gamma|, \gamma y]. \end{aligned} \tag{4.3}$$

On the other hand, we have

$$\begin{aligned}&\psi([(\mu, g, \beta), (\alpha, h, \gamma) \cdot \gamma y])\psi([(\alpha, h, \gamma), \gamma y]) \\ &\quad= \psi([(\mu, g, \beta), \alpha(h \cdot y)])\psi([(\alpha, h, \gamma), \gamma y]) \\ &\quad= \psi([(\mu, g, \beta), \beta\alpha'(h \cdot y)])\psi([(\beta\alpha', h, \gamma), \gamma y]) \\ &\quad= [\mu(g \cdot (\alpha'(h \cdot y))), |\mu|, g, |\beta|, \beta\alpha'(h \cdot y)][\beta\alpha'(h \cdot y), |\beta\alpha'|, h, |\gamma|, \gamma y] \\ &\quad= [\mu(g \cdot (\alpha'(h \cdot y))), |\mu| + |\beta\alpha'|, \varphi(g, \beta\alpha'(h \cdot y)(|\beta|, |\beta| + |\beta\alpha'|)) \\ &\qquad\quad \varphi(h, \gamma y(|\gamma|, |\gamma| + |\beta|)), |\beta| + |\gamma|, \gamma y]. \end{aligned} \tag{4.4}$$

Equations (4.3) and (4.4) ensure that $\mu(g \cdot \alpha')(\varphi(g, \alpha')h \cdot y) = \mu(g \cdot (\alpha'(h \cdot y)))$, and

$$|\mu(g\cdot\alpha')|-|\gamma| = |\mu|+|\alpha'|-|\gamma| = |\mu|+(|\alpha|-|\beta|)-|\gamma| = (|\mu|+|\alpha|)-(|\beta|)+|\gamma|). = (|\mu|+|\beta\alpha'|)-(|\beta|)+|\gamma|).$$

Let $M := |\gamma| + |\beta\alpha'| \geq \max\{|\gamma|, |\beta| + |\gamma|\}$. Then,

$$\begin{aligned}&\varphi(\varphi(g, \beta\alpha'(h \cdot y)(|\beta|, |\beta| + |\beta\alpha'|))\varphi(h, \gamma y(|\gamma|, |\gamma| + |\beta|)), \gamma y(|\beta| + |\gamma|, M)) \\ &= \varphi(\varphi(g, \beta\alpha'(h \cdot y)(|\beta|, |\beta| + |\beta\alpha'|)), \varphi(h, \gamma y(|\gamma|, |\gamma| + |\beta|) \cdot \gamma y(|\beta| + |\gamma|, M)) \\ &\qquad \varphi(\varphi(h, \gamma y(|\gamma|, |\gamma| + |\beta|)), \gamma y(|\beta| + |\gamma|, M)) \\ &= \varphi(\varphi(g, \beta\alpha'(h \cdot y)(|\beta|, |\beta| + |\beta\alpha'|)), \varphi(h, \gamma y(|\gamma|, |\gamma| + |\beta|)) \cdot \gamma y(|\beta| + |\gamma|, M)) \\ &\qquad \varphi(h, \gamma y(|\gamma|, M)) \\ &= \varphi(\varphi(g, \beta\alpha'(h \cdot y)(|\beta|, |\beta| + |\beta\alpha'|)), \beta\alpha'(h \cdot y)(|\alpha| + |\beta|, |\alpha| + |\beta| + |\alpha'|)) \\ &\qquad \varphi(h, \gamma y(|\gamma|, M)) \\ &= \varphi(g, \beta\alpha'(h \cdot y)(|\beta|, |\beta| + |\alpha| + |\alpha'|))\varphi(h, \gamma y(|\gamma|, M)) \\ &= \varphi(\varphi(g, \alpha'), h \cdot \gamma y(|\gamma|, M))\varphi(h, \gamma y(|\gamma|, M)) \\ &= \varphi(\varphi(g, \alpha')h, \gamma y(|\gamma|, M)).\end{aligned}$$

Equation (4.2) follows. Similar computations yield the same result for $\beta = \alpha\beta'$. Hence, ψ is multiplicative, and therefore a groupoid isomorphism. □

5. An Étale Self-Similar Graph Groupoid

By [17], $S(\mathcal{G}, E) \ltimes E^\infty$ is an étale groupoid with unit space E^∞ (locally compact Hausdorff). Lemma 4.1 provides an isomorphism onto $\mathcal{G}_{(\mathcal{G},E)}$, enabling us to transfer the topology to make $\mathcal{G}_{(\mathcal{G},E)}$ locally Hausdorff and étale. The following theorem partly parallels [3, Lemma 6.4].

Theorem 5.1 *For each $(\mu, g, \beta) \in S(\mathcal{G}, E)$, set $Z(\mu, g, \beta) := \{[\mu(g \cdot x), |\mu|, g, |\beta|, \beta x] \mid x \in Z[s(\beta))\}$. The family $Z(\mu, g, \beta)$ provides a basis of compact open sets for a locally Hausdorff topology on $\mathcal{G}_{(\mathcal{G},E)}$, making $\mathcal{G}_{(\mathcal{G},E)}$ an étale groupoid.*

Proof: To begin with, we introduce the following notation. For $(\mu, g, \beta) \in S(\mathcal{G}, E)$ and open set $U \subseteq Z[\beta) \in E^\infty$, define

$$N((\mu, g, \beta), U) := \{[\mu(g \cdot x), |\mu|, g, |\beta|, \beta x] \mid \beta x \in U\}.$$

Lemma 5.1.5 together with Proposition 4.14 in [17] shows that $\mathcal{G}_{(\mathcal{G},E)}$ becomes a topological groupoid whose topology is generated by the sets $N((\mu, g, \beta), U)$. Fix $[\mu(g \cdot x), |\mu|, g, |\beta|, \beta x] \in N((\mu, g, \beta), U)$. From the topology on E^∞, we may choose $\beta' = x(0, m)$ for some $m \in \mathbb{N}$ such that $\beta x \in Z[\beta\beta') \subseteq U$. Hence, for some $y \in E^\infty$

$$\begin{aligned}[\mu(g \cdot x), |\mu|, g, |\beta|, \beta x] &= [\mu(g \cdot \beta')(\varphi(g, \beta') \cdot y), |\mu(g \cdot \beta')|, \varphi(g, \beta'), |\beta\beta'|, \beta\beta' y] \\ &\in N(\mu(g \cdot \beta'), \varphi(g, \beta'), \beta\beta')) = N((\mu, g, \beta), Z[\beta\beta')) \subseteq N((\mu, g, \beta), U).\end{aligned}$$

Hence, the sets $N(\mu, g, \beta)$ generate the same topology on $\mathcal{G}_{(\mathcal{G},E)}$ as the sets $N((\mu, g, \beta), U)$.

From [17, Proposition 4.15], the maps $r : N(\mu, g, \beta) \to Z[\mu)$ and $s : N(\mu, g, \beta) \to Z[\beta)$ are homeomorphisms. Since $Z[\mu)$ and $Z[\beta)$ are compact Hausdorff spaces, the sets $N(\mu, g, \beta)$ inherit these properties. Hence, $\mathcal{G}_{G,E}$ is locally Hausdorff and étale.

□

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Isnie Yusnitha,
Mathematics Study Program, FPMIPA
Universitas Pendidikan Indonesia,
Indonesia.
E-mail address: isnieyusnitha@upi.edu