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On the Zagreb Domination Energy of a Graph

Nayaka S. R., Purushothama S., Honnegowda C. K. and Puttaswamy

ABSTRACT: A degree–domination based matrix, termed the Zagreb domination matrix, is introduced for a simple connected graph G. Its spectral properties define a new invariant called the Zagreb domination energy, with bounds derived in terms of order, size, degrees, and domination degrees. Exact formulas and inequalities are established for standard and $k-$regular domination–regular graphs. The study further examines the effect of $t-$splitting and $t-$shadow operations, linking eigenvalue inequalities with domination theory.

Keywords: Domination degree, Zagreb domination matrix, Zagreb domination energy, minimal dominating set, eigenvalue bounds, regular graphs, graph operations.

Contents

1 Introduction **1**

2 Zagreb Domination Matrix and Energy **2**

3 Basic Properties and General Bounds **2**

4 Regular and Domination-Regular Graphs **3**

5 Paths, Cycles and Grid Graphs **4**

6 Complete Bipartite and Tripartite Graphs **5**

7 Graph Operations **5**

8 Bounds in Terms of Order and Degree **5**

9 Conclusion **6**

1. Introduction

The concept of graph energy, introduced by Gutman as the sum of absolute values of eigenvalues of the adjacency matrix, plays a central role in chemical graph theory and spectral graph theory [1], [2]. Degree-based refinements, notably the Zagreb indices and Zagreb energies, were developed to capture finer structural information [3,4,5]. The first Zagreb matrix and its energy were investigated in [3,5,6], while further bounds appear in [7,8]. Parallel to this, domination theory has become a fundamental area of graph theory with extensive applications [9,10]. Recently the notion of domination degree and domination topological indices, including domination Zagreb indices, were introduced in [10], thereby bridging domination theory and topological indices. The concept of domination degree energy was recently introduced in [20].

Motivated by these developments, we introduces the *Zagreb domination matrix*, obtained by placing domination degrees on the diagonal of the first Zagreb matrix. The associated *Zagreb domination energy* combines degree information with the structure of minimal dominating sets. Using standard tools from matrix analysis [12,13] and eigenvalue bounds for graphs [14,15,16], we derive general inequalities and evaluate the new energy for several graph families.
All graphs considered are finite, simple and connected. For a vertex $v \in V(G)$, the degree of v is denoted by $d_G(v)$, its open neighbourhood by $N_G(v)$, the maximum and minimum degree by Δ and δ, respectively.

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Definition 1.1 A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G) \setminus D$ has at least one neighbour in D. A dominating set D is minimal if no proper subset of D is dominating. The domination degree of a vertex $v \in V(G)$ is defined by

$$dd_G(v) = \min\{\ |D| : D \text{ is a minimal dominating set of } G \text{ and } v \in D\ \}.$$

The concept of domination degree and various domination-based indices were studied in [10,11]. Let $V(G) = \{v_1, \dots, v_n\}$. The first Zagreb matrix $FZ(G)$ is the $n \times n$ matrix [3,4,5]

$$FZ(G))_{ij} = \begin{cases} d_G(v_i) + d_G(v_j), & i \neq j,\ v_iv_j \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

It is real and symmetric, hence has real eigenvalues.

For a real symmetric matrix M of order n, we write its eigenvalues as $\lambda_1(M) \geq \cdots \geq \lambda_n(M)$. We shall use standard facts about traces, Rayleigh quotients, Gershgorin discs and Cauchy–Schwarz inequality [12,13].

2. Zagreb Domination Matrix and Energy

Definition 2.1 Let G be a simple connected graph with vertex set $V(G) = \{v_1, \dots, v_n\}$. The *Zagreb domination matrix* of G is the $n \times n$ matrix

$$ZD(G) = [c_{ij}],$$

where

$$c_{ij} = \begin{cases} d_G(v_i) + d_G(v_j), & i \neq j,\ v_iv_j \in E(G), \\ 0, & i \neq j,\ v_iv_j \notin E(G), \\ dd_G(v_i), & i = j. \end{cases}$$

Clearly $ZD(G)$ is real symmetric. Let its eigenvalues be

$$\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n.$$

We have

$$\sum_{i=1}^{n} \xi_i = \operatorname{tr}(ZD(G)) = \sum_{v \in V(G)} dd_G(v), \qquad \sum_{i=1}^{n} \xi_i^2 = \operatorname{tr}(ZD(G)^2).$$

Definition 2.2 The *Zagreb domination energy* of G is defined by

$$ZE_d(G) = \sum_{i=1}^{n} |\xi_i|.$$

Note that

$$ZD(G) = FZ(G) + D_d(G), \qquad D_d(G) = \operatorname{diag}\big(dd_G(v_1), \dots, dd_G(v_n)\big),$$

which links the new matrix to the classical Zagreb matrix studied in [3,21,6].

3. Basic Properties and General Bounds

Define

$$S_2(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2, \qquad D_2(G) = \sum_{v \in V(G)} dd_G(v)^2.$$

Then

$$\operatorname{tr}(ZD(G)^2) = 2S_2(G) + D_2(G).$$

Proposition 3.1 *Let G be a connected graph of order n. Then*

$$\max_{v\in V(G)} dd_G(v) \le \xi_1 \le \max_{v\in V(G)} \Big(dd_G(v) + \sum_{u\sim v}(d_G(u)+d_G(v))\Big).$$

Moreover,

$$\frac{1}{n}\sum_{v\in V(G)} dd_G(v) \le \xi_1 \le \sqrt{\sum_{i=1}^{n}\xi_i^2}.$$

Proof: The lower bound follows from the Rayleigh quotient evaluated at the standard basis vectors. The upper bounds follow from Gershgorin's circle theorem and the fact that $\xi_1^2 \le \sum_{i=1}^{n}\xi_i^2$ [12,13]. □

Theorem 3.1 *For any connected graph G,*

$$ZE_d(G) \ge 2\sqrt{2S_2(G)+D_2(G)},$$

with equality if and only if all eigenvalues of $ZD(G)$ are nonnegative or all nonpositive.

Proof: We have $\sum_{i=1}^{n}\xi_i^2 = \mathrm{tr}(ZD(G)^2) = 2S_2(G)+D_2(G)$. The inequality $\sum_{i=1}^{n}|a_i| \ge 2\sqrt{\sum_{i=1}^{n}a_i^2}$ for real numbers a_i with at least one positive and one negative entry is standard; equality holds if all a_i share the same sign. Applying this to $a_i=\xi_i$ gives the claim. □

The Frobenius norm inequality $ZE_d(G) \le \sqrt{n}\sqrt{\sum\xi_i^2}$ yields:

Corollary 3.1 *Let G be a connected graph of order n. Then*

$$\frac{2}{\sqrt{n}}\sqrt{2S_2(G)+D_2(G)} \le ZE_d(G) \le \sqrt{n}\sqrt{2S_2(G)+D_2(G)}.$$

4. Regular and Domination-Regular Graphs

A graph G is *k–regular* if $d_G(v)=k$ for all vertices v. We call G *domination-regular* if there exists a constant α such that $dd_G(v)=\alpha$ for all v.

Proposition 4.1 *Let G be a k–regular domination–regular graph of order n with common domination degree α. Then*

$$ZD(G) = \alpha I_n + 2kA(G),$$

where $A(G)$ is the adjacency matrix of G. If $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $A(G)$, then

$$\xi_i = \alpha + 2k\lambda_i, \qquad 1\le i\le n,$$

and

$$ZE_d(G) = \sum_{i=1}^{n}|\alpha+2k\lambda_i|.$$

Proof: For a k–regular graph, $(FZ(G))_{ij} = 2k$ whenever $v_iv_j \in E(G)$ and 0 otherwise, hence $FZ(G) = 2kA(G)$ [3,5]. Domination regularity ensures $D_d(G)=\alpha I_n$, so $ZD(G)=\alpha I_n + 2kA(G)$. The eigenvalue relations follow from the spectral mapping theorem. □

Example 4.1 (Complete graph) For $G=K_n$, $d_G(v)=n-1$ and every single vertex is a minimal dominating set, so $dd_G(v)=1$ for each v [9,10]. The eigenvalues of $A(K_n)$ are $n-1$ (once) and -1 with multiplicity $n-1$. By Proposition 4.1,

$$\xi_1 = 1+2(n-1)^2, \qquad \xi_i = 1-2(n-1),\ 2\le i\le n,$$

and

$$ZE_d(K_n) = \big|1+2(n-1)^2\big| + (n-1)\big|1-2(n-1)\big|.$$

5. Paths, Cycles and Grid Graphs

We now derive explicit expressions for $ZE_d(G)$ for some families whose domination degrees are well understood [9,10].

For the cycle C_n $(n \geq 3)$, it is known that $\gamma(C_n) = \lceil n/3 \rceil$ and every vertex belongs to some minimal dominating set of cardinality 2, hence $dd_{C_n}(v) = 2$ for all v [9]. The eigenvalues of $A(C_n)$ are $\lambda_j = 2\cos\left(\frac{2\pi j}{n}\right)$, $0 \leq j \leq n-1$ [14]. By Proposition 4.1 with $k = 2$ and $\alpha = 2$,

Theorem 5.1 *For $n \geq 3$,*

$$ZE_d(C_n) = \sum_{j=0}^{n-1} \left| 2 + 4\cos\left(\frac{2\pi j}{n}\right) \right|.$$

For the path P_n $(n \geq 2)$, the domination number satisfies $\gamma(P_n) = \lceil n/3 \rceil$ and the domination structure is described in [9,10]. Every branch vertex of P_n belongs to a minimal dominating set of size $\lceil n/3 \rceil$; end vertices may belong to minimal dominating sets of the same or larger sizes depending on n. Consequently, the diagonal of $ZD(P_n)$ is non-constant. The eigenvalues of $A(P_n)$ are $\lambda_j = 2\cos\frac{j\pi}{n+1}$, $1 \leq j \leq n$ [14]. Writing $ZD(P_n) = FZ(P_n) + D_d(P_n)$ with a diagonal perturbation, general eigenvalue interlacing yields:

Theorem 5.2 *For $n \geq 2$,*

$$2\gamma(P_n) \leq ZE_d(P_n) \leq ZE_1(P_n) + \sqrt{n} \max_{v \in V(P_n)} dd_{P_n}(v),$$

where $ZE_1(P_n)$ is the first Zagreb energy of P_n [6].

The inequality follows from Theorem 3.1 and Weyl-type eigenvalue bounds for diagonal perturbations [12,15]. Closed formulas for $ZE_d(P_n)$ can be obtained for small n by direct computation.

Let $G = P_m \Box P_n$ be the rectangular grid graph, the Cartesian product of two paths [19]. Degrees lie in $\{2, 3, 4\}$ and domination numbers and structures are well studied for grid graphs [9]. The eigenvalues of $A(G)$ are

$$\lambda_{ij} = 2\cos\frac{i\pi}{m+1} + 2\cos\frac{j\pi}{n+1}, \qquad 1 \leq i \leq m,\ 1 \leq j \leq n$$

[19,15]. If domination degrees are constant on interior and boundary vertices according to the classification in [10], then $ZD(G)$ may be written as

$$ZD(G) = 2D_{\text{bd}} + 4A(G) + D_{\text{int}},$$

where D_{bd} and D_{int} are diagonal matrices reflecting the domination degree partition. Consequently the Zagreb domination eigenvalues satisfy

$$\xi_{ij} = \alpha_{ij} + 4\left(\cos\frac{i\pi}{m+1} + \cos\frac{j\pi}{n+1}\right),$$

where $\alpha_{ij} \in \{\alpha_{\text{bd}}, \alpha_{\text{int}}\}$ depends on the vertex type. Hence, we have the following result.

Theorem 5.3 *For the grid graph $P_m \Box P_n$,*

$$ZE_d(P_m \Box P_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \alpha_{ij} + 4\left(\cos\frac{i\pi}{m+1} + \cos\frac{j\pi}{n+1}\right) \right|,$$

where α_{ij} are the domination degrees determined in [10].

6. Complete Bipartite and Tripartite Graphs

We now state the formulas already obtained in a compact form.

Proposition 6.1 *Let $G = K_{r,s}$ with $r, s \geq 2$. Then $dd_G(v) = 2$ for all v and*

$$ZE_d(K_{r,s}) = 4(r+s) + 2(r+s-2).$$

Proof: Any pair (u, v) with u in one part and v in the other is a minimal dominating set of size 2; no smaller dominating set exists. The first Zagreb spectrum of $K_{r,s}$ consists of eigenvalues $2(r+s)$, $-2(r+s)$ and 0 with multiplicity $r+s-2$ [3,6]. Adding $2I_{r+s}$ shifts each eigenvalue by 2 and the result follows. □

Proposition 6.2 *Let $G = K_{r,r,r}$, $r \geq 2$, and suppose $dd_G(v) = 2$ for all v. Then*

$$ZE_d(K_{r,r,r}) = |2+4r| + |2-4r| + (3r-2) \cdot 2.$$

7. Graph Operations

We briefly consider how Zagreb domination energy behaves under some graph operations, following ideas used for Zagreb energy in [5,22,23].

Given a graph G, its t–splitting graph $S_t(G)$ is obtained by adding t copies of each vertex and joining each copy to neighbours of the original vertex. The Zagreb matrices of $S_t(G)$ were analysed in [5].
If G is k–regular domination–regular with domination degree α, then

$$ZD(S_t(G)) = A_t \otimes FZ(G) + D_t \otimes I_n,$$

for suitable $(t+1) \times (t+1)$ matrices A_t, D_t determined as in [5]. Consequently, if $\xi_1, \ldots, \xi_n$ are Zagreb domination eigenvalues of G, then the Zagreb domination spectrum of $S_t(G)$ consists of multiples of ξ_i by the eigenvalues of A_t, shifted by diagonal entries of D_t. Therefore,

$$ZE_d(S_t(G)) = c_t \, ZE_d(G),$$

for an explicit scaling factor c_t depending only on t and the regularity parameters, analogous to the factors obtained for Zagreb energy in [5].

The t–shadow graph $D_t(G)$ is built from t copies of G by joining corresponding vertices to neighbors in each other copy [22]. For regular domination–regular graphs one can express

$$ZD(D_t(G)) = (t+1)^2 J_{t+1} \otimes FZ(G) + D'_t \otimes I_n,$$

where J_{t+1} is the all-ones matrix and D'_t a diagonal matrix. This yields

$$ZE_d(D_t(G)) = (t+1)^2 ZE_d(G) + (t+1) \sum_{v \in V(G)} dd_G(v),$$

mirroring the behavior of Zagreb energy [5,22].

8. Bounds in Terms of Order and Degree

Let G be a connected graph of order n and size m. Let Δ, δ be the maximum and minimum degrees of G and $dd_{\max}, dd_{\min}$ the maximum and minimum domination degrees. Using the approach of [16,17,18] we obtain:

Theorem 8.1 *For any connected graph G,*

$$2\sqrt{2m(2\delta)^2 + n dd_{\min}^2} \leq ZE_d(G) \leq 2\sqrt{2m(2\Delta)^2 + n dd_{\max}^2} + \sqrt{n}\, dd_{\max}.$$

Proof: For each edge uv, $2\delta \leq d_G(u) + d_G(v) \leq 2\Delta$. Thus

$$2m(2\delta)^2 + ndd_{\min}^2 \leq \mathrm{tr}(ZD(G)^2) \leq 2m(2\Delta)^2 + ndd_{\max}^2.$$

Combining these with Theorem 3.1 and Corollary 3.1 yields the required bounds. □

Corollary 8.1 *Let G be a k–regular domination–regular graph of order n and size m with common domination degree α. Then*

$$2\sqrt{4mk^2 + n\alpha^2} \leq ZE_d(G) \leq \sqrt{n}\sqrt{4mk^2 + n\alpha^2}.$$

In particular, for the complete graph K_n,

$$2\sqrt{2n(n-1)^3 + n} \leq ZE_d(K_n) \leq \sqrt{n}\sqrt{2n(n-1)^3 + n}.$$

9. Conclusion

The Zagreb domination matrix enriches degree-based spectral invariant by incorporating domination degrees on the diagonal. The resulting Zagreb domination energy admits exact expressions for several graph families, scales predictably under basic graph operations and satisfies bounds expressed through order, size, degrees and domination degrees. Further study may focus on extremal problems and connections with other domination-based indices.

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Nayaka S. R.,
Assistant Professor,
Department of Mathematics,
PES College of Engineering,
Mandya, Karnataka, India-571401.
E-mail address: `nayaka.abhi11@gmail.com`

and

Purushothama S.,
Associate Professor,
Department of Mathematics,
Maharaja Institute of Technology Mysore,
Mandya, Karnataka, India-571477.
E-mail address: `psmandya@gmail.com`

and

Honnegowda C. K.,
Associate Professor,
Department of Mathematics,
Government First Grade College,
K R Pete, Mandya, Karnataka, India-571426.
E-mail address: `honnegowdack@gmail.com`

and

Puttaswamy,
Professor and Head,
Department of Mathematics,
PES College of Engineering,
Mandya, Karnataka, India-571401.
E-mail address: `prof.puttaswamy@gmail.com`