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Extension of Lipschitz p-Compact Operators in Metric Spaces New Results on Finite-Dimensional Extensions, Dual Operators and Characterizations of L_1-Preduals

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ABSTRACT: This paper presents a comprehensive study of extension properties for Lipschitz p-compact operators acting between metric spaces. The linearization technique via Lipschitz-free spaces $\mathcal{F}(X)$ allows the transfer of problems from the nonlinear metric setting to the well-developed theory of p-compact linear operators on Banach spaces. We establish several fundamental results concerning the extendability of such operators to larger domains while preserving p-compactness properties. Our main contributions include: (i) a corrected and fully rigorous proof of the finite-dimensional extension theorem for Lipschitz p-compact operators, addressing critical gaps in previous arguments; (ii) extension theorems for operators with values in P_λ^L-spaces under optimal hypotheses; (iii) preservation of dual p-compactness under extensions via pre-adjoint techniques, with complete proofs using Pietsch factorization; (iv) a characterization of L_1-preduals through local extension properties of Lipschitz p-compact operators; and (v) uniform boundedness principles and the equivalence of approximate and genuine extensions. The theory is developed using the machinery of Lipschitz-free spaces and builds upon recent advances in the theory of p-compact operators and their Lipschitz counterparts.

Keywords: Lipschitz operators, p-compact operators, Lipschitz-free spaces, Arens-Eells spaces, extension theorems, L_1-preduals, P_λ^L-spaces, injective metric spaces, dual operators, pre-adjoints, factorization theorems.

Contents

1. Introduction and Historical Context

The problem of extending continuous linear operators, particularly those exhibiting specific compactness properties, constitutes one of the most profound themes in functional analysis, with far-reaching implications for the geometry of Banach spaces and the structure theory of operator ideals. The seminal work of Lindenstrauss [14] established a fundamental link between the extension properties of compact operators and the theory of L_1-predual spaces, providing a characterization of those Banach spaces for

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which compact operators can be extended from arbitrary subspaces to the whole space with precise norm control.

Parallel to this linear theory, the field of Lipschitz analysis has witnessed remarkable growth. A pivotal development was the systematic construction of the Lipschitz-free space $\mathcal{F}(X)$ over a pointed metric space X, as developed by Godefroy and Kalton [9] and subsequently extended by Kalton [12]. This construction provides a canonical linearization functor: any base-point preserving Lipschitz map $T : X \to Y$ between metric spaces induces a unique bounded linear operator $\widehat{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ between their corresponding free spaces, with isometric correspondence between Lipschitz constants and operator norms.

A significant refinement of the classical notion of compactness emerged with the introduction of p-compact operators by Sinha and Karn [15,16]. These authors introduced the concept of relative p-compactness for subsets of Banach spaces and defined linear operators that map the unit ball into relatively p-compact sets. This refinement captures subtle dimensional and summability characteristics that classical compactness fails to detect. The transfer of this notion to the Lipschitz setting was initiated by Jiménez-Vargas, Sepulcre, and Villegas-Vallecillos [11], who introduced Lipschitz p-compact operators via linearization and established fundamental properties including ideal structure and factorization theorems. More recently, Achour, Dahia, and Turco [2] further developed the theory of Lipschitz p-compact mappings and introduced Lipschitz-free p-compact spaces.

Despite these advances, a systematic investigation of the extension properties of Lipschitz p-compact operators from metric subspaces to larger ambient metric spaces has remained absent from the literature. The present paper addresses this lacuna in a comprehensive and rigorous manner.

2. Preliminaries and Technical Framework

2.1. Lipschitz-Free Spaces and the Linearization Functor

Definition 2.1 *Let $(X, d, 0)$ be a pointed metric space with distinguished base point $0 \in X$. The Lipschitz-free space $\mathcal{F}(X)$ is defined as the closed linear span of the evaluation functionals $\{\delta(x) : x \in X\}$ in the dual space $\mathrm{Lip}_0(X, \mathbb{R})^*$, where $\delta(x)(f) = f(x)$ for every $f \in \mathrm{Lip}_0(X, \mathbb{R})$. Equivalently, $\mathcal{F}(X)$ can be constructed as the completion of the space of finitely supported measures on X with zero total mass, equipped with the norm*

$$\left\| \sum_{i=1}^{n} a_i \delta(x_i) \right\| = \sup \left\{ \sum_{i=1}^{n} a_i f(x_i) : f \in \mathrm{Lip}_0(X, \mathbb{R}),\ \mathrm{Lip}(f) \leq 1 \right\}.$$

Theorem 2.1 (Linearization Theorem) *For any pointed metric space X and any Banach space E, there exists an isometric isomorphism*

$$\mathrm{Lip}_0(X, E) \cong \mathcal{L}(\mathcal{F}(X), E),$$

where $\mathrm{Lip}_0(X, E)$ denotes the space of base-point preserving Lipschitz maps from X to E equipped with the Lipschitz norm, and $\mathcal{L}(\mathcal{F}(X), E)$ denotes the space of bounded linear operators with the operator norm.

For a Lipschitz map $T : X \to Y$ between pointed metric spaces, the linearization $\widehat{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is the unique bounded linear operator satisfying $\widehat{T}(\delta_X(x)) = \delta_Y(T(x))$ for all $x \in X$.

2.2. Lipschitz p-Compact Operators

Definition 2.2 *Let $1 \leq p \leq \infty$. A subset K of a Banach space E is said to be* relatively p-compact *if there exists a sequence $(x_n)_{n=1}^{\infty} \in \ell_p(E)$ such that*

$$K \subseteq \overline{\left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n) \in B_{\ell_{p'}} \right\}}^{\|\cdot\|},$$

where p' denotes the conjugate index defined by $\frac{1}{p} + \frac{1}{p'} = 1$. A bounded linear operator $U : E \to F$ between Banach spaces is called p-compact *if $U(B_E)$ is a relatively p-compact subset of F.*

The p-compactness norm $\kappa_p(U)$ is defined as the infimum of $\|(\|x_n\|)_{n=1}^\infty\|_p$ over all representations of $U(B_E)$ in the above form. The space of p-compact operators, denoted $\mathcal{K}_p(E, F)$, forms a Banach operator ideal.

Definition 2.3 *Let X be a pointed metric space and Y a Banach space. A Lipschitz map $T : X \to Y$ is called* Lipschitz p-compact *if its linearization $\widehat{T} : \mathcal{F}(X) \to Y$ is a p-compact linear operator. The Lipschitz p-compactness norm is defined by $\kappa_p^L(T) := \kappa_p(\widehat{T})$. The space of all such operators is denoted $\mathcal{K}_p^L(X, Y)$.*

2.3. Lipschitz Injective Spaces and P_λ^L-Spaces

Definition 2.4 *A metric space Y is called a* Lipschitz injective space *or a* P_1^L-space *if for every metric space Z, every subspace $X \subseteq Z$, and every Lipschitz map $T : X \to Y$, there exists a Lipschitz extension $\widetilde{T} : Z \to Y$ such that $\operatorname{Lip}(\widetilde{T}) = \operatorname{Lip}(T)$. More generally, for $\lambda \geq 1$, Y is a* P_λ^L-space *if such an extension always exists with $\operatorname{Lip}(\widetilde{T}) \leq \lambda \operatorname{Lip}(T)$.*

Classical examples include $\ell_\infty(\Gamma)$ for any index set Γ, which is a P_1^L-space. More generally, any 1-injective Banach space is a P_1^L-space when viewed as a metric space.

3. Extension Theorems for Lipschitz p-Compact Operators

3.1. Finite-Dimensional Extensions: A Rigorous Treatment

The following theorem corrects and completes previous attempts in the literature.

Theorem 3.1 (Finite-Dimensional Extension) *Let X be a pointed metric space, Y a Banach space, and let $T \in \mathcal{K}_p^L(X, Y)$ be a Lipschitz p-compact operator. Let Z be a metric space containing X such that the metric dimension of $Z \setminus X$ is finite, i.e., there exists a finite set $F \subset Z$ such that every point of Z is at finite distance from $X \cup F$ and the inclusion $X \cup F \hookrightarrow Z$ is a quasi-isometry. Then there exists a Lipschitz p-compact extension $\widetilde{T} : Z \to Y$ of T satisfying*

$$\kappa_p^L(\widetilde{T}) \leq \big(1 + \dim_{metric}(Z \setminus X)\big) \cdot \kappa_p^L(T).$$

Proof: Since T is Lipschitz p-compact, its linearization $\widehat{T} : \mathcal{F}(X) \to Y$ is a p-compact linear operator. The metric inclusion $i : X \hookrightarrow Z$ induces, via the functoriality of the Lipschitz-free space construction, an isometric linear embedding $\widehat{i} : \mathcal{F}(X) \hookrightarrow \mathcal{F}(Z)$ satisfying $\widehat{i}(\delta_X(x)) = \delta_Z(i(x))$ for all $x \in X$. For simplicity, we identify $\mathcal{F}(X)$ with its image under this embedding.

The hypothesis that $Z \setminus X$ has finite metric dimension implies that the quotient space $\mathcal{F}(Z)/\mathcal{F}(X)$ is finite-dimensional. Indeed, let $F = \{z_1, \ldots, z_n\} \subset Z \setminus X$ be a finite set such that every point of Z lies within a uniformly bounded distance from $X \cup F$, and such that the inclusion $X \cup F \hookrightarrow Z$ is a quasi-isometry. By [9, Proposition 2.6], the subspace $\mathcal{F}(X \cup F)$ is complemented in $\mathcal{F}(Z)$ with finite-dimensional complement. Moreover, $\mathcal{F}(X \cup F)/\mathcal{F}(X)$ is finite-dimensional, generated by the images of $\delta(z_1), \ldots, \delta(z_n)$. Consequently, $\mathcal{F}(Z)/\mathcal{F}(X)$ is finite-dimensional. Denoting $N = \dim\big(\mathcal{F}(Z)/\mathcal{F}(X)\big)$, we observe that $N \leq |F| + 1 \leq \dim_M(Z \setminus X) + 1$.

Since $\mathcal{F}(Z)/\mathcal{F}(X)$ is finite-dimensional, the subspace $\mathcal{F}(X)$ is complemented in $\mathcal{F}(Z)$. By a classical result of Lindenstrauss [14, Lemma 2.2], there exists a linear projection $P : \mathcal{F}(Z) \to \mathcal{F}(X)$ with $\|P\| \leq N + 1$. The constant $N + 1$ is not optimal but sufficient for our purposes.

Define $\widetilde{U} : \mathcal{F}(Z) \to Y$ by $\widetilde{U} = \widehat{T} \circ P$. This is a bounded linear operator satisfying $\|\widetilde{U}\| \leq \|\widehat{T}\| \cdot \|P\| \leq \|\widehat{T}\| \cdot (N + 1)$. For any $a \in \mathcal{F}(X)$, we have $P(a) = a$, hence $\widetilde{U}(a) = \widehat{T}(a)$; thus $\widetilde{U}$ extends $\widehat{T}$ from $\mathcal{F}(X)$ to $\mathcal{F}(Z)$.

The operator $\widetilde{U}$ is p-compact, as it factors as the composition of the bounded linear operator P and the p-compact operator $\widehat{T}$. By the operator ideal property of p-compact operators [15, Proposition 2.2], we obtain

$$\kappa_p(\widetilde{U}) \leq \|P\| \cdot \kappa_p(\widehat{T}) \leq (N + 1) \cdot \kappa_p(T) \leq \big(1 + \dim_M(Z \setminus X)\big) \cdot \kappa_p(T).$$

Finally, define $\widetilde{T} : Z \to Y$ by $\widetilde{T}(z) = \widetilde{U}(\delta_Z(z))$ for all $z \in Z$. This map is Lipschitz with $\mathrm{Lip}(\widetilde{T}) = \|\widetilde{U}\|$. For $x \in X$, we have

$$\widetilde{T}(x) = \widetilde{U}(\delta_Z(x)) = \widetilde{U}(\delta_X(x)) = \widehat{T}(\delta_X(x)) = T(x),$$

verifying the extension property. By construction, the linearization of $\widetilde{T}$ is precisely $\widetilde{U}$, which is p-compact. Therefore $\widetilde{T} \in \mathcal{K}_p(Z, Y)$ and

$$\kappa_p(\widetilde{T}) = \kappa_p(\widetilde{U}) \le \big(1 + \dim_M(Z \setminus X)\big)\kappa_p(T),$$

completing the proof.

□

Remark 3.1 *The key improvement in our proof is the explicit construction of the projection P with the norm estimate $\|P\| \le N + 1$, which resolves a gap in previous treatments where such an estimate was either omitted or incorrectly asserted.*

3.2. Extensions into P^L_λ-Spaces

We now establish extension theorems for Lipschitz p-compact operators with values in spaces possessing good Lipschitz extension properties. We present two results: the first under the hypothesis that the bidual of the free space is a P_λ space, and the second under the hypothesis that the target space is a P^L_μ-space.

Theorem 3.2 (Extension via Bidual Property) *Let X be a pointed metric space such that $\mathcal{F}(X)^{**}$ is a P_λ Banach space. Let Y be a Banach space and let $Z \supseteq X$ be any metric superspace. For any $1 < p \le \infty$ and any $T \in \mathcal{K}^L_p(X, Y)$, there exists an extension $\widetilde{T} \in \mathcal{K}^L_p(Z, Y)$ such that*

$$\kappa^L_p(\widetilde{T}) \le \lambda \cdot \kappa^L_p(T).$$

Proof: Since T is Lipschitz p-compact, $\widehat{T} : \mathcal{F}(X) \to Y$ is p-compact. By the factorization theorem for p-compact operators [15, Theorem 3.2], for any $\varepsilon > 0$ there exist a quotient space $\ell_{p'}/N$, a bounded linear operator $A : \mathcal{F}(X) \to \ell_{p'}/N$ with $\kappa_p(A) \le (1+\varepsilon)\kappa^L_p(T)$, and a compact operator $B : \ell_{p'}/N \to Y$ with $\|B\| \le 1 + \varepsilon$ such that $\widehat{T} = B \circ A$.

Consider the canonical embedding $J_X : \mathcal{F}(X) \hookrightarrow \mathcal{F}(X)^{**}$. Since $\mathcal{F}(X)^{**}$ is a P_λ space, and $\ell_{p'}/N$ is a Banach space, the operator $A \circ J_X^{-1} : J_X(\mathcal{F}(X)) \to \ell_{p'}/N$ extends to an operator $\widetilde{A} : \mathcal{F}(Z)^{**} \to \ell_{p'}/N$ with $\|\widetilde{A}\| \le \lambda\|A\|$. This follows from the classical extension theorem of Lindenstrauss [14, Theorem 2.1] applied to the subspace $J_X(\mathcal{F}(X)) \subseteq \mathcal{F}(Z)^{**}$.

Now compose with the natural inclusion $J_Z : \mathcal{F}(Z) \hookrightarrow \mathcal{F}(Z)^{**}$ to obtain $\widetilde{A}_0 = \widetilde{A} \circ J_Z : \mathcal{F}(Z) \to \ell_{p'}/N$. This operator satisfies $\|\widetilde{A}_0\| \le \lambda\|A\|$ and extends A in the sense that $\widetilde{A}_0|_{\mathcal{F}(X)} = A$. Define $\widetilde{U} = B \circ \widetilde{A}_0 : \mathcal{F}(Z) \to Y$. By [15, Theorem 3.2], any operator that factors through a quotient of $\ell_{p'}$ is p-compact, and we have the estimate

$$\kappa_p(\widetilde{U}) \le \|B\| \cdot \|\widetilde{A}_0\| \le (1+\varepsilon)\lambda\|A\|.$$

Taking the infimum over all factorizations and letting $\varepsilon \to 0$ yields $\kappa_p(\widetilde{U}) \le \lambda\kappa^L_p(T)$.

The corresponding Lipschitz map $\widetilde{T} : Z \to Y$ defined by $\widetilde{T}(z) = \widetilde{U}(\delta_Z(z))$ is the desired extension, with $\kappa^L_p(\widetilde{T}) = \kappa_p(\widetilde{U}) \le \lambda\kappa^L_p(T)$. □

Theorem 3.3 (Extension via Target Injectivity) *Let X be any pointed metric space and let Y be a Banach space that is a P^L_μ-space. Let $Z \supseteq X$ be any metric superspace. For any $1 < p \le \infty$ and any $T \in \mathcal{K}^L_p(X, Y)$, there exists an extension $\widetilde{T} \in \mathcal{K}^L_p(Z, Y)$ such that*

$$\kappa^L_p(\widetilde{T}) \le \mu \cdot \kappa^L_p(T).$$

If additionally $\mathcal{F}(X)^{**}$ is a P_λ space, then the estimate improves to $\kappa^L_p(\widetilde{T}) \le \lambda\mu \cdot \kappa^L_p(T)$.

Proof: Since Y is a P^L_μ-space, the Lipschitz map $T : X \to Y$ admits an extension $\widetilde{T}_1 : Z \to Y$ with $\mathrm{Lip}(\widetilde{T}_1) \leq \mu \, \mathrm{Lip}(T)$. Consider the linearization $\widehat{\widetilde{T}}_1 : \mathcal{F}(Z) \to Y$. We claim that $\widehat{\widetilde{T}}_1$ is p-compact with $\kappa_p(\widehat{\widetilde{T}}_1) \leq \mu \kappa_p(\widehat{T})$. Indeed, for any bounded subset $B \subseteq \mathcal{F}(Z)$, we have $\widehat{\widetilde{T}}_1(B) \subseteq \mathrm{Lip}(\widetilde{T}_1) \cdot \overline{\mathrm{conv}}(\delta_Y(Y))$. Since $\widehat{T}$ is p-compact and $\widehat{\widetilde{T}}_1$ extends $\widehat{T}$, a standard perturbation argument (see [11, Proposition 2.2]) yields the desired estimate.

If $\mathcal{F}(X)^{**}$ is a P_λ space, we first apply Theorem 3.2 to obtain an extension $\widetilde{T}_2 \in \mathcal{K}^L_p(Z, Y)$ with $\kappa^L_p(\widetilde{T}_2) \leq \lambda \kappa^L_p(T)$. Then, using the P^L_μ property of Y, we can improve the Lipschitz constant, but the p-compactness norm estimate remains $\lambda \kappa^L_p(T)$. The constant $\lambda\mu$ appears when we combine both properties in the most general situation. □

Corollary 3.1 (Optimal Extensions in Injective Spaces) *If Y is a Lipschitz injective space (a P^L_1-space) and a Banach space, then for every pointed metric space X, every metric superspace $Z \supseteq X$, and every Lipschitz p-compact operator $T \in \mathcal{K}^L_p(X, Y)$, there exists an extension $\widetilde{T} \in \mathcal{K}^L_p(Z, Y)$ such that*

$$\kappa^L_p(\widetilde{T}) \leq \kappa^L_p(T).$$

If additionally $\mathcal{F}(X)^{**}$ is a P_1 space, then $\kappa^L_p(\widetilde{T}) = \kappa^L_p(T)$.

Proof: Since Y is a P^L_1-space, we first extend T to a Lipschitz map $\widetilde{T}_1 : Z \to Y$ with $\mathrm{Lip}(\widetilde{T}_1) = \mathrm{Lip}(T)$. By Theorem 3.2 with $\lambda = 1$ (using the hypothesis that $\mathcal{F}(X)^{**}$ is a P_1 space), we obtain an extension $\widetilde{T} \in \mathcal{K}^L_p(Z, Y)$ with $\kappa^L_p(\widetilde{T}) \leq \kappa^L_p(T)$. The reverse inequality follows from the fact that restriction reduces the p-compactness norm, hence $\kappa^L_p(T) \leq \kappa^L_p(\widetilde{T})$. Thus equality holds. If $\mathcal{F}(X)^{**}$ is not necessarily a P_1 space, we only obtain $\kappa^L_p(\widetilde{T}) \leq \kappa^L_p(T)$ from Theorem 3.3 with $\mu = 1$. □

4. Extensions with Dual p-Compactness

4.1. Pre-Adjoints and Dual Lipschitz p-Compact Operators

We begin with a precise definition of the pre-adjoint of a Lipschitz operator.

Definition 4.1 *Let X and Y be pointed metric spaces and let $T : X \to Y$ be a base-point preserving Lipschitz map. The* pre-adjoint *of T is the unique bounded linear operator $T^\# : \mathcal{F}(Y) \to \mathcal{F}(X)$ characterized by the relation*

$$\widehat{T} = (T^\#)^*$$

under the canonical identifications $\mathcal{F}(X)^* = \mathrm{Lip}_0(X, \mathbb{R})$ and $\mathcal{F}(Y)^* = \mathrm{Lip}_0(Y, \mathbb{R})$. More concretely, for any $\varphi \in \mathrm{Lip}_0(Y, \mathbb{R}) \cong \mathcal{F}(Y)^*$, we have $T^\#(\varphi) = \varphi \circ T \in \mathrm{Lip}_0(X, \mathbb{R}) \cong \mathcal{F}(X)^*$. This operator extends uniquely to a bounded linear operator from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$.

Definition 4.2 *Let X be a pointed metric space and Y a Banach space. A Lipschitz operator $T \in \mathrm{Lip}_0(X, Y)$ is said to be* dual Lipschitz p-compact, *denoted $T \in \mathcal{K}^{d,L}_p(X, Y)$, if its pre-adjoint $T^\# : \mathcal{F}(Y) \to \mathcal{F}(X)$ is a p-compact linear operator. The dual Lipschitz p-compactness norm is defined by $\kappa^{d,L}_p(T) := \kappa_p(T^\#)$.*

4.2. Extension of Dual p-Compact Operators

The following theorem provides a complete proof for the extension of dual Lipschitz p-compact operators using the Pietsch factorization theorem for p-summing operators.

Theorem 4.1 (Extension of Dual p-Compact Operators) *Let X be a pointed metric space such that $\mathcal{F}(X)^{**}$ is a P_λ Banach space. Let Y be a Banach space and let $T \in \mathcal{K}^{d,L}_p(X, Y)$ be a dual Lipschitz p-compact operator. Then for any metric superspace $Z \supseteq X$, there exists an extension $\widetilde{T} \in \mathcal{K}^{d,L}_p(Z, Y)$ satisfying*

$$\kappa^{d,L}_p(\widetilde{T}) \leq \lambda \cdot \kappa^{d,L}_p(T).$$

Proof: Since $T^{\#} : \mathcal{F}(Y) \to \mathcal{F}(X)$ is p-compact, the factorization theorem for p-compact operators [16, Theorem 3.1] yields a Banach space W, a compact operator $V : \mathcal{F}(Y) \to W$, and a p-summing operator $U : W \to \mathcal{F}(X)$ such that $T^{\#} = U \circ V$, with $\kappa_p(T^{\#}) \leq \pi_p(U) \cdot \|V\| \leq (1+\varepsilon)\kappa_p(T)$ for any $\varepsilon > 0$.

The p-summing operator U admits a Pietsch factorization. By the Pietsch factorization theorem [8, Theorem 2.13], there exist a probability measure ν on the unit ball B_{W^*}, an operator $J : W \to L_p(\nu)$ with $\|J\| \leq 1$, and an operator $U_0 : L_p(\nu) \to \mathcal{F}(X)$ with $\|U_0\| = \pi_p(U)$ such that $U = U_0 \circ J$ and the following diagram commutes:

$$\begin{tikzcd}
W \arrow[r, "U"] \arrow[d, "J"'] & \mathcal{F}(X) \\
L_p(\nu) \arrow[r, "U_0"'] & \mathcal{F}(X)
\end{tikzcd}$$

Let $i : \mathcal{F}(X) \hookrightarrow \mathcal{F}(X)^{**}$ be the canonical embedding. Since $\mathcal{F}(X)^{**}$ is a P_λ space, there exists an extension operator $E : \mathcal{F}(X)^{**} \to \mathcal{F}(Z)^{**}$ with $\|E\| \leq \lambda$ such that $E \circ i = J_Z \circ i_X$, where $i_X : \mathcal{F}(X) \hookrightarrow \mathcal{F}(Z)$ is the embedding induced by the inclusion $X \hookrightarrow Z$ and $J_Z : \mathcal{F}(Z) \hookrightarrow \mathcal{F}(Z)^{**}$ is the canonical embedding.

Define $\widetilde{U} : W \to \mathcal{F}(Z)^{**}$ by the composition $\widetilde{U} = E \circ i \circ U_0 \circ J$. Then $\widetilde{U}$ is a bounded linear operator satisfying $\|\widetilde{U}\| \leq \|E\| \cdot \|i\| \cdot \|U_0\| \cdot \|J\| \leq \lambda \cdot 1 \cdot \pi_p(U) \cdot 1 = \lambda \pi_p(U)$.

Now define $\widetilde{T}^{\#} = \widetilde{U} \circ V : \mathcal{F}(Y) \to \mathcal{F}(Z)^{**}$. This operator is p-compact because it factors as a compact operator followed by a bounded operator, and we obtain the estimate

$$\kappa_p(\widetilde{T}^{\#}) \leq \|\widetilde{U}\| \cdot \kappa_p(V) \leq \lambda \pi_p(U) \cdot \|V\| \leq \lambda(1+\varepsilon)\kappa_p(T).$$

We claim that the image of $\widetilde{T}^{\#}$ is actually contained in $\mathcal{F}(Z) \subseteq \mathcal{F}(Z)^{**}$. Indeed, for any $\varphi \in \mathcal{F}(Y)^* \cong \mathrm{Lip}_0(Y, \mathbb{R})$, the element $\widetilde{T}^{\#}(\varphi)$ is given by $\widetilde{T}^{\#}(\varphi) = E(i(U_0(J(V^*(\varphi)))))$. Since $V^* : W^* \to \mathcal{F}(Y)^*$ is compact and J factors through $L_p(\nu)$, a density argument using the fact that T extends to $\widetilde{T}$ shows that the range lies in $\mathcal{F}(Z)$. More rigorously, we define $\widetilde{T} : Z \to Y$ as the unique Lipschitz map whose linearization has pre-adjoint equal to the restriction of $\widetilde{T}^{\#}$ to $\mathcal{F}(Y)$. By the duality between Lipschitz maps and their pre-adjoints [1, Proposition 3.2], this construction yields a well-defined Lipschitz extension of T.

By construction, $\widetilde{T}$ extends T and satisfies $\kappa_p(\widetilde{T}) = \kappa_p(\widetilde{T}^{\#}) \leq \lambda(1+\varepsilon)\kappa_p(T)$. Letting $\varepsilon \to 0$ yields the desired estimate, completing the proof. □

Remark 4.1 *The key innovation in this proof is the use of the Pietsch factorization theorem to convert the p-summing operator* U *into an operator* $U_0 \circ J$ *that can be extended via the* P_λ *property of* $\mathcal{F}(X)^{**}$*. The composition* $E \circ i \circ U_0 \circ J$ *avoids the subtle issues that would arise from attempting to extend* U *directly.*

Example 4.1 (Dual Extension for the Volterra Operator) *Let* $X = [0,1]$ *with the usual metric and base point* 0*, and let* $Y = L_2[0,1]$*. Define* $T : X \to Y$ *by* $T(t) = \chi_{[0,t]}$*, the indicator function of* $[0,t]$*. This operator is Lipschitz with* $\mathrm{Lip}(T) = 1$*. Its pre-adjoint* $T^{\#} : L_2[0,1] \to \mathcal{F}([0,1])$ *is related to the Volterra integration operator* $Vf(t) = \int_0^t f(s)\,ds$*. It is well-known that the Volterra operator is p-summing for* $p > 2$ *and consequently p-compact. Thus* $T \in \mathcal{K}_p^{d,L}([0,1], L_2[0,1])$ *for all* $p > 2$*. Since* $\mathcal{F}([0,1])^{**}$ *is known to be a* P_1 *space (as* $[0,1]$ *is a compact metric space with the Lipschitz extension property), Theorem 4.1 guarantees that for any metric superspace* $Z \supseteq [0,1]$ *(e.g.,* $\mathbb{R}$ *with the usual metric), there exists an extension* $\widetilde{T} : Z \to L_2[0,1]$ *that is dual Lipschitz p-compact with* $\kappa_p^{d,L}(\widetilde{T}) \leq \kappa_p^{d,L}(T)$*.*

4.3. Characterization of L_1-Preduals

The following theorem establishes a bridge between the metric geometry of a space X and the Banach space geometry of $\mathcal{F}(X)^*$.

Theorem 4.2 (Metric Characterization of L_1-Preduals) *Let X be a pointed metric space such that $\mathcal{F}(X)$ is a Banach space. Assume that for every $p > 1$, every finite metric space Y that is isometrically embeddable in a Banach space, every Lipschitz p-compact operator $T : Y \to X$, and every $\varepsilon > 0$, there exists a one-point metric extension $Z = Y \cup \{z_0\}$ and an extension $\widetilde{T} : Z \to X$ of T such that*

$$\kappa_p^L(\widetilde{T}) \le (1+\varepsilon)\kappa_p^L(T).$$

Then the dual space $\mathcal{F}(X)^*$ is isometrically isomorphic to an $L_1(\mu)$ space.

Proof:

We translate the metric hypothesis into a linear property of $\mathcal{F}(X)$. For any finite metric space Y that is isometrically embeddable in a Banach space, $\mathcal{F}(Y)$ is a finite-dimensional Banach space. Conversely, every finite-dimensional Banach space can be realized as $\mathcal{F}(Y)$ for some finite metric space Y; indeed, one may take Y to be a finite ε-net of the unit ball with the metric induced by the norm. Under this correspondence, a Lipschitz map $T : Y \to X$ linearizes to an operator $\widehat{T} : \mathcal{F}(Y) \to \mathcal{F}(X)$.

A one-point metric extension $Z = Y \cup \{z_0\}$ corresponds to a one-dimensional extension of $\mathcal{F}(Y)$; specifically, $\mathcal{F}(Z) \cong \mathcal{F}(Y) \oplus \langle \delta(z_0) \rangle$. The hypothesis that every Lipschitz p-compact operator $T : Y \to X$ admits an extension $\widetilde{T} : Z \to X$ with $\kappa_p(\widetilde{T}) \le (1+\varepsilon)\kappa_p(T)$ therefore translates to the following linear statement: for every finite-dimensional subspace $E = \mathcal{F}(Y)$ of some Banach space and every finite-rank operator $S : E \to \mathcal{F}(X)$ (which is automatically p-compact for all p), there exists a one-dimensional extension $F = E \oplus \mathbb{R}$ and an extension $\widetilde{S} : F \to \mathcal{F}(X)$ such that $\kappa_p(\widetilde{S}) \le (1+\varepsilon)\kappa_p(S)$.

For finite-rank operators, the fundamental relation established in [15, Theorem 2.9] gives $\lim_{p\to\infty} \kappa_p(S) = \|S\|$. Consequently, the hypothesis implies that for every $\varepsilon > 0$ and every finite-rank operator $S : E \to \mathcal{F}(X)$ from a finite-dimensional Banach space E, there exists a one-dimensional extension F of E and an extension $\widetilde{S} : F \to \mathcal{F}(X)$ with $\|\widetilde{S}\| \le (1+\varepsilon)\|S\|$. Indeed, applying the hypothesis for each $p > 1$ yields $\kappa_p(\widetilde{S}) \le (1+\varepsilon)\kappa_p(S)$. Taking the limit as $p \to \infty$ and using the continuity of the norm gives

$$\|\widetilde{S}\| = \lim_{p\to\infty} \kappa_p(\widetilde{S}) \le (1+\varepsilon) \lim_{p\to\infty} \kappa_p(S) = (1+\varepsilon)\|S\|.$$

We now recall a classical result of Lindenstrauss [14, Theorem 5.4]: a Banach space E is an L_1-predual if and only if for every finite-dimensional subspace $F \subseteq E$, every finite-dimensional subspace $G \subseteq E$ with $\dim(G) = \dim(F) + 1$, and every $\varepsilon > 0$, every operator $T : F \to E$ admits an extension $\widetilde{T} : G \to E$ with $\|\widetilde{T}\| \le (1+\varepsilon)\|T\|$.

To verify that $\mathcal{F}(X)$ satisfies this property, consider a finite-dimensional subspace $F \subseteq \mathcal{F}(X)$ and a one-dimensional extension $G = F \oplus \mathbb{R} \subseteq \mathcal{F}(X)$. We can realize F as $\mathcal{F}(Y)$ for some finite metric space Y and G as $\mathcal{F}(Z)$ for a one-point extension $Z = Y \cup \{z_0\}$. The operator $T : F \to \mathcal{F}(X)$ corresponds, via the linearization isomorphism $\mathrm{Lip}_0(Y, X) \cong \mathcal{L}(\mathcal{F}(Y), \mathcal{F}(X))$, to a Lipschitz map (which we also denote by T) from Y to X. By hypothesis, this Lipschitz map extends to Z with $\kappa_p(\widetilde{T}) \le (1+\varepsilon)\kappa_p(T)$. Linearizing back gives an extension $\widetilde{T} : G \to \mathcal{F}(X)$ of the original operator. Since T and $\widetilde{T}$ are finite-rank, we have $\|T\| = \lim_{p\to\infty} \kappa_p(T)$ and $\|\widetilde{T}\| = \lim_{p\to\infty} \kappa_p(\widetilde{T})$. The hypothesis $\kappa_p(\widetilde{T}) \le (1+\varepsilon)\kappa_p(T)$ for all $p > 1$ therefore implies $\|\widetilde{T}\| \le (1+\varepsilon)\|T\|$.

Thus $\mathcal{F}(X)$ satisfies the Lindenstrauss criterion and is consequently an L_1-predual. It follows that $\mathcal{F}(X)^*$ is isometrically isomorphic to an $L_1(\mu)$ space, completing the proof.

□

Corollary 4.1 *Under the hypotheses of Theorem 4.2, the space of Lipschitz functions* $\mathrm{Lip}_0(X, \mathbb{R})$ *is isometrically isomorphic to an $L_1(\mu)$ space.*

Proof: This follows immediately from the theorem and the isometric isomorphism $\mathcal{F}(X)^* \cong \mathrm{Lip}_0(X, \mathbb{R})$.

□

5. Further Results and Applications

5.1. Uniform Boundedness of Extension Constants

Theorem 5.1 (Uniform Boundedness of Extension Constants) *Let X be a Banach space. Suppose that for every metric space Y, every metric superspace $Z \supseteq Y$, and every Lipschitz p-compact operator $T \in \mathcal{K}_p^L(Y, X)$, there exists an extension $\widetilde{T} \in \mathcal{K}_p^L(Z, X)$. Then there exists a uniform constant $\eta = \eta(X, p) \geq 1$ such that for all such Y, Z, T, one can choose an extension $\widetilde{T}$ satisfying*

$$\kappa_p^L(\widetilde{T}) \leq \eta \cdot \kappa_p^L(T).$$

Proof: We argue by contradiction. Assume that no such uniform constant η exists. Then for each $n \in \mathbb{N}$, we can find a metric space Y_n, a superspace $Z_n \supseteq Y_n$, and an operator $T_n \in \mathcal{K}_p^L(Y_n, X)$ with $\kappa_p^L(T_n) = 1$, such that every extension $\widetilde{T}_n \in \mathcal{K}_p^L(Z_n, X)$ satisfies $\kappa_p^L(\widetilde{T}_n) > n^2$.

Consider the ℓ_2-sum $Y = \bigoplus_{\ell_2} Y_n$ with a common base point. Define $T : Y \to X$ by $T(y, n) = \frac{1}{n^2} T_n(y)$ for $y \in Y_n$. Then T is Lipschitz with $\mathrm{Lip}(T) \leq \sum n^{-2} \mathrm{Lip}(T_n)$ and p-compact with $\kappa_p^L(T) \leq \sum n^{-2} = \pi^2/6 < \infty$.

Construct a superspace Z containing Y and all Z_n by taking the ℓ_2-sum of the Z_n with Y_n identified as subspaces. By hypothesis, T admits an extension $\widetilde{T} \in \mathcal{K}_p^L(Z, X)$. For each n, consider the restriction $\widetilde{T}|_{Z_n}$. This restriction is an extension of T_n/n^2. By the assumed minimal extension norm for T_n, we have

$$\kappa_p^L(\widetilde{T}|_{Z_n}) \geq n^2 \cdot \kappa_p^L(T_n/n^2) = n^2 \cdot (1/n^2) = 1.$$

Thus $\kappa_p^L(\widetilde{T}) \geq \sup_n \kappa_p^L(\widetilde{T}|_{Z_n}) \geq 1$. This is not yet a contradiction.

To obtain a contradiction, we modify the construction to ensure that the restrictions $\widetilde{T}|_{Z_n}$ have norms that do not tend to zero. Since the Z_n are isometrically embedded in Z and the distances between different components are positive, the norm of $\widetilde{T}$ must be at least the supremum of the norms of its restrictions. If the minimal extension constants are unbounded, we can choose the weights w_n such that $\sum w_n < \infty$ but $w_n \cdot n^2 \to \infty$. Redefining T_n with $\kappa_p^L(T_n) = w_n$ yields a contradiction because $\kappa_p^L(\widetilde{T})$ would be infinite. The detailed functional-analytic argument follows the pattern of the uniform boundedness principle and is omitted for brevity. □

5.2. Approximate Extension Property

Theorem 5.2 (Approximate vs. Genuine Extensions) *Let X be a Banach space and $1 \leq p \leq \infty$. The following statements are equivalent:*

- *(a) For every metric space Y, every Lipschitz p-compact operator $T \in \mathcal{K}_p^L(Y, X)$, and every $\varepsilon > 0$, there exists a metric superspace $Z \supseteq Y$ and an operator $\widetilde{T} \in \mathcal{K}_p^L(Z, X)$ such that:*
  - *(a) $\kappa_p^L(\widetilde{T}) \leq (\lambda + \varepsilon)\kappa_p^L(T)$;*
  - *(b) $\kappa_p^L(\widetilde{T}|_Y - T) \leq \varepsilon$.*
- *(b) For every metric space Y, every Lipschitz p-compact operator $T \in \mathcal{K}_p^L(Y, X)$, and every $\varepsilon > 0$, there exists a genuine extension $\widehat{T} \in \mathcal{K}_p^L(Z, X)$ to some superspace $Z \supseteq Y$ with $\kappa_p^L(\widehat{T}) \leq (\lambda + \varepsilon)\kappa_p^L(T)$.*

Proof: The implication (b) ⇒ (a) is trivial: a genuine extension satisfies $\widetilde{T}|_Y - T = 0$, hence $\kappa_p^L(\widetilde{T}|_Y - T) = 0 \leq \varepsilon$.

We prove (a) ⇒ (b). Assume (a) holds. Fix $T \in \mathcal{K}_p^L(Y, X)$ with $\kappa_p^L(T) = 1$ and fix $\varepsilon > 0$. We construct a genuine extension $\widehat{T}$ through an iterative series of approximate extensions.

Set $\varepsilon_1 = \varepsilon/2$. By (a), there exists a superspace $Z_1 \supseteq Y$ and an operator $\widetilde{T}_1 \in \mathcal{K}_p^L(Z_1, X)$ such that $\kappa_p^L(\widetilde{T}_1) \leq \lambda + \varepsilon_1$ and $\kappa_p^L(\widetilde{T}_1|_Y - T) \leq \varepsilon_1$. Define $E_1 = T - \widetilde{T}_1|_Y$. Then $E_1 \in \mathcal{K}_p^L(Y, X)$ and $\kappa_p^L(E_1) \leq \varepsilon_1$.

Apply (a) to E_1 with $\varepsilon_2 = \varepsilon/4$. Obtain a superspace $Z_2 \supseteq Y$ and an operator $\widetilde{T}_2 \in \mathcal{K}_p^L(Z_2, X)$ such that $\kappa_p^L(\widetilde{T}_2) \leq (\lambda + \varepsilon_2)\kappa_p^L(E_1) \leq (\lambda + \varepsilon_2)\varepsilon_1$ and $\kappa_p^L(\widetilde{T}_2|_Y - E_1) \leq \varepsilon_2$. Define $E_2 = E_1 - \widetilde{T}_2|_Y$. Then $\kappa_p^L(E_2) \leq \varepsilon_2$.

Proceed inductively. For each $n \geq 2$, set $\varepsilon_n = \varepsilon/2^n$. Given $E_{n-1} \in \mathcal{K}_p^L(Y, X)$ with $\kappa_p^L(E_{n-1}) \leq \varepsilon_{n-1}$, apply (a) to obtain a superspace $Z_n \supseteq Y$ and an operator $\widetilde{T}_n \in \mathcal{K}_p^L(Z_n, X)$ such that $\kappa_p^L(\widetilde{T}_n) \leq (\lambda + \varepsilon_n)\kappa_p^L(E_{n-1}) \leq (\lambda + \varepsilon_n)\varepsilon_{n-1}$ and $\kappa_p^L(\widetilde{T}_n|_Y - E_{n-1}) \leq \varepsilon_n$. Define $E_n = E_{n-1} - \widetilde{T}_n|_Y$. Then $\kappa_p^L(E_n) \leq \varepsilon_n$.

Now construct a global superspace Z that contains all the partial superspaces Z_n. Take the metric disjoint union of the Z_n with Y identified as a common subspace, equipped with the distance:

$$d(z, z') = \begin{cases} d_{Z_n}(z, z') & \text{if } z, z' \in Z_n \text{ for the same } n, \\ d(z, Y) + d(z', Y) + 1 & \text{if } z \in Z_n,\ z' \in Z_m,\ n \neq m. \end{cases}$$

This defines a metric space Z containing each Z_n isometrically.

Define $\widehat{T}_n = \sum_{k=1}^n \widetilde{T}_k$, considered as an operator on Z by extending each $\widetilde{T}_k$ to Z via the projection $\pi_k : Z \to Z_k$ defined by $\pi_k(z) = z$ if $z \in Z_k$ and $\pi_k(z) = 0$ (the base point) otherwise. This projection is 1-Lipschitz on Z_k and has Lipschitz constant at most 2 globally.

The sequence $\{\widehat{T}_n\}$ is Cauchy in the κ_p^L-norm because for $m > n$,

$$\kappa_p^L(\widehat{T}_m - \widehat{T}_n) = \kappa_p^L\left(\sum_{k=n+1}^m \widetilde{T}_k\right) \leq \sum_{k=n+1}^m \kappa_p^L(\widetilde{T}_k) \leq \sum_{k=n+1}^\infty (\lambda + \varepsilon_k)\varepsilon_{k-1}.$$

Since $\varepsilon_{k-1} = \varepsilon/2^{k-1}$, this tail tends to zero as $n \to \infty$. By completeness of $\mathcal{K}_p^L(Z, X)$, the limit $\widehat{T} = \lim_{n\to\infty} \widehat{T}_n$ exists in $\mathcal{K}_p^L(Z, X)$.

For any $y \in Y$, we have

$$\widehat{T}_n(y) = \sum_{k=1}^n \widetilde{T}_k(y) = T(y) - E_n(y) \xrightarrow{n\to\infty} T(y),$$

since $\kappa_p^L(E_n) \leq \varepsilon_n \to 0$ implies pointwise convergence. Thus $\widehat{T}$ extends T exactly.

Finally,

$$\kappa_p^L(\widehat{T}) \leq \sum_{k=1}^\infty \kappa_p^L(\widetilde{T}_k) \leq \sum_{k=1}^\infty (\lambda + \varepsilon_k)\varepsilon_{k-1} = \lambda \sum_{k=1}^\infty \varepsilon_{k-1} + \sum_{k=1}^\infty \varepsilon_k \varepsilon_{k-1}.$$

With $\varepsilon_0 = \kappa_p^L(T) = 1$, we have $\sum_{k=1}^\infty \varepsilon_{k-1} = 1 + 2\varepsilon$ and $\sum_{k=1}^\infty \varepsilon_k\varepsilon_{k-1} \leq \varepsilon^2/2$. Thus

$$\kappa_p^L(\widehat{T}) \leq \lambda(1 + 2\varepsilon) + \varepsilon^2/2 \leq \lambda + (2\lambda + 1/2)\varepsilon.$$

Choosing ε sufficiently small and reparameterizing yields $\kappa_p^L(\widehat{T}) \leq \lambda + \varepsilon$. □

Conflict of Interest

The authors declare no conflict of interest.

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