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Solution of Non Linear Matrix Equation Using θ-Hyperbolic Sine Distance Functions

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ABSTRACT: In this paper we introduce Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction and discuss the existence and uniqueness of fixed point of a mapping satisfying Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction in the setting of orbitally complete metric spaces. We furnish the result by suitable example. Further we apply the results to find solution of non liner matrix equation.

Key Words: Metric spaces, hyperbolic functions, θ-hyperbolic sine distance functions, fixed point.

Contents

1. Introduction and Preliminaries

A distance function or a metric on a set $\mathcal{Q}$ is defined as a mapping $\rho : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ satisfying the following axioms:

(i) $\rho(u, v) = 0$ if and only if $u = v$,

(ii) $\rho(u, v) = \rho(v, u)$,

(iii) $\rho(u, v) = \rho(u, w) + \rho(w, v)$ (triangle inequality)

for every $u, v, w \in \mathcal{Q}$. Along with the metric ρ defined on $\mathcal{Q}$, the pair $(\mathcal{Q}, \rho)$ forms a metric space. Metric spaces are foundational in various fields of mathematics and have numerous applications in computer science, data analysis, optimization, neuroscience, image restoration, and medical image classification and more (see, e.g., [1–3]).
However, there are areas of applications, where one of the axioms (i)–(iii) is not suitable. Like in denotational semantic, the metric space topology is not appropriate (see, e.g., [4]). Motivated by the fact many researchers proposed various concepts generalizing the notion of metric spaces. For example, Smyth [5] generalized the metric notion by dropping the symmetry condition (ii). Matthews [4] provided the notion of partial metric spaces, where the distance of an element to itself not necessarily be zero. Notion of G-metric spaces, where $G : Z \times Z \times Z \to [0, \infty)$ is a mapping satisfying certain axioms, is introduced by Mustafa and Sims [8]. The literature includes many other interesting extensions of the notion of metric spaces.
The importance of Fixed point theory lies in some of the most important topics in pure and applied mathematics. Indeed, the problem of existence of solutions to a nonlinear problem is often reduced to a fixed point problem. The literature also includes several generalizations and extensions of very well celebrated Banach's contraction principle. These generalizations and extensions can be classified into two major categories. The one which is dealing with weakening the contractive nature of the mapping such as Kannan's contraction [9], quasi-contractions [10], Meir-Keeler-type contractions [11,12], contractions involving simulation functions [13,14],(α, ψ) -contractions [15,16] and many other. The other one category is dealing with the study of fixed points when the space under study is equipped with a generalized metric,

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see, e.g., the above mentioned references related to generalized metric spaces and [17]. In addition, some of the recent results of fixed point in Hilbert spaces can be found be in [18,19]. Some related studies and applications can be referred from [20,21,22,23].
On the other hand, hyperbolic functions has many applications in mathematics, physics, chemistry, and engineering. Very recently, Mohamed Jleli and Bessem Samet[6] introduced the notion of θ-hyperbolic sine distance functions associated to a certain metric, where $\theta : [0,\infty) \to [0,\infty)$ satisfies the condition

$$\theta(r) \geq cr^{\tau}$$

for all $r \geq 0$, for some constants $c, \tau > 0$. Given a metric space $(\mathcal{Q}, \delta)$, the mapping $\delta_\theta : \mathcal{Q} \times \mathcal{Q} \to [0,\infty)$ defined by $\delta_\theta(u,v) = \theta(sinh(\delta(u,v)))$ for all $u, v \in \mathcal{Q}$, is called the θ-hyperbolic sine distance function associated to the metric δ.
Inspired by this study, we define Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction in the present paper. The motive behind defining Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction lies in the exploration of fixed point theorems and the extension of existing results in the field of fixed point theory. This definition aims to expand the theoretical framework and provide a basis for further research and applications in various mathematical and applied contexts. The concept of Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction is particularly significant in the study of metric spaces and contraction mappings, as it allows for the existence of fixed points and the development of related fixed point theorems.
Our study in this paper is to obtain a unique fixed point for the single valued mapping in the setting of orbitally complete metric space. We use Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction in order to obtain the desired solution.
The following has been established by Mohamed Jleli and Bessem Samet[6] for θ-hyperbolic sine distance function.

Definition 1.1 [6] For all $\tau > 0$, we denote by θ_τ the set of functions $\theta : [0,\infty) \to [0,\infty)$ satisfying the condition

$$\theta(r) \geq cr^{\tau} \tag{1.1}$$

for all $r \geq 0$, where $c > 0$ is a constant.

Definition 1.2 [6] Let $(\mathcal{Q}, \delta)$ be a metric space. For all $\tau > 0$ and $\theta \in \Theta_\tau$, the mapping $\delta_\theta : \mathcal{Q} \times \mathcal{Q} \to [0,\infty)$ is defined by

$$\delta_\theta(u,v) = \theta(\sinh(\delta(u,v))), u, v \in \mathcal{Q}$$

where sinh is the hyperbolic sine function defined by

$$\sinh r = \frac{e^r + e^{-r}}{2}, \ r \in R$$

The mapping δ_θ is called the θ-hyperbolic sine distance function associated to the metric δ.
We consider the following:

Definition 1.3 Let $\mathcal{Z}$ be the family of all mappings $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that

(i) $\zeta(0,0) = 0$;

(ii) $\zeta(\mu,\nu) < \nu - \mu \ \forall \mu,\nu > 0$;

(iii) for any sequence $\{\mu_n\}, \{\nu_n\} \subset [0,\infty\}$

$$\lim_{n\to\infty} \sup \zeta(\mu_n,\nu_n) < 0, whenever \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \nu_n > 0.$$

We say that $\zeta \in \mathcal{Z}$ is a simulation function.

Definition 1.4 [7] A mapping $\mathfrak{T} : X \to X$ is called Ciric-type(I) $\mathcal{Z}$- contraction with respect to $\zeta \in \mathcal{Z}$ if

$$\zeta(m(a, x, y)), d(x, y)) \geq 0$$

for all $x, y \in X$ where

$$m(a, x, y) = d(\mathfrak{T}x, \mathfrak{T}y) - \min d(x, \mathfrak{T}y), d(y, \mathfrak{T}x), d(x, \mathfrak{T}x), d(y, \mathfrak{T}y)$$

in the setting of generating space of quasi metric family.

Following propositions have also been established by[6].

Proposition 1.1 *Let $(\mathcal{Q}, \delta)$ be a metric space and $\theta \in \Theta_\tau$ for some $\tau > 0$. Then, for all $u, v \in \mathcal{Q}$, we have*

(i) $\delta_\theta(u, v) = 0 \implies u = v$;

(ii) *If* $\theta(0) = 0$, *then* $\delta_\theta(u, u) = 0$;

(iii) $\delta_\theta(u, v) = \delta_\theta(v, u)$.

They emphasis on the fact that a θ-hyperbolic sine distance function is not necessarily a metric, even if $\theta(0) = 0$.

Proposition 1.2 *Let $(\mathcal{Q}, \delta)$ be a metric space.*

(i) Let δ_θ be the θ-hyperbolic sine distance function associated to δ, where $\theta \in \Theta_\tau$ for some $\tau > 0$. Then, for all $l > 0$, we have $l\delta_\theta = \delta_{\theta_l}$, where $\theta_l = l\theta$.

(ii) Let $\theta_1, \theta_2 \in \Theta_\tau$ for some $\tau > 0$. Then

$$\delta\theta_1 + \delta\theta_2 = \delta\theta$$

where $\theta = \theta_1 + \theta_2$.

Proposition 1.3 *Let $\theta \in \Theta_\tau$ for some $\tau > 0$. Assume that the following conditions hold:*

(i) $\theta(0) = 0$.

(ii) There exists $r^* > 0$ such that $\theta(\sinh r^*) = r^*$

Then, for every nonempty set $\mathcal{Q}$, there exists a metric δ on $\mathcal{Q}$ such that the θ-hyperbolic sine distance function associated to δ coincides with δ, i.e., $\delta_\theta = \delta$.

We will furnish the results with examples. The obtained results are also applied to solve the non-linear Matrix equation.

2. Main results

On the lines of Definition 1.5, we rewrite the following definition of Ciric-type(I) θ-hyperbolic $\mathcal{Z}$-contraction for the mapping under study.

Definition 2.1 A mapping $\mathfrak{T} : X \to X$ is called Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction with respect to $\zeta \in \mathcal{Z}$ if

$$\zeta(\mathcal{M}(x, y), \delta_\theta(x, y)) \geq 0 \tag{2.1}$$

for all $x, y \in X$ where

$$\mathcal{M}(x, y) = k\delta_\theta(\mathfrak{T}x, \mathfrak{T}y) - \min\{\delta_\theta(x, \mathfrak{T}y), \delta_\theta(y, \mathfrak{T}x), \delta_\theta(x, \mathfrak{T}x), \delta_\theta(y, \mathfrak{T}y)\}$$

for $k \in (1, \infty)$

Theorem 2.1 *Let $(\mathcal{Q}, \delta)$ be a metric space and $\theta \in \Theta_\tau$ for some $\tau > 0$ and $\theta(0) = 0$. Let $\mathfrak{T} : \mathcal{Q} \to \mathcal{Q}$ be a mapping such that*

(i) $\mathfrak{T}$ is a Ciric´-type (I) θ-hyperbolic $\mathcal{Z}-$contraction on $\mathcal{Q}$.

(ii) $\mathcal{Q}$ is $\mathfrak{T}$- orbitally complete.

Then $\mathfrak{T}$ possesses one and only one fixed point. Moreover, for all $\mu_0 \in \mathcal{Q}$ the sequence $\{\mathfrak{T}^n \mu_0\}$ converges to this unique fixed point.

Proof: Start with defining a sequence μ_n in $\mathcal{Q}$, where $\mu_n = \mathfrak{T}^n \mu_0 \forall n \in \mathbb{N}$.

Without the loss of generality, we can assume that $\mu_n \neq \mu_{n+1}$ for all $n \in \mathbb{N}$. Since $\mathfrak{T}$ is a Ciric´-type (I) θ-hyperbolic $\mathcal{Z}$ - contraction with respect to $\zeta \in \mathcal{Z}$ for any $n \in \mathbb{N}$, putting $x = \mu_{n-1}$ and $y = \mu_n$ in (2), we get

$$\begin{aligned}
&M(\mu_{n-1}, \mu_n)\\
&= k\delta_\theta(\mathfrak{T}\mu_{n-1}, \mathfrak{T}\mu_n) - \min\{\delta_\theta(\mu_{n-1}, \mathfrak{T}\mu_n), \delta_\theta(\mu_n, \mathfrak{T}\mu_{n-1}), (\delta_\theta\mu_{n-1}, \mathfrak{T}u_{n-1}), \delta_\theta(\mu_n, \mathfrak{T}\mu_n)\}\\
&= k\delta_\theta(\mathfrak{T}\mu_{n-1}, \mathfrak{T}\mu_n) - \min\{\delta_\theta(\mu_{n-1}, \mathfrak{T}\mu_n), 0, \delta_\theta(\mathfrak{T}\mu_{n-1}, \mathfrak{T}\mu_{n-1}), \delta_\theta(\mu_n, \mathfrak{T}\mu_n)\}\\
&= k\delta_\theta(\mathfrak{T}\mu_{n-1}, \mathfrak{T}\mu_n)\\
&= k\delta_\theta(\mu_n, \mu_{n+1})
\end{aligned}$$

It follows that

$$\begin{aligned}
0 &\le \zeta(M(\mu_{n-1}, \mu_n), \delta_\theta(\mu_{n-1}, \mu_n))\\
&\le \zeta(k\delta_\theta(\mu_n, \mu_{n+1}), \delta_\theta(\mu_{n-1}, \mu_n)\\
&< \delta_\theta(\mu_{n-1}, \mu_n) - k\delta_\theta(\mu_n, \mu_{n+1})
\end{aligned}$$

implying

$$\delta_\theta(\mu_n, \mu_{n+1}) < \frac{1}{k}\delta_\theta(\mu_{n-1}, \mu_n) \tag{2.2}$$

which is equivalent to

$$\theta(\sinh(\delta(\mu_n, \mu_{n+1}))) < \frac{1}{k}\delta_\theta(\mu_{n-1}, \mu_n), n \ge 0$$

Also we have

$$\theta(\sinh(\delta(\mu_n, \mu_{n+1}))) \ge c[\sinh(\delta(\mu_n, \mu_{n+1}))]^\tau, n \ge 0$$

Furthermore, making use of the elementary inequality

$$\frac{\sinh r}{r} > 1, r > 0$$

we get

$$c[\sinh(\delta(\mu_n, \mu_{n+1}))]^\tau > c\delta^\tau(\mu_n, \mu_{n+1}), n \ge 0.$$

Thus we obtain

$$c\delta^\tau(\mu_n, \mu_{n+1}) \le \frac{1}{k}\delta_\theta(\mu_{n-1}, \mu_n), n \ge 0$$

or

$$\delta(\mu_n, \mu_{n+1}) \le [\frac{\delta_\theta(\mu_{n-1}, \mu_n)}{ck}]^{\frac{1}{\tau}}, n \ge 0$$

On repeated application of (2.2), we get

$$\delta(\mu_n, \mu_{n+1}) \le [\frac{\delta_\theta(\mu_0, \mu_1)}{c}]^{\frac{1}{\tau}}(\frac{1}{k})^{\frac{n}{\tau}}, n \ge 0 \tag{2.3}$$

Then, using (2.3) and the triangle inequality, we obtain that for all $n \geq 0$ and $m \geq 1$,

$$\begin{aligned}\delta(\mu_n, \mu_{n+m}) &\leq \delta(\mu_n, \mu_{n+1}) + ... + \delta(\mu_{n+m-1}, \mu_{n+m})\\ &\leq [\frac{\delta_\theta(\mu_0, \mu_1)}{c}]^{\frac{1}{\tau}}((\frac{1}{k})^{\frac{n}{\tau}} + ...(\frac{1}{k})^{\frac{n+m-1}{\tau}})\\ &= [\frac{\delta_\theta(\mu_0, \mu_1)}{c}]^{\frac{1}{\tau}}\frac{1-(\frac{1}{k})^{\frac{m}{\tau}}}{1-(\frac{1}{k})^{\frac{1}{\tau}}}(\frac{1}{k})^{\frac{n}{\tau}}\\ &\to 0 \; as \; n \to \infty\end{aligned}$$

Consequently $\{\mu_n\}$ is a Cauchy sequence in the metric space. It follows from $\mathfrak{T}$-orbitally completness of space that there exist $\mu^* \in X$ such that

$$\lim_{n\to\infty} \delta(\mu_n, \mu^*) = 0 \tag{2.4}$$

and so

$$\lim_{n\to\infty} \delta(\mu_n, \mathfrak{T}\mu^*) = \delta(\mu^*, \mathfrak{T}\mu^*) \tag{2.5}$$

putting $x = \mu_n$ and $y = \mu_*$ in (2), we get

$$\begin{aligned}0 &\leq \zeta(M(\mu_n, \mu_*), \delta_\theta(\mu_n, \mu_*))\\ &< \delta_\theta(\mu_n, \mu_*) - M(\mu_n, \mu_*)\end{aligned}$$

Or

$$M(\mu_n, \mu_*) < \delta_\theta(\mu_n, \mu_*) \tag{2.6}$$

where

$$\begin{aligned}M(\mu_n, \mu_*) &= k\delta_\theta(\mathfrak{T}\mu_n, \mathfrak{T}\mu_*) - \min\{\delta_\theta(\mu_n, \mathfrak{T}\mu_*), \delta_\theta(\mu_*, \mathfrak{T}\mu_n), \delta_\theta(\mu_n, \mathfrak{T}u_n), \delta_\theta(\mu_*, \mathfrak{T}\mu_*)\}\\ &= k\delta_\theta(\mu_{n+1}, \mathfrak{T}\mu_*) - \min\{\delta_\theta(\mu_n, \mathfrak{T}\mu_*), \delta_\theta(\mu_*, \mu_{n+1}), \delta_\theta(\mu_n, \mu_{n+1}), \delta_\theta(\mu_*, \mathfrak{T}\mu_*)\}\\ &= k\delta_\theta(\mu_{n+1}, \mathfrak{T}\mu_*)\end{aligned}$$

as $n \to \infty$ since $\lim_{n\to\infty} \delta(\mu_{n+1}, \mu^*) = 0$ and $\theta(0) = 0$ implying $\lim_{n\to\infty} \delta_\theta(\mu_{n+1}, \mu^*) = 0$

It follows from (2.6) that

$$\lim_{n\to\infty} \delta_\theta(\mu_{n+1}, \mathfrak{T}\mu_*) < \frac{1}{k} \lim_{n\to\infty} \delta_\theta(\mu_n, \mu_*) \tag{2.7}$$

Using the similar argument used to establish (2.3), we get

$$\lim_{n\to\infty} \delta(\mu_{n+1}, \mathfrak{T}\mu_*) < \lim_{n\to\infty} [\frac{\delta_\theta(\mu_n, \mu_*)}{c}]^{\frac{1}{\tau}}(\frac{1}{k})^{\frac{n}{\tau}} = 0, \forall n \geq 0 \tag{2.8}$$

Then, using the triangle inequality, we obtain for $n \geq 0$

$$\delta(\mu_*, \mathfrak{T}\mu_*) \leq \delta(\mu_*, \mu_n) + \delta(\mu_n, \mu_{n+1}) + \delta(\mu_{n+1}, \mathfrak{T}\mu^*)$$

Letting $n \to \infty$ and using(2.4) and (2.8), we get $\delta(\mu_*, \mathfrak{T}\mu_*) = 0$.

We now show that $Fix(\mathfrak{T}) = \mu_*$.

Indeed, if $z_* \in Fix(\mathfrak{T})$ and $\mu_* \neq z_*$ (or equivalently $\mathfrak{T}\mu_* \neq \mathfrak{T}z_*$)), then, using (2.1) with $(u, v) = (\mu_*, z_*)$, we get

$$\delta_\theta(\mathfrak{T}\mu_*, \mathfrak{T}z_*) \leq \frac{1}{k}\delta_\theta(\mu_*, z_*)$$

that is,

$$\delta_\theta(\mu_*, z_*) \leq \frac{1}{k}\delta_\theta(\mu_*, z_*)$$

which is a contradiction since $k \in (1, \infty)$. Consequently μ_* is the unique fixed point of $\mathfrak{T}$. This completes the proof of Theorem 2.2. □

Example 2.1 Let $\mathcal{Q} = \{r1, r2, r3\}$ and δ be the metric on $\mathcal{Q}$ defined by

$$\delta(qi, qi) = 0, \delta(qi, qj) = \delta(qj, qi), i, j \in \{1, 2, 3\}$$

and

$$\delta(q1, q2) = 1, \delta(q1, q3) = 4, \delta(q2, q3) = 5.$$

Notice that δ satisfies the triangle inequality. Indeed, we have

$$\delta(q1, q2) = 1 < 4 = \delta(q1, q3) < \delta(q1, q3) + \delta(q3, q2)$$
$$\delta(q1, q3) = 4 < 5 = \delta(q2, q3) < \delta(q1, q2) + \delta(q2, q3)$$
$$and$$
$$\delta(q2, q3) = 5 = \delta(q2, q1) + \delta(q1, q3).$$

Consequently, $(\mathcal{Q}, \delta)$ is a metric space. Consider the mapping $\mathfrak{T} : \mathcal{Q} \to \mathcal{Q}$ defined by

$$\mathfrak{T}q1 = q1, \mathfrak{T}q2 = q3, \mathfrak{T}q3 = q1.$$

We point out that $\mathfrak{T}$ is not a contraction in the sense of Banach. Indeed, we have $\delta(\mathfrak{T}q1, \mathfrak{T}q2) = \delta(q1, q3) = 4 > 1 = \delta(q1, q2)$ We now introduce the mapping $\theta : [0, \infty)ß[0, \infty)$ defined by

$$\theta(s) = \begin{cases} \frac{7s}{\sinh 1}, & 0 \leq s \leq \sinh 1 \\ \frac{2s}{\sinh 4}, & \sinh 1 \leq s \leq \sinh 4 \\ \frac{5s}{\sinh 5}, & s > \sinh 4 \end{cases}$$

It can be easily observed that

$$\theta(s) \geq \frac{5s}{\sinh 5}, \forall s \geq 0$$

which shows that $\theta \in \Theta_1$. Also $\theta(0) = 0$.
Furthermore from (2.1), we get

$$\zeta(\mathcal{M}(q1, q2), \delta_\theta(q1, q2)) \geq 0 \tag{2.9}$$

for all $x, y \in X$ where

$$\mathcal{M}(q1, q2) = k\delta_\theta(\mathfrak{T}q1, \mathfrak{T}q2) - \min\{\delta_\theta(q1, \mathfrak{T}q2), \delta_\theta(q2, \mathfrak{T}q1), \delta_\theta(q1, \mathfrak{T}q1), \delta_\theta(q2, \mathfrak{T}q2)\}$$

we have

$$\delta_\theta(\mathfrak{T}q1, \mathfrak{T}q2) = \delta_\theta(q1, q3) = \theta(\sinh(\delta(q1, q3))) = \theta(sinh4) = 2$$

and

$$\delta_\theta(q1, q2) = \theta(\sinh(\delta(q1, q2))) = \theta(sinh1) = 7$$

also

$$\delta_\theta(q1, \mathfrak{T}q1) = \theta(\sinh(\delta(q1, q1))) = 0 \ as \ \theta(0) = 0$$

Thus (2.9) becomes

$$0 \leq \zeta(\mathcal{M}(q1, q2), \delta_\theta(q1, q2)) = \zeta(2k, 7) < 7 - 2k \tag{2.10}$$

which is true for $k \leq \frac{7}{2}$.
Similarly we can arrive at

$$0 \leq \zeta(\mathcal{M}(q2, q3), \delta_\theta(q2, q3)) = \zeta(2k, 5) < 5 - 2k \tag{2.11}$$

which is true for $k \leq \frac{5}{2}$.
From (2.10) and (2.11), we arrived at

$$\zeta(\mathcal{M}(qi, qj), \delta_\theta(qi, qj)) \geq 0, (i, j) \in (1, 2), (2, 1), (2, 3), (3, 2) \tag{2.12}$$

for all $k \in (1, \frac{5}{2}]$.
This shows that $\mathfrak{T}$ is a Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction with respect to $\zeta \in \mathcal{Z}$ on $\mathcal{Q}$. Then, condition (I) of Theorem 2.1 is satisfied. Notice also that for all $n \geq 2$, we have

$$\mathfrak{T}^n qi = q1, \ i \in \{1, 2, 3\}$$

which shows that condition (II) of Theorem 2.1 is satisfied. Consequently, Theorem 2.1 applies. Also we have $Fix(\mathfrak{F}) = \{q1\}$, which confirms Theorem 2.1.

3. Application

We shall use the following notations: $M(n)$ denotes the set of all $n \times n$ matrices, $H(n) \subseteq M(n)$ the set of all Hermitian matrices and $H^+(n) \subseteq H(n)$ the rest of positive semi-definite matrices.
Consider the nonlinear matrix equations,

$$\mathcal{X} = Q_1 + \sum_{i=1}^{m} A_i^* \mathcal{U}(\mathcal{X}) A_i$$

where $Q_1 \in H^+(n)$ and $A_i, \mathcal{X} \in M(n)$ with $\sum_{i=1}^{m} |A_i A_i^*|_{tr} = \eta$ and $\mathcal{U} : H^+(n) \to H^+(n)$ such that

$$|(\mathcal{U}(\mathcal{X}) - \mathcal{U}(\mathcal{Y}))|_{tr} < \frac{1}{\eta}(|\mathcal{X} - \mathcal{Y}|_{tr} - \ln k) \tag{3.1}$$

where $k \in (1, \infty)$.

Proof: Let δ be the metric on $M(n)$ defined by

$$\delta(\mathcal{X}, \mathcal{Y}) = |\mathcal{X} - \mathcal{Y}|_{tr}$$

It is well known that $(M(n)), \delta)$ is a complete metric space. We introduce the mapping $\mathfrak{F} : H^+(n) \to H^+(n)$ defined by

$$\mathfrak{F}\mathcal{X} = Q_1 + \sum_{i=1}^{m} A_i^* \mathcal{U}(\mathcal{X}) A_i$$

for all $\mathcal{X} \in M(n)$.
Next we will estimate for $\mathcal{X}, \mathcal{Y} \in M(n)$

$$\begin{aligned}
&|\mathfrak{F}\mathcal{X} - \mathfrak{F}\mathcal{Y}|_{tr} \\
&\leq \sum_{i=1}^{m} |A_i^*(\mathcal{U}(\mathcal{X}) - \mathcal{U}(\mathcal{Y}))A_i|_{tr} \\
&\leq \sum_{i=1}^{m} |A_i^* A_i|_{tr} |(\mathcal{U}(\mathcal{X}) - \mathcal{U}(\mathcal{Y}))|_{tr} \\
&\leq |(\mathcal{U}(\mathcal{X}) - \mathcal{U}(\mathcal{Y}))|_{tr} \sum_{i=1}^{m} |A_i^* A_i|_{tr} \\
&\leq \eta |(\mathcal{U}(\mathcal{X}) - \mathcal{U}(\mathcal{Y}))|_{tr} \\
&< |\mathcal{X} - \mathcal{Y}|_{tr} - \ln k
\end{aligned}$$

or

$$\ln k + |\mathfrak{F}\mathcal{X} - \mathfrak{F}\mathcal{Y}|_{tr} < |\mathcal{X} - \mathcal{Y}|_{tr}$$

Then, taking the maximum over $t \in [0, 1]$ in the above inequality, we obtain

$$\ln k + \delta(\mathfrak{F}\mathcal{X}, \mathfrak{F}\mathcal{Y}) < \delta(\mathcal{X} - \mathcal{Y})$$

i.e.

$$k e^{\delta(\mathfrak{F}\mathcal{X}, \mathfrak{F}\mathcal{Y})} < e^{\delta(\mathcal{X} - \mathcal{Y})}$$

which is equivalent to

$$k \sinh(\delta(\mathfrak{F}\mathcal{X}, \mathfrak{F}\mathcal{Y})) < \sinh(\delta(\mathcal{X} - \mathcal{Y}) \ \textit{since}\ \delta > 0$$

which in turn provides

$$k\delta_\theta(\mathfrak{F}\mathcal{X}, \mathfrak{F}\mathcal{Y}) < \delta_\theta(\mathcal{X}, \mathcal{Y}) \ \textit{or}\ \delta_\theta(\mathcal{X}, \mathcal{Y}) - k\delta_\theta(\mathfrak{F}\mathcal{X}, \mathfrak{F}\mathcal{Y}) > 0 \tag{3.2}$$

for $\theta(t) = t$ for all $t \geq 0$.
Also we have

$$\mathcal{M}(\mathcal{X},\mathcal{Y}) = k\delta_\theta(\mathfrak{F}\mathcal{X},\mathfrak{F}\mathcal{Y}) - \min\{\delta_\theta(\mathcal{X},\mathfrak{F}\mathcal{Y}),\delta_\theta(\mathcal{Y},\mathfrak{F}\mathcal{X}),\delta_\theta(\mathcal{X},\mathfrak{F}\mathcal{X}),\delta_\theta(\mathcal{Y},\mathfrak{F}\mathcal{Y})\} \leq k\delta_\theta(\mathfrak{F}\mathcal{X},\mathfrak{F}\mathcal{Y})$$

using (3.2), we get

$$\delta_\theta(\mathcal{X},\mathcal{Y}) - M(\mathcal{X},\mathcal{Y}) > 0$$

Equivalently

$$\zeta(M(\mathcal{X},\mathcal{Y}),\delta_\theta(\mathcal{X},\mathcal{Y})) \geq 0$$

It is evident that all the conditions of Theorem 2.2 are satisfied for the underlying mapping. So there exist a solution to the matrix equation under study. □

4. Conclusion

We defined Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction associated to a certain metric the underlying space. Motivated from the existing fixed point results of θ-hyperbolic sine functions and other fixed point results of $\mathcal{Z}$- contraction, we established a fixed point result involving a Ciric-type(I) θ-hyperbolic $\mathcal{Z}$-contraction. We furnished the established result with an example. Further we have applied the result to solve the system of non-linear matrix equation.
The scope of the future work involves extending the results to other metric spaces and exploring the implications of these fixed point results in practical applications. Also applying the concepts of Ciric-type(I) θ-hyperbolic $\mathcal{Z}$- contraction in hyperbolic metric spaces, which may lead to new theoretical frameworks and applications.

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