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Symmetry Analysis and Exact Similarity Solutions for Maxwellian Equation

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ABSTRACT: In this paper, we study the structure of the symmetry algebra associated with the Maxwellian equation which is solvable, non-semi-simple, and non-nilpotent. By adopting Ovsiannikov's approach which relies on two essential concepts: the adjoint representation and the Killing form. We establish the optimal system in one dimension based on this and the notion of the normalizer of a subalgebra. We then construct the optimal systems in two and three dimensions. By exploiting the structurally significant information derived from these systems, we derive several reduction equations and obtain some exact solutions.

Keywords: Invariant solutions, Lie algebra classification, symmetry reduction, optimal system.

Contents

1. Introduction

Understanding natural and physical phenomena requires modeling them using partial differential equations. Unfortunately, the complexity faced when studying these phenomena results in their nonlinearity; generally, their resolution is not an easy task. In the mathematical literature, several methods and techniques had been developed to construct exact and numerical solutions. Recently, a class of solutions for various types of partial differential equations have been proposed and discovered in the literature by different authors, through specializing of forms for particular choices of a similarity variables. The laters transform the equation into a simplified one, for example the singularity theory for semi linear waves. Lie analysis has been a crucial instrument and It provides a powerful mathematical tool for extracting exact solutions for a large family of both integer and fractional order ordinary and partial differential equations [8,10,11]. It relies on the exploitation of admitted symmetries, which allows these equations to be transformed in terms of reducing their complexity, either through decreasing the number of independent variables or through lowering their order. This simplification based on identifying invariance properties under a specific group of point transformations proposed firstly by Sophus Lie and afterwards, several applications were developed under the guidance of many authors in a multiple works found in the literature [6,1,9,13].

The Maxwellian equation is one of the most important equations in physics, specifically, in the classical kinetic theory field. It results on the relaxation of nonequilibrium distribution functions, in addition to

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the modeling of the problem of energy distribution. It suggests to understand the behavior in the high-energy tail of the velocity distribution, and to calculate values of certain gas phase reaction rates at a given kinetic temperature [14]. The local existence and uniqueness of its solutions have been extensively studied in the literature.

The discovery and construction of new particular solutions each time has led many authors to develop a special interest by the classification problem. To address this question, it was necessary to establish the different dimension optimal systems associated for the symmetry algebra admitted by the studied equation. Ovsiannikov relies on two main concepts, the adjoint representation and invariants particularly, the Killing form [6]. The Ovsiannikov's approach has been extensively developed by Patera, Winternitz, and Zassenhaus leading to the formulation of numerous optimal subgroup systems for Lie groups used in mathematical physics [2,3]. Galas has extended this work by eliminating equivalent subalgebras and addressing the case of non-solvable algebras which are more complex to handle than solvable algebras [5]. Additional examples of optimal systems can also be found among Ibragimov researches [7]. In this work, we study the algebraic structure of the symmetry algebra admitted by the Maxwellian equation and we list an inequivalent reduced equations family. Furthermore, we construct some exact solutions of the studied equation.

The paper is structured as follows: Section 2, is dedicated to analyze the structural properties of the symmetry algebra, to determine both, the global matrix of the adjoint representation and some invariant functions. In Section 3, we construct the optimal systems for subalgebras of different dimensions. Finally, section 4, deals with symmetry reductions and construction of invariant solutions.

2. Maxwellian Equation and Invariance Analysis

2.1. Study of the structure of the Lie symmetry algebra for Maxwellian equation

The Maxwellian Equation (M) is given by the following expression

$$\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial u}{\partial t} + u^2 = 0. \tag{2.1}$$

The Lie algebra of infinitesimal symmetries for this equation, denoted by $\mathcal{G}$, is spanned by four vector fields [4]

$$\begin{aligned} &Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \\ &Z_4 = e^x \frac{\partial}{\partial x} - e^x u \frac{\partial}{\partial u}. \end{aligned} \tag{2.2}$$

The commutator table of $\mathcal{G}$ is given by

Table 1: Commutator Table

$[Z_i, Z_j]$	Z_1	Z_2	Z_3	Z_4
Z_1	0	0	$-Z_1$	0
Z_2	0	0	0	Z_4
Z_3	Z_1	0	0	0
Z_4	0	$-Z_4$	0	0

Based on Table (1), we have the following remark.

Remark 2.1

1. The Lie algebra $\mathcal{G}$ is not semi-simple. Indeed,

$$D(\mathcal{G}) = [\mathcal{G}, \mathcal{G}] = \text{Span}(Z_1, Z_4) \neq \mathcal{G}.$$

2. The Lie algebra $\mathcal{G}$ is solvable. Indeed,

$$\mathcal{G}^{(2)} = [\mathcal{G}', \mathcal{G}'] = \{0\}.$$

Proposition 2.1 *The Lie algebra $\mathcal{G}$ is not nilpotent.*

Proof:
By simple induction, we can show that for $k \geq 2$, $\mathcal{G}^k = \text{Span}(Z_1, Z_4)$, where $(\mathcal{G}^k)$ denotes the central series of $\mathcal{G}$. □

2.2. Adjoint representation and invariants

In this section, we will determine the global adjoint matrix. To do this, we will recall the definition of the adjoint representation.

Definition 2.1
Let G be a Lie group and $\mathcal{G}$ its associated Lie algebra. The adjoint representation is denoted by Ad and it is defined by

$$\forall g \in G, \ \forall w \in \mathcal{G}, \quad Ad_g(w) = g^{-1}wg.$$

Theorem 2.1 *Let G be a Lie groups and $\mathcal{G}$ its associated Lie algebra. We have*

$$Ad(\exp(\epsilon v))(w) = \sum_{k=0}^{+\infty} \frac{\epsilon^k}{k!}(\text{ad}(v))^k(w),$$

with ad *given by*

$$\text{ad}(v) : \mathcal{G} \to \mathcal{G}, \quad w \mapsto \text{ad}(v)(w) = [w, v].$$

The previous theorem allows us to construct the following table of the adjoint representation.

Table 2: Adjoint representation table

$Ad(exp(\epsilon\star)\star)$	Z_1	Z_2	Z_3	Z_4
Z_1	Z_1	Z_2	$Z_3 + \epsilon Z_1$	Z_4
Z_2	Z_1	Z_2	Z_3	$e^{-\epsilon}Z_4$
Z_3	$e^{-\epsilon}Z_1$	Z_2	Z_3	Z_4
Z_4	Z_1	$Z_2 + \epsilon Z_4$	Z_3	Z_4

We can now address the notion of invariants.

Definition 2.2 *Let $\mathcal{G}$ be a Lie algebra. A real function ϕ on $\mathcal{G}$ is called an invariant if $\phi\left(Ad_g(Z)\right) = \phi(Z)$ for all $Z \in \mathcal{G}$ and all $g \in G$. Two vectors Z and T are equivalent under the adjoint action, if $\phi(Z) = \phi(T)$, for any invariant function ϕ.*

Remark 2.2 In general, it is not easy to determine invariant functions. The well-known invariant function is the one related to the Killing form [1], and we also have the following theorem

Theorem 2.2 *A real function $\phi(a_1, \ldots, a_4)$ is an invariant function of the Lie algebra $\mathcal{G}$ if and only if ϕ is a solution of the system:*

$$\begin{cases} a_1 \dfrac{\partial \phi}{\partial a_1} = 0, \\ a_4 \dfrac{\partial \phi}{\partial a_4} = 0, \\ -a_2 \dfrac{\partial \phi}{\partial a_4} = 0, \\ -a_3 \dfrac{\partial \phi}{\partial a_1} = 0. \end{cases} \tag{2.3}$$

Remark 2.3 If $a_2 \neq 0$ and $a_3 \neq 0$ we have $\phi(a_1, a_2, a_3, a_4) = F(a_2, a_3)$, with F an arbitrary function.

Proposition 2.2 *The function defined on $\mathcal{G}$ by $C(Z) = K(Z, Z)$ is an invariant function, where K is the Killing form on $\mathcal{G}$ given by*

$$K : \mathcal{G} \times \mathcal{G} \to \mathbb{R}, \quad (Z, Y) \mapsto \mathrm{Tr}(\mathrm{ad}(Z)\,\mathrm{ad}(Y)).$$

In our case we have the following result

Proposition 2.3 $C(Z) = K(Z, Z) = a_2^2 + a_3^2.$

Proof: We have $K(Z, Z) = \mathrm{Tr}(\mathrm{ad}(Z)\,\mathrm{ad}(Z))$, and

$$\mathrm{ad}(Z) = \begin{pmatrix} -a_3 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_4 & 0 & -a_2 \end{pmatrix},$$

then, $C(Z) = K(Z, Z) = a_2^2 + a_3^2.$ □

We can determine additional invariants that help us confirm optimality, that is, the completeness and mutual inequivalence of the representatives of the subalgebras. From the adjoint action table we can construct the following table

Table 3: Table for construction of invariant functions.

$Ad(\exp(\epsilon Z_i)Z)$	coef Z_1	coef Z_2	coef Z_3	coef Z_4
$Ad(\exp(\epsilon Z_1)Z)$	$a_1 + \epsilon a_3$	a_2	a_3	a_4
$Ad(\exp(\epsilon Z_2)Z)$	a_1	a_2	a_3	$a_4 e^{-\epsilon}$
$Ad(\exp(\epsilon Z_3)Z)$	$a_1 e^{-\epsilon}$	a_2	a_3	a_4
$Ad(\exp(\epsilon Z_4)Z)$	a_1	a_2	a_3	$\epsilon a_2 + a_4$

According to Table (3), we have the following proposition.

Proposition 2.4 *The following functions are invariants*

$$L(a_1, a_2, a_3, a_4) = a_2, \quad A(a_1, a_2, a_3, a_4) = a_3,$$

and

$$B(a_1, a_2, a_3, a_4) = \begin{cases} \mathrm{sgn}(a_1), & \text{if } a_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: For L and A it is clearly observed from Table (3).

For B the coefficient of Z_1, i.e. a_1 remains unchanged under the action of $Ad(\exp(\epsilon Z_i))$ for $i = 2$ and 4. Therefore, we investigate both the adjoint actions under $Ad(\exp(\epsilon Z_1))$ and $Ad(\exp(\epsilon Z_3))$ and the invariance condition $B(Y) = B(Ad(Y))$. Then, the $\mathrm{sgn}(a_1)$ maps to $\mathrm{sgn}(a_1 e^{-\epsilon})$, which gives positive, negative or zero, depending on the sign of a_1. □

2.3. Adjoint matrix of the Maxwellian equation

In order to construct the global adjoint matrix, let us first apply the action of Z_2 to $Z = \sum_{i=1}^{4} a_i Z_i$ and using Table (2), we get

$$Ad_{\exp(\epsilon_2 Z_2)} Z = a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 e^{-\epsilon_2} Z_4 = (a_1, a_2, a_3, a_4) M_2 (Z_1, Z_2, Z_3, Z_4)^T,$$

where

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\epsilon_2} \end{pmatrix}.$$

Similarly, M_1, M_3 and M_4 are found to be

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \epsilon_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} e^{-\epsilon_3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \epsilon_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$Z = a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4,$$

and

$$g = \exp(\epsilon_1 Z_1)\exp(\epsilon_2 Z_2)\exp(\epsilon_3 Z_3)\exp(\epsilon_4 Z_4) \in G.$$

As

$$Ad(g) = Ad(\exp(\epsilon_1 Z_1)) \circ Ad(\exp(\epsilon_2 Z_2)) \circ Ad(\exp(\epsilon_3 Z_3)) \circ Ad(\exp(\epsilon_4 Z_4)),$$

then, the global adjoint transformation matrix M is obtained to be of the form

$$M = M_1 M_2 M_3 M_4 = \begin{pmatrix} e^{-\epsilon_3} & 0 & 0 & 0 \\ 0 & 1 & 0 & \epsilon_4 \\ \epsilon_1 e^{-\epsilon_3} & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\epsilon_2} \end{pmatrix}. \tag{2.4}$$

3. Optimal System of Symmetry Algebra Admitted by the Maxwellian Equation

In order to determine the optimal system of one dimension, we will use an approach developed by Ovsiannikov [6], furthermore the automorphisms of finite-dimensional Lie transformation groups are inner automorphisms.

3.1. A one-dimensional optimal system of symmetry algebra admitted by Maxwellian equation

Proposition 3.1 *The optimal system of dimension* 1 *is given by*

$$M_1 = \langle Z_2 + Z_3 \rangle, \quad M_2 = \langle Z_2 - Z_3 \rangle, \quad M_3 = \langle Z_1 + Z_2 \rangle, \quad M_4 = \langle -Z_1 + Z_2 \rangle$$
$$M_5 = \langle Z_3 + Z_4 \rangle, \quad M_6 = \langle -Z_3 + Z_4 \rangle, \quad M_7 = \langle Z_1 + Z_4 \rangle, \quad M_8 = \langle -Z_1 + Z_4 \rangle,$$

Proof:

As every element X of $\mathcal{G}$ can be written in the form $Z = \sum_{i=1}^{4} a_i Z_i$. Then, we will simplify the form of X as much as possible. To do this, we search a simple form $\tilde{Z} = \sum_{i=1}^{4} \tilde{a}_i Z_i$ of Z. Using the global adjoint matrix, we get

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = (a_1, a_2, a_3, a_4)\, A \quad \Leftrightarrow \quad \begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-2\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \tag{3.1}$$

According to the invariant function given by the Killing form, we are led to discuss the following cases
Case 1: $a_2 = 1$ **and** $a_3 = 1$

Select a representative element $\tilde{X}$. Substituting $\epsilon_1 = -a_1, \epsilon_2 = 0$, $\epsilon_4 = -a_4$, $a_2 = 1$ and $a_3 = 1$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \Leftrightarrow \begin{cases} \tilde{a}_1 = 0, \\ \tilde{a}_2 = 1, \\ \tilde{a}_3 = 1, \\ \tilde{a}_4 = 0. \end{cases}$$

That is to say, that all expressions $a_1 Z_1 + Z_2 + Z_3 + a_4 Z_4$ are equivalent to $Z_2 + Z_3$.

Case 2: $a_2 = 1$ **and** $a_3 = -1$

Select a representative element $\tilde{Z}$. Substituting $\epsilon_1 = a_1, \epsilon_2 = 0$, $\epsilon_4 = -a_4$, $a_2 = 1$ and $a_3 = -1$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-2\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \Leftrightarrow \begin{cases} \tilde{a}_1 = 0, \\ \tilde{a}_2 = 1, \\ \tilde{a}_3 = -1, \\ \tilde{a}_4 = 0. \end{cases}$$

That is to say, that all expressions $a_1 Z_1 + Z_2 - Z_3 + a_4 Z_4$ are equivalent to $Z_2 - Z_3$.

Case 3: $a_2 = 1$ **and** $a_3 = 0$

If $a_1 > 0$,
select a representative element $\tilde{X}$. Substituting $\epsilon_3 = \ln(a_1), \epsilon_2 = 0$, $\epsilon_4 = -a_4$, $a_2 = 1$ and $a_3 = 0$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \Leftrightarrow \begin{cases} \tilde{a}_1 = 1, \\ \tilde{a}_2 = 1, \\ \tilde{a}_3 = 0, \\ \tilde{a}_4 = 0. \end{cases}$$

Then, the simple form is

$$\tilde{Z} = Z_1 + Z_2.$$

If $a_1 < 0$,
select a representative element $\tilde{X}$. Substituting $\epsilon_3 = \ln(-a_1), \epsilon_2 = 0$, $\epsilon_4 = -a_4$, $a_2 = 1$ and $a_3 = 0$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \Leftrightarrow \begin{cases} \tilde{a}_1 = -1, \\ \tilde{a}_2 = 1, \\ \tilde{a}_3 = 0, \\ \tilde{a}_4 = 0. \end{cases}$$

Then, the simple form is

$$\tilde{Z} = -Z_1 + Z_2.$$

Case 4: $a_2 = 0$ and $a_3 = 1$.

If $a_4 > 0$,

select a representative element $\tilde{X}$. Substituting $\epsilon_2 = \ln(a_4), \epsilon_1 = -a_1$, $\epsilon_4 = -a_4$, $a_2 = 0$ and $a_3 = 1$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \Leftrightarrow \begin{cases} \tilde{a}_1 = 0, \\ \tilde{a}_2 = 0, \\ \tilde{a}_3 = 1, \\ \tilde{a}_4 = 1. \end{cases}$$

Then, the simple form in this case is

$$\tilde{Z} = Z_3 + Z_4.$$

If $a_4 < 0$,

select a representative element $\tilde{Z}$. Substituting $\epsilon_2 = \ln(-a_4), \epsilon_1 = -a_1, \,, a_2 = 0$ and $a_3 = 1$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \Leftrightarrow \begin{cases} \tilde{a}_1 = 0, \\ \tilde{a}_2 = 0, \\ \tilde{a}_3 = 1, \\ \tilde{a}_4 = -1. \end{cases}$$

Then, the simple form in this case is

$$\tilde{Z} = Z_3 + Z_4.$$

Case 5: $a_2 = 0$ and $a_3 = 0$.

If $a_1 > 0$ and $a_4 > 0$,

select a representative element $\tilde{Z}$. Substituting $\epsilon_2 = \ln(a_4), \epsilon_3 = \ln(a_1)$, $a_2 = 0$ and $a_3 = 0$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \Leftrightarrow \begin{cases} \tilde{a}_1 = 1, \\ \tilde{a}_2 = 0, \\ \tilde{a}_3 = 0, \\ \tilde{a}_4 = 1. \end{cases}$$

Then, the simple form in this case is

$$\tilde{Z} = Z_1 + Z_4.$$

If $a_1 > 0$ and $a_4 < 0$,

select a representative element $\tilde{Z}$. Substituting $\epsilon_2 = \ln(-a_4), \epsilon_3 = \ln(a_1)$, $a_2 = 0$ and $a_3 = 0$ into equation (3.1), we obtain the solution

$$\begin{cases} \tilde{a}_1 = a_1 e^{-\epsilon_3} + a_3 \epsilon_1 e^{-\epsilon_3}, \\ \tilde{a}_2 = a_2, \\ \tilde{a}_3 = a_3, \\ \tilde{a}_4 = \epsilon_4 a_2 + a_4 e^{-\epsilon_2}. \end{cases} \Leftrightarrow \begin{cases} \tilde{a}_1 = 1, \\ \tilde{a}_2 = 0, \\ \tilde{a}_3 = 0, \\ \tilde{a}_4 = -1. \end{cases}$$

Then, the simple form in this case is

$$\tilde{Z} = -Z_1 + Z_4.$$

□

Remark 3.1 The following table summarizes the evaluation of the different obtained invariants C, L, A, and B on the one-dimensional subalgebras M_i, $i = 1, \ldots, 8$.

Table 4: Evaluation of invariants.

	C	L	A	B
M_1	2	1	1	0
M_2	2	1	-1	0
M_3	1	1	0	1
M_4	1	1	0	-1
M_5	1	0	1	0
M_6	1	0	-1	0
M_7	1	0	0	1
M_8	0	0	0	-1

Table 4 confirms the mutual inequivalence of the representatives of the subalgebras.

3.2. A two-dimensional optimal system of symmetry algebra admitted by Maxwellian equation

To construct the two-dimensional optimal subalgebras [6], let $\langle Z,\ Y\rangle$ be a two-dimensional subalgebra such that

$$[Z,Y] = \alpha Z + \beta Y, \text{ for } Z = M_i,\ i = 1, \ldots, 8 \text{ and } Y = \sum_{i=1}^{4} b_i Z_i.$$

The idea is to select Y from the normaliser $Nor_{\mathcal{G}}(Z)$ of Z in $\mathcal{G}$ given by

$$Nor_{\mathcal{G}}(Z) = \{Y \in \mathcal{G}; [Z,Y] \in \langle Z\rangle\}, \text{ i.e. } [Z,Y] = \alpha Z,$$

where α is an arbitrary constant. After some algebraic manipulations, we get

Proposition 3.2 *The optimal system of dimension* 2 *is given by*

$$N_1 = \langle Z_2,\ Z_3\rangle, \quad N_2 = \langle Z_1,\ Z_2\rangle, \quad N_3 = \langle Z_3,\ Z_4\rangle, \quad N_4 = \langle Z_1,\ Z_4\rangle, \tag{3.2}$$

all N_i, $i = 1, \ldots, 4$ *are abelian sub-algebras.*

3.3. A three-dimensional optimal system of symmetry algebra admitted by Maxwellian equation

The construction of the three-dimensional optimal system is based on the extension of the obtained two-dimensional optimal system. To achieve this, we consider all two-dimensional subalgebras $N_i = \langle Y_r, Y_s\rangle$, $i = 1, \ldots, 5$ and find a vector field $a_1Z_1 + \cdots + a_4Z_4$ such that the triple $\{Y_r, Y_s, Y\}$ of generators form a basis of a three-dimensional subalgebra. For that, it is necessary and sufficient that the vector field Y satisfies the equations:

$$[Y_r, Y] = \alpha_1 Y + \beta_1 Y_r + \gamma_1 Y_s, \quad [Y_s, Y] = \alpha_2 Y + \beta_2 Y_r + \gamma_2 Y_s. \tag{3.3}$$

According to the above equations, we obtain

$$C^i_{jk}\mu^j_r a_k = \alpha_1 a_i + \beta_1 \mu^i_r + \gamma_1 \mu^i_s, \quad C^i_{jk}\mu^j_s a_k = \alpha_2 a_i + \beta_2 \mu^i_r + \gamma_2 \mu^i_s, \tag{3.4}$$

where $Y_l = \sum_{i=1}^{4} \mu^i_l Z_i$, $l \in \{r, s\}$ and $[Z_j, Z_k] = \sum_{i=1}^{4} C^i_{jk} Z_i$.

A solution of the system Y from (3.4), linearly independent of the vector fields Y_r and Y_s, generates a three-dimensional subalgebra. By applying the same procedure to the other subalgebras N_i, we obtain a three-dimensional optimal system.

In our case, by applying this method to all pairs of vector fields in 3.2, we conclude that $Y = \lambda_1 Y_r + \lambda_2 Y_s$. Through an appropriate change of basis, we can assume that $Y = 0$, which shows that $\mathcal{G}$ is not a three-dimensional subalgebra.

Consequently, we have the following proposition.

Proposition 3.3 *The Lie algebra of the symmetries $\mathcal{G}$ admitted by the Maxwellian equation has no three-dimensional optimal system.*

4. Similarity and Invariant Solutions

In this section, we will search for some exact solutions of equation (2.1). This equation is expressed in the coordinates (t, x, u). Therefore, we need to determine the form of this equation in adapted coordinates in order to reduce it. These coordinates will be constructed by identifying independent invariants (z, r) corresponding to certain obtained generators.

Reduction with Z_3: The vector field Z_3 is given by

$$Z_3 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

In this case, the characteristic system is

$$\frac{dt}{-t} = \frac{du}{u}.$$

The corresponding invariants are $z = x$ and $r = tu$. Hence, a similarity solution will be of the form $u(t, x) = \frac{1}{t}f(x)$. Substituting into (2.1), we obtain the reduced differential equation

$$f' + f - f^2 = 0.$$

It is a Bernoulli-type equation, whose solutions are given by

$$f(z) = \frac{1}{1 + ce^{-z}}, \quad c \in \mathbb{R}.$$

Consequently,

$$u(t, x) = \frac{1}{t(1 + ce^{-x})}, \quad c \in \mathbb{R}.$$

Reduction with $A = Z_2 - Z_3$: We have

$$Z_2 - Z_3 = t\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \frac{\partial}{\partial u},$$

the associated characteristic equation is given by

$$\frac{dt}{t} = \frac{dx}{1} = \frac{du}{-u}.$$

The associated invariants are: $z = te^{-x}$ and $r = tu$. Considering the last invariant, we assume that a similarity solution is of the form:

$$u(t, x) = \frac{1}{t}f(te^{-x}).$$

By substituting the above expression into equation (2.1), we get that f is a solution to the following reduced differential equation

$$-z^2 f'' + zf' - f + f^2 = 0. \tag{4.1}$$

Remark 4.1 The invariants z_j, r_j, along with some similarity solutions, are presented in Table **??**. The reduced forms obtained for equation (2.1) are presented in Table (5).

Table 5: Lie Invariants and Similarity Solutions

$\mathbf{Y_j}$	z_j	f_j	u_j
$\mathbf{Z_3}$	x	tu	$u(t,x) = \dfrac{1}{t\,(+ce^{-x})}, \quad c \in \mathbb{R}$
$\mathbf{Z_1 + Z_2}$	$t - x$	u	$u(t,x) = f(z)$
$\mathbf{Z_1 + Z_4}$	$t{+}e^{-x}$	ue^{x}	$u(t,x) = e^{-x} f(z)$
$\mathbf{Z_2 + Z_3}$	te^{x}	ue^{-x}	$u(t,x) = e^{x} f(z)$
$\mathbf{-Z_2 - Z_3}$	te^{-x}	tu	$u(t,x) = \dfrac{1}{t} f(z)$
$\mathbf{Z_3 + Z_4}$	$te^{-e^{-x}}$	$ue^{x}e^{e^{-x}}$	$u(t,x) = e^{-\left(x+e^{-x}\right)} f(z)$
$\mathbf{Z_3 - Z_4}$	$te^{e^{-x}}$	$ue^{-e^{-x}}e^{x}$	$u(t,x) = e^{-x}e^{e^{-x}} f(z)$
$\mathbf{Z_1 - Z_4}$	$t{-}e^{-x}$	ue^{x}	$u(t,x) = e^{-x} f(z)$
$\mathbf{-Z_2 + Z_2}$	$t + x$	u	$u(t,x) = f(z)$

The reduced equations obtained from the Maxwellian equation are presented in the table below.

Table 6: Reduced equations

Y_j	Similarity Reduced Equations
Z_3	$f'(z) + f(z) - (f(z))^2 = 0$
$Z_1 + Z_2$	$-f''(z) + f'(z) + (f(z))^2 = 0$
$Z_1 + Z_4$	$f''(z) - (f(z))^2 = 0$
$Z_2 + Z_3$	$zf''(z) + 3f'(z) + (f(z))^2 = 0$
$Z_2 - Z_3$	$-z^2 f''(z) - f(z) + zf'(z) + (f(z))^2 = 0$
$Z_3 + Z_4$	$z^3 f''(z) + 2z^2 f'(z) + z^2\,(f(z))^2 = 0$
$Z_3 - Z_4$	$zf''(z) + 2f'(z) - (f(z))^2 = 0$
$Z_1 - Z_4$	$f''(z) +' (f(z))^2 = 0$
$-Z_1 + Z_2$	$f''(z) + f'(z) + f(z) + (f(z))^2 = 0$

Remark 4.2

- Among the solutions of the reduced equation $f''(z) - (f(z))^2 = 0$ given in Table (6): $f(z) = \frac{6}{(z+c)^2}$, $c \in \mathbb{R}$, so using the data from Table (4), the functions defined by

  $$u(t,x) = \frac{6e^{-x}}{(t + e^{-x} + c)^2}, \quad c \in \mathbb{R}$$

  ,

  form a family of solutions to the Maxwellian equation.
- The function defined by $f(z) = \frac{1}{z}$, is a solution of the reduced equation $zf''(z) + 3f'(z) + (f(z))^2 = 0$, consequently, the function defined by

  $$u(t,x) = \frac{1}{t},$$

  is also a solution of the Maxwellian equation.

5. Conclusion

In this paper, we show that the symmetry algebra associated to the Maxwellian equation is solvable, but neither semi-simple nor nilpotent. We applied the Ovsiannikov method for classification by identifying the optimal system subalgebras of various dimensions. Using the one-dimensional optimal system subalgebras, we reduced the equation under study to a set of distinct and non-equivalent ordinary differential equations. This study is interesting insofar as it can be extended to other nonlinear partial or fractional differential equations.

Declarations

Conflict of interest. The authors declare that they have no competing of interests regarding the publication of this paper.
Availability of data and materials. No data was used for the research described in the article.
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