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# Extended Parametric $A_b$-Metric Space and Some Fixed Point Results with an Application to a System of Integral Equations

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ABSTRACT: In this paper, we consider the generalized framework of an extended parametric $A_b$-metric space and prove some results on common fixed points and fixed points in the space with a condition on symmetry. Our results are generalized enough to contain many interesting results which are generalizations, extensions and analogs of Banach Contraction Principle, Kannan fixed point Theorem and Jaggi fixed point Theorem in extended parametric $A_b$-metric space and other lower spaces. We also apply our result to solve a system of integral equations for common solution.

Keywords: Extended parametric $A_b$-metric space, complete, symmetric, common fixed point, fixed point.

## Contents



## 1. Introduction

The study of metric fixed point theory continues to hold significant importance in contemporary mathematical research, primarily due to its applications in nonlinear analysis. Since the introduction of the celebrated Banach Contraction Principle in 1922, researchers have pursued various generalizations in the context of space and contraction mapping. The concept of metric space has been extended to $b$-metric space, as examined by I.A. Bakhtin[3] and S. Czerwik[4]. Sedghi et al.[6] introduced $S$-metric space, while Souayah and Mlaiki[7] proposed $S_b$-metric space. Further generalizations to $A$-metric and $A_b$-metric spaces were made by Mujahid Abbas et al.[14] and Manoj Ughade et al.[15], respectively. K. Anthony Singh et al.[2] also contributed another generalization in the form of a cone $A_b$-metric space in another line of generalizations consisting of cone metric space, cone $b$-metric space, cone $S$-metric space, cone $S_b$-metric space and cone $A$-metric space etc. Hussain et al.[9] developed parametric metric space, which was subsequently extended to parametric $b$-metric space[10]. Tas and Ozgur[11] introduced parametric $S$-metric space, which was further expanded to parametric $S_b$-metric space[12]. N. Priyobarta et al.[17] introduced the parametric $A$-metric space. Mlaiki[13] introduced the extended $S_b$-metric space, while Naveen Mani et al.[16] described the extended parametric $S_b$-metric space. Notable contributions include studies by Beniwal et al.[5] and Naveen et al.[18]. K. Anthony Singh et al.[1] introduced the extended parametric $A_b$-metric space, which encompasses all previous spaces.

This paper explores this framework of an extended parametric $A_b$-metric space and establishes some common fixed point and fixed point theorems by incorporating a symmetry condition in the space. Our study is motivated by the results of Naveen Mani et al.[18]. We also demonstrate an application of our result in solving a system of integral equations for common solution.

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## 2. Preliminaries

**Definition 2.1.**([4]) Consider $X \neq \phi$ and a real-valued function $\Omega : X \times X \to [0, \infty)$ and $b \geq 1$. This $\Omega$ is called a $b$-metric on $X$ if:
1. $\Omega(p, q) = 0 \Longleftrightarrow p = q$;
2. $\Omega(p, q) = \Omega(q, p)$;
3. $\Omega(p, q) \leq b[\Omega(p, r) + \Omega(r, q)]$, $\forall p, q, r \in X$.

Then, $(X, \Omega)$ is said to be a $b$-metric space.

Sedghi et al. [6] defined a new form of generalized metric space, the $S$-metric space.
**Definition 2.2.**([6]) Consider $X \neq \phi$ and a function $S : X^3 \to [0, \infty)$. This function $S$ is called an $S$-metric on $X$ if:
1. $S(p, q, r) \geq 0$;
2. $S(p, q, r) = 0 \Longleftrightarrow p = q = r$;
3. $S(p, q, r) \leq S(p, p, s) + S(q, q, s) + S(r, r, s)$ $\forall p, q, r, s \in X$.

Then, $(X, S)$ is called an $S$-metric space.

**Example 2.3.**([6]) Let $X = \mathbb{R}^n$ and $||.||$ be a norm on $X$; then $S(\alpha, \beta, \gamma) = ||\beta + \gamma - 2\alpha|| + ||\beta - \gamma||$ is an $S$-metric on $X$.

Souayah and Mlaiki [7] defined the notion of an $S_b$-metric space which was later modified by Rohen et al. [8].
**Definition 2.4.**([8]) Consider $X \neq \phi$ and a function $S_b$: $X^3 \to [0, \infty)$. This function $S_b$ is said to be an $S_b$-metric on $X$ if:
1. $S_b(\daleth, \chi, \psi) = 0 \Longleftrightarrow \daleth = \chi = \psi$;
2. $S_b(\daleth, \chi, \psi) \leq b[S_b(\daleth, \daleth, s) + S_b(\chi, \chi, s) + S_b(\psi, \psi, s)]$ $\forall \daleth, \chi, \psi, s \in X$ and $b \geq 1$.

Then, $(X, S_b)$ is said to be an $S_b$-metric space.

Hussain et al.[9] conducted research on parametric metric spaces. As a follow-up, they went a step further when they presented the parametric $b$-metric space[10]. Later, Tas and Ozgur[11] defined parametric $S$-metric space as a generalized form of the parametric metric space, as defined below.
**Definition 2.5.**([11]) Consider $X \neq \phi$ and a function $\theta_t : X^3 \times (0, \infty) \to [0, \infty)$. This function $\theta_t$ is said to be a parametric $S$-metric on $X$ if:
1. $\theta_t(p, q, r, \lambda) = 0 \Longleftrightarrow p = q = r$;
2. $\theta_t(p, q, r, \lambda) \leq \theta_t(p, p, \sigma, \lambda) + \theta_t(q, q, \sigma, \lambda) + \theta_t(r, r, \sigma, \lambda), \forall p, q, r, \sigma \in X$ and $\lambda > 0$.

Then, $(X, \theta_t)$ is said to be a parametric $S$-metric space.

In 2018, Tas and Ozgur [12] advanced their earlier work by proposing the concept of a parametric $S_b$ metric space.
**Definition 2.6.**([12]) Consider $X \neq \phi$ and a function $H : X^3 \times (0, \infty) \to [0, \infty)$. This function $H$ is called a parametric $S_b$-metric on $X$ if:
1. $H(p, q, r, \lambda) = 0 \Longleftrightarrow p = q = r$;
2. $H(p, q, r, \lambda) \leq b[H(p, p, \sigma, \lambda) + H(q, q, \sigma, \lambda) + H(r, r, \sigma, \lambda)]$

where $b \geq 1$ and $\forall p, q, r, \sigma \in X, \lambda > 0$.
Then, $(X, H)$ is called a parametric $S_b$-metric space.

**Example 2.7.**([12]) Consider a collection $X = \{t | t : (0, \infty) \to \mathbb{R}\}$ and a function $H : X^3 \times (0, \infty) \to [0, \infty)$ by

$$H(t, i, o, \alpha) = \tfrac{1}{9}(|t(\alpha) - i(\alpha)| + |t(\alpha) - o(\alpha)| + |i(\alpha) - o(\alpha)|)^2$$

for each $\alpha > 0$ and $\forall t, i, o \in X$. Then $(X, H)$ is a parametric $S_b$-metric space; however, it does not qualify as a parametric $S$-metric space.

As a continuation of Rohen et al. [8]'s work, the notion of extended $S_b$-metric space was formulated by Mlaiki [13] as given below:

**Definition 2.8.**([13]). Consider $X \neq \phi$ and a function $H : X^3 \to [1, \infty)$. A function $\Sigma_H : X^3 \to [0, \infty)$ satisfying:

1. $\Sigma_H(p, q, r) = 0 \iff p = q = r$;
2. $\Sigma_H(p, q, r) \leq H(p, q, r)[\Sigma_H(p, p, \sigma) + \Sigma_H(q, q, \sigma) + \Sigma_H(r, r, \sigma)]$

$\forall p, q, r, \sigma \in X$, is said to be an extended $S_b$-metric on $X$ and $(X, \Sigma_H)$ is said to be an extended $S_b$-metric space.

**Remark 2.9.**([13]) While every $S_b$-metric space can be considered as an extended $S_b$-metric space when $H(p, q, r) = b \geq 1$, the converse isn't always the case.

The notion of an extended parametric $S_b$-metric space was established by Naveen Mani et al.[16].

**Definition 2.10.**([16]) Consider $M \neq \phi$ and a function $H : M^3 \to [1, \infty)$. A function $\Sigma_H : M^3 \times (0, \infty) \to [0, \infty)$ satisfying:

1. $\Sigma_H(p, q, r, \lambda) = 0 \iff p = q = r$;
2. $\Sigma_H(p, q, r, \lambda) \leq H(p, q, r)[\Sigma_H(p, p, \sigma, \lambda) + \Sigma_H(q, q, \sigma, \lambda) + \Sigma_H(r, r, \sigma, \lambda)]$

$\forall p, q, r, \sigma \in M$ and $\lambda > 0$ is called an extended parametric $S_b$-metric on $M$.
Then, $(M, \Sigma_H)$ is called an extended parametric $S_b$-metric space.

**Example 2.11.**([16]) Consider $M = \mathbb{R}$ and a function $H : M^3 \to [1, \infty)$ which is defined by

$$H(\phi, \chi, \psi) = 1 + |\phi| + |\chi|.$$

Again, consider another function $\Sigma_H : M^3 \times (0, \infty) \to [0, \infty)$ which is defined by

$$\Sigma_H(\phi, \chi, \psi, \lambda) = \lambda^2 \big[|\phi - \chi| + |\chi - \psi| + |\phi - \psi|\big]$$

$\forall \phi, \chi, \psi \in \mathbb{R}$ and $\lambda > 0$. Then, $\Sigma_H$ is an extended parametric $S_b$-metric on $M$.

In 2015, as a generalization of $S$-metric space, the concept of $A$-metric space was introduced by Abbas et al. [14].

**Definition 2.12.**([14]) Consider $X \neq \phi$ and a function $A : X^n \to [0, \infty)$. This function $A$ is said to be an $A$-metric on $X$ if:

1. $A(a_1, a_2, ..., a_n) \geq 0$;
2. $A(a_1, a_2, ..., a_n) = 0 \iff a_1 = a_2 = ... = a_n$;
3. $A(a_1, a_2, ..., a_n) \leq A(a_1, a_1, ..., a_{1(n-1)}, a) + A(a_2, a_2, ..., a_{2(n-1)}, a) + ...+$
$A(a_n, a_n, ..., a_{n(n-1)}, a)$, $\forall a_1, a_2, ..., a_n, a \in X$.

Then, $(X, A)$ is called an $A$-metric space.

The concept of parametric $A$-metric space was introduced by N. Priyobarta et al. [17] and the definition is as follows:

**Definition 2.13.**([17]) Consider $Y \neq \phi$ and a function $P_A : Y^n \times (0, \infty) \to [0, \infty)$. This function $P_A$ is said to be parametric $A$-metric on $Y$ if:

1. $P_A(y_1, y_2, ..., y_n, t) = 0 \iff y_1 = y_2 = ... = y_n$,
2. $P_A(y_1, y_2, ..., y_n, t) \leq P_A\big(y_1, y_1, ..., (y_1)_{n-1}, y, t\big) +$
$P_A\big(y_2, y_2, ..., (y_2)_{n-1}, y, t\big)$
$+ \quad . \quad . \quad . \quad +$
$P_A\big(y_n, y_n, ..., (y_n)_{n-1}, y, t\big)$, $\forall y_1, y_2, ..., y_n, y \in X$ and $t > 0$.

Then, $(Y, P_A)$ is said to be a parametric $A$-metric space.

**Example 2.14.**([17]) Consider a set $X = \mathbb{R}$ and a function $P_A : X^n \times (0, \infty) \to [0, \infty)$ which is defined by $P_A(\psi_1, \psi_2, ..., \psi_n, t) = h(t)\big(\big|\psi_1 - \psi_2\big| + \big|\psi_2 - \psi_3\big| + ... + \big|\psi_n - \psi_1\big|\big)$ $\forall \psi_1, \psi_2, ..., \psi_n \in \mathbb{R}$ and all $t > 0$. Here $h : (0, \infty) \to (0, \infty)$ is a continuous function. Then, $P_A$ is a parametric $A$-metric on $X$ and $(X, P_A)$ is a parametric $A$-metric space.

The concept of $A_b$-metric space was defined by Manoj et al.[15] as given below:

**Definition 2.15.**([15]) Consider a set $Y \neq \phi$ and a mapping $A_b : Y^n \to [0, \infty)$. This mapping $A_b$ is called an $A_b$-metric if:

1. $A_b(y_1, y_2, ..., y_n) \geq 0$;
2. $A_b(y_1, y_2, ..., y_n) = 0 \iff y_1 = y_2 = ... = y_n$;
3. $A_b(y_1, y_2, ..., y_n) \leq m[A_b(y_1, y_1, ..., y_{1_{(n-1)}}, y) + A_b(y_2, y_2, ..., y_{2_{(n-1)}}, y) + ...+ A_b(y_n, y_n, ..., y_{n_{(n-1)}}, y)]$, where $m \geq 1, \forall y_1, y_2, ..., y_n, y \in Y$.

Then, $(Y, A_b)$ is called an $A_b$-metric space.

**Remark 2.16.**([15]) (i) An $S_b$-metric space is essentially an $A_b$-metric space where the value of $n$ is specially 3.
(ii) Every $A$-metric space is a specific type of $A_b$-metric space when the parameter $m$ is equal to 1.

The notion of an extended parametric $A_b$-metric space was introduced by K. Anthony Singh et al.[1] as a generalization of almost all the spaces as follows:
**Definition 2.17.**[1] Let $\mathbb{K} \neq \phi$ and $\Psi : \mathbb{K}^n \to [1, \infty)$ be a function. Then, a function $A : \mathbb{K}^n \times (0, \infty) \to [0, \infty)$ is said to be extended parametric $A_b$-metric on $\mathbb{K}$ if the following conditions are satisfied for all $u_1, u_2, ..., u_n, u \in \mathbb{K}$ and $\lambda > 0$:
1. $A(u_1, u_2, ..., u_n, \lambda) = 0 \iff u_1 = u_2 = ... = u_n$;
2. $A(u_1, u_2, ..., u_n, \lambda) \leq \Psi(u_1, u_2, ..., u_n)\big[A(u_1, ..., u_1, u, \lambda) + A(u_2, ..., u_2, u, \lambda) + ...+ A(u_n, ..., u_n, u, \lambda)\big]$.

The pair $(\mathbb{K}, A)$ is said to be an extended parametric $A_b$-metric space.

**Remark 2.18.**[1] The extended parametric $A_b$-metric space is
(a) an extended parametric $S_b$-metric space when $n = 3$.
(b) an extended $S_b$-metric space when $n = 3$ and $A(u_1, u_2, u_3, \lambda) = A^0(u_1, u_2, u_3)$, $\forall u_1, u_2, u_3 \in \mathbb{K}$ and $\lambda > 0$.
(c) a parametric $A_b$-metric space when $\Psi(u_1, u_2, \ldots, u_n) = b \geq 1, \forall u_1, u_2, \ldots, u_n \in \mathbb{K}$.
(d) a parametric $A$-metric space when $\Psi(u_1, u_2, \ldots, u_n) = 1, \forall u_1, u_2, \ldots, u_n \in \mathbb{K}$.
(e) an extended $A_b$-metric space when $A(u_1, u_2, ..., u_n, \lambda) = A^0(u_1, u_2, ..., u_n)$, $\forall u_1, u_2, ..., u_n \in \mathbb{K}$ and $\lambda > 0$.
(f) an $A_b$-metric space when $A(u_1, u_2, ..., u_n, \lambda) = A^0(u_1, u_2, ..., u_n), \forall u_1, u_2, ..., u_n \in \mathbb{K}$ and $\lambda > 0$ and $\Psi(u_1, u_2, ..., u_n) = b \geq 1, \forall u_1, u_2, ..., u_n \in \mathbb{K}$ etc.

**Definition 2.19.**[1] Let $(\mathbb{K}, A)$ be an extended parametric $A_b$-metric space and let $\{u_n\}$ be a sequence in $\mathbb{K}$. Then,
1. $\{u_m\}$ is said to converge to $u \in \mathbb{K}$ if given any $\epsilon > 0, \exists\, N \in \mathbb{N}$ such that $A(u_m, ..., u_m, u, \lambda) < \epsilon\ \forall\, m \geq N$ and $\lambda > 0$.
2. $\{u_m\}$ is called Cauchy sequence if given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $A(u_l, ..., u_l, u_m, \lambda) < \epsilon$ $\forall l, m \geq N$ and $\lambda > 0$.
3. $(\mathbb{K}, A)$ is complete if each Cauchy sequence converges to a point in $\mathbb{K}$.

**Definition 2.20.**[1] An extended parametric $A_b$-metric space $(\mathbb{K}, A)$ is said to be symmetric if
$$A(u, ..., u, v, \lambda)=A(v, ..., v, u, \lambda) \text{ for all } u, v \in \mathbb{K} \text{ and } \lambda >0.$$

**Lemma 2.21.**[1] *Let $(\mathbb{K}, A)$ be a symmetric extended parametric $A_b$-metric space. Then, the limit of a convergent sequence in $\mathbb{K}$ is unique.*

**Lemma 2.22.**[1] *Let $(\mathbb{K}, A)$ be a symmetric extended parametric $A_b$-metric space. If a convergent sequence $\{u_m\}$ in $\mathbb{K}$ be such that $\lim\limits_{l,m\to\infty} \Psi(u_l, ..., u_l, u_m) < \infty$, then the sequence $\{u_m\}$ is Cauchy.*

## 3. Main Results

Here, we consider the extended parametric $A_b$-metric space introduced by K. Anthony Singh et al.[1]. We first prove a lemma which is used in the proof of our main theorem.

**Lemma 3.1.** *Let $(X, A)$ be a symmetric extended parametric $A_b$-metric space and $\{u_m\}$ be a sequence in $X$ and $k \in (0,1)$ be such that*

$$A(u_m, ..., u_m, u_{m+1}, \lambda) \leq kA(u_{m-1}, ..., u_{m-1}, u_m, \lambda), \forall m \in \mathbb{N} \text{ and } \lambda > 0. \tag{3.1}$$

*Also, let* $\lim_{l,m\to\infty} \Psi(u_l, ..., u_l, u_m) < \frac{1}{k}$. *Then, $\{u_m\}$ is a Cauchy sequence in $X$.*

*Proof*: By repeated application of (3.1), we have

$A(u_m, ..., u_m, u_{m+1}, \lambda) \leq k^m A(u_0, ..., u_0, u_1, \lambda)$.

Here, we have $\forall l \in \mathbb{N}$ and for any $p = 1, 2, 3, ...,$

$$\begin{aligned}
A(u_l, ..., u_l, u_{l+p}, \lambda) &\leq \Psi(u_l, ..., u_l, u_{l+p})\big[(n-1)A(u_l, ..., u_l, u_{l+1}, \lambda) + A(u_{l+p}, ..., u_{l+p}, u_{l+1}, \lambda)\big] \\
&= \Psi(u_l, ..., u_l, u_{l+p})\big[(n-1)A(u_l, ..., u_l, u_{l+1}, \lambda) + A(u_{l+1}, ..., u_{l+1}, u_{l+p}, \lambda)\big], \\
&\quad \{\text{since the space is symmetric, we have } A(u_{l+p}, ..., u_{l+p}, u_{l+1}, \lambda) = \\
&\quad\ A(u_{l+1}, ..., u_{l+1}, u_{l+p}, \lambda)\} \\
&\leq (n-1)\Psi(u_l, ..., u_l, u_{l+p})k^l A(u_0, ..., u_0, u_1, \lambda) + \Psi(u_l, ..., u_l, u_{l+p}) \\
&\quad \Psi(u_{l+1}, ..., u_{l+1}, u_{l+p})\big[(n-1)A(u_{l+1}, ..., u_{l+1}, u_{l+2}, \lambda) + A(u_{l+2}, ..., u_{l+2}, u_{l+p}, \lambda)\big] \\
&\leq (n-1)\Psi(u_l, ..., u_l, u_{l+p})k^l A(u_0, ..., u_0, u_1, \lambda) + (n-1)\Psi(u_l, ..., u_l, u_{l+p}) \\
&\quad \Psi(u_{l+1}, ..., u_{l+1}, u_{l+p})k^{l+1} A(u_0, ..., u_0, u_1, \lambda) + \\
&\qquad\quad ... \quad ... \quad ... \\
&\qquad\quad ... \quad ... \quad ... \\
&\qquad\quad ... \quad ... \quad ... + \\
&\quad (n-1)\Psi(u_l, ..., u_l, u_{l+p})\Psi(u_{l+1}, ..., u_{l+1}, u_{l+p})...\Psi(u_{l+p-1}, ..., u_{l+p-1}, u_{l+p}) \\
&\quad k^{l+p-1} A(u_0, ..., u_0, u_1, \lambda) \\
&\leq (n-1)A(u_0, ..., u_0, u_1, \lambda)\big[\Psi(u_l, ..., u_l, u_{l+p})k^l + \Psi(u_l, ..., u_l, u_{l+p}) \\
&\quad \Psi(u_{l+1}, ..., u_{l+1}, u_{l+p})k^{l+1} + ... + \Psi(u_l, ..., u_l, u_{l+p})\Psi(u_{l+1}, ..., u_{l+1}, u_{l+p})... \\
&\quad \Psi(u_{l+p-1}, ..., u_{l+p-1}, u_{l+p})k^{l+p-1}\big] \\
&\leq (n-1)A(u_0, ..., u_0, u_1, \lambda)\Big[\prod_{i=1}^{l} \Psi(u_i, ..., u_i, u_{l+p})k^l + \prod_{i=1}^{l+1} \Psi(u_i, ..., u_i, u_{l+p})k^{l+1} + \\
&\quad ... + \prod_{i=1}^{l+p-1} \Psi(u_i, ..., u_i, u_{l+p})k^{l+p-1}\Big] \\
&= (n-1)A(u_0, ..., u_0, u_1, \lambda) \sum_{j=l}^{l+p-1} k^j \prod_{i=1}^{j} \Psi(u_i, ..., u_i, u_{l+p}).
\end{aligned}$$

Let us now define $u_l^{(l+p)} = k^l \prod_{i=1}^{l} \Psi(u_i, ..., u_i, u_{l+p})$. Then, for any $p = 1, 2, 3, ...,$ we have

$$\begin{aligned}
\lim_{l\to\infty} \frac{u_{l+1}^{(l+p)}}{u_l^{(l+p)}} &= \lim_{l\to\infty} \frac{k^{l+1} \prod_{i=1}^{l+1} \Psi(u_i, ..., u_i, u_{l+p})}{k^l \prod_{i=1}^{l} \Psi(u_i, ..., u_i, u_{l+p})} \\
&= \lim_{l\to\infty} k\Psi(u_{l+1}, ..., u_{l+1}, u_{l+p}) \\
&= k \lim_{l\to\infty} \Psi(u_{l+1}, ..., u_{l+1}, u_{l+p}) \\
&< k.\frac{1}{k} \\
&= 1,
\end{aligned}$$

since it is given that $\lim_{l,m\to\infty} \Psi(u_l, ..., u_l, u_m) < \frac{1}{k}$.

Thus, for any $p = 1, 2, 3, ...,$ $\lim_{l\to\infty} \frac{u_{l+1}^{(l+p)}}{u_l^{(l+p)}} < 1$.

Therefore, by the Ratio Test, we have for any $p = 1, 2, 3, ...,$

$$\sum_{j=l}^{l+p-1} k^j \prod_{i=1}^{j} \Psi(u_i, ..., u_i, u_{l+p}) \to 0 \text{ as } l \to \infty.$$

This implies that, for any $p = 1, 2, 3, ...,$ $A(u_l, ..., u_l, u_{l+p}, \lambda) \to 0$ as $l \to \infty$ for any $\lambda > 0$.

Therefore, the sequence $\{u_m\}$ is Cauchy.

**Theorem 3.2.** *Suppose that $F$ and $G$ are two self-maps on a complete symmetric extended parametric $A_b$-metric space $(X, A)$ such that for each $u, v \in X$ and $\lambda > 0$, the following condition is satisfied:*

$$A(Fu, ...Fu, Gv, \lambda) \leq kM(u, ..., u, v, \lambda), \text{ and } M(u, ..., u, v, \lambda) =$$
$$max\{A(u, ..., u, v, \lambda), A(u, ..., u, Fu, \lambda), A(v, ..., v, Gv, \lambda), \frac{A(v, ..., v, Fu, \lambda)A(u, ..., u, Gv, \lambda)}{1 + A(u, ..., u, v, \lambda)}\}.$$

*Further, for any $u_0 \in X$, let $\lim_{l,m\to\infty} \Psi(u_l, ..., u_l, u_m) < \frac{1}{k}$, where $k \in (0, 1)$ and $u_1 = Fu_0, u_2 = (GF)u_0$, $u_3 = (FGF)u_0, ...$ Then, $F$ and $G$ have a unique common fixed point(UCFP).*

*Proof*: We first prove that if $u$ is a fixed point of $F$, then it is also a fixed point of $G$.
Now, we have

$$\begin{aligned} A(u, ..., u, Gu, \lambda) &= A(Fu, ..., Fu, Gu, \lambda) \\ &\leq kM(u, ..., u, u, \lambda), \end{aligned}$$

where

$$\begin{aligned} &M(u, ..., u, u, \lambda) \\ &= max\{A(u, ..., u, u, \lambda), A(u, ..., u, Fu, \lambda), A(u, ..., u, Gu, \lambda), \frac{A(u, ..., u, Fu, \lambda)A(u, ..., u, Gu, \lambda)}{1 + A(u, ..., u, u, \lambda)}\} \\ &= max\{A(u, ..., u, u, \lambda), A(u, ..., u, u, \lambda), A(u, ..., u, Gu, \lambda), \frac{A(u, ..., u, u, \lambda)A(u, ..., u, Gu, \lambda)}{1 + A(u, ..., u, u, \lambda)}\} \\ &= A(u, ..., u, Gu, \lambda). \end{aligned}$$

Therefore, we have $A(u, ..., u, Gu, \lambda) \leq kA(u, ..., u, Gu, \lambda)$
$\implies A(u, ..., u, Gu, \lambda) = 0$, since $0 < k < 1$
$\implies Gu = u$.
In a similar way, we can show that if $Gu = u$, then $Fu = u$.
Therefore, to prove our theorem, it is enough to show that either $F$ or $G$ has a unique fixed point.
Let $u_0 \in X$ be arbitrary and let a sequence $\{u_n\}$ in $X$ be constructed in such a way that

$$u_{2n+1} = Fu_{2n}, u_{2n+2} = Gu_{2n+1}, n = 0, 1, 2, 3, ....$$

Let us assume that $u_{2n} \neq u_{2n+1}$ and $u_{2n+1} \neq u_{2n+2}$ for any $n = 0, 1, 2, 3, ...$
Now, we have

$$\begin{aligned} A(u_{2n}, ..., u_{2n}, u_{2n+1}, \lambda) &= A(u_{2n+1}, ..., u_{2n+1}, u_{2n}, \lambda) \\ &= A(Fu_{2n}, ..., Fu_{2n}, Gu_{2n-1}, \lambda) \\ &\leq kM(u_{2n}, ..., u_{2n}, u_{2n-1}, \lambda), \end{aligned}$$

where

$$\begin{aligned} M(u_{2n}, ..., u_{2n}, u_{2n-1}, \lambda) &= max\{A(u_{2n}, ..., u_{2n}, u_{2n-1}, \lambda), A(u_{2n}, ..., u_{2n}, Fu_{2n}, \lambda), \\ &\quad A(u_{2n-1}, ..., u_{2n-1}, Gu_{2n-1}, \lambda), \\ &\quad \frac{A(u_{2n-1}, ..., u_{2n-1}, Fu_{2n}, \lambda)A(u_{2n}, ..., u_{2n}, Gu_{2n-1}, \lambda)}{1 + A(u_{2n}, ..., u_{2n}, u_{2n-1}, \lambda)}\} \\ &= max\{A(u_{2n}, ..., u_{2n}, u_{2n-1}, \lambda), A(u_{2n}, ..., u_{2n}, u_{2n+1}, \lambda), \\ &\quad A(u_{2n-1}, ..., u_{2n-1}, u_{2n}, \lambda), \\ &\quad \frac{A(u_{2n-1}, ..., u_{2n-1}, u_{2n+1}, \lambda)A(u_{2n}, ..., u_{2n}, u_{2n}, \lambda)}{1 + A(u_{2n}, ..., u_{2n}, u_{2n-1}, \lambda)}\} \\ &= max\{A(u_{2n-1}, ..., u_{2n-1}, u_{2n}, \lambda), A(u_{2n}, ..., u_{2n}, u_{2n+1}, \lambda)\}. \end{aligned}$$

We can consider two cases.

Case 1. $M(u_{2n}, ..., u_{2n}, u_{2n-1}, \lambda) = A(u_{2n}, ..., u_{2n}, u_{2n+1}, \lambda)$.
Case 2. $M(u_{2n}, ..., u_{2n}, u_{2n-1}, \lambda) = A(u_{2n-1}, ..., u_{2n-1}, u_{2n}, \lambda)$.

Let us suppose that case 1 is true. Then, we have
$A(u_{2n}, ..., u_{2n}, u_{2n+1}, \lambda) \leq kA(u_{2n}, ..., u_{2n}, u_{2n+1}, \lambda)$
$\implies A(u_{2n}, ..., u_{2n}, u_{2n+1}, \lambda) = 0$, since $0 < k < 1$
$\implies u_{2n} = u_{2n+1}$, which contradicts our assumption that $u_{2n} \neq u_{2n+1}$, for any $n = 0, 1, 2, 3, ....$
Therefore, case 2 must hold.
Thus, we have

$$A(u_{2n}, ..., u_{2n}, u_{2n+1}, \lambda) \leq kA(u_{2n-1}, ..., u_{2n-1}, u_{2n}, \lambda). \tag{3.2}$$

Again, we have

$$\begin{aligned}A(u_{2n+1},...,u_{2n+1},u_{2n+2},\lambda) &= A(Fu_{2n},...,Fu_{2n},Gu_{2n+1},\lambda)\\ &\leq kM(u_{2n},...,u_{2n},u_{2n+1},\lambda),\end{aligned}$$

where

$$\begin{aligned}M(u_{2n},...,u_{2n},u_{2n+1},\lambda) &= max\{A(u_{2n},...,u_{2n},u_{2n+1},\lambda), A(u_{2n},...,u_{2n},Fu_{2n},\lambda),\\ &\quad A(u_{2n+1},...,u_{2n+1},Gu_{2n+1},\lambda),\\ &\quad \frac{A(u_{2n+1},...,u_{2n+1},Fu_{2n},\lambda)A(u_{2n},...,u_{2n},Gu_{2n+1},\lambda)}{1+A(u_{2n},...,u_{2n},u_{2n+1},\lambda)}\}\\ &= max\{A(u_{2n},...,u_{2n},u_{2n+1},\lambda), A(u_{2n},...,u_{2n},u_{2n+1},\lambda),\\ &\quad A(u_{2n+1},...,u_{2n+1},u_{2n+2},\lambda),\\ &\quad \frac{A(u_{2n+1},...,u_{2n+1},u_{2n+1},\lambda)A(u_{2n},...,u_{2n},u_{2n+2},\lambda)}{1+A(u_{2n},...,u_{2n},u_{2n+1},\lambda)}\}\\ &= max\{A(u_{2n},...,u_{2n},u_{2n+1},\lambda), A(u_{2n+1},...,u_{2n+1},u_{2n+2},\lambda)\}.\end{aligned}$$

Here, we can consider two cases.

Case 1. $M(u_{2n},...,u_{2n},u_{2n+1},\lambda) = A(u_{2n+1},...,u_{2n+1},u_{2n+2},\lambda)$.
Case 2. $M(u_{2n},...,u_{2n},u_{2n+1},\lambda) = A(u_{2n},...,u_{2n},u_{2n+1},\lambda)$.

Let us suppose that case 1 is true. Then, we have
$A(u_{2n+1},...,u_{2n+1},u_{2n+2},\lambda) \leq kA(u_{2n+1},...,u_{2n+1},u_{2n+2},\lambda)$
$\Longrightarrow$ $A(u_{2n+1},...,u_{2n+1},u_{2n+2},\lambda) = 0$, since $0 < k < 1$
$\Longrightarrow$ $u_{2n+1} = u_{2n+2}$, which again contradicts our assumption that $u_{2n+1} \neq u_{2n+2}$, for any $n = 0,1,2,3,...$ .
Therefore, case 2 must hold.
Thus, we have

$$A(u_{2n+1},...,u_{2n+1},u_{2n+2},\lambda) \leq kA(u_{2n},...,u_{2n},u_{2n+1},\lambda). \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$A(u_n,...,u_n,u_{n+1},\lambda) \leq kA(u_{n-1},...,u_{n-1},u_n,\lambda), \forall n = 0,1,2,3,... \tag{3.4}$$

By repeated application of (3.4), we get

$$A(u_n,...,u_n,u_{n+1},\lambda) \leq k^n A(u_0,...,u_0,u_1,\lambda).$$

Now, by Lemma 3.1., the sequence $\{u_n\}$ is Cauchy. Since $X$ is complete, $\exists\ u \in X$ such that $u_n \to u$. Also, we have

$$\lim_{n\to\infty} Fu_{2n} = \lim_{n\to\infty} Gu_{2n+1} = u.$$

Next, we show that $Gu = u$.
Here, we have

$$A(Fu_{2n},...,Fu_{2n},Gu,\lambda) \leq kM(u_{2n},...,u_{2n},u,\lambda),$$

where

$$\begin{aligned}M(u_{2n},...,u_{2n},u,\lambda) &= max\{A(u_{2n},...,u_{2n},u,\lambda), A(u_{2n},...,u_{2n},Fu_{2n},\lambda),\\ &\quad A(u,...,u,Gu,\lambda), \frac{A(u,...,u,Fu_{2n},\lambda)A(u_{2n},...,u_{2n},Gu,\lambda)}{1+A(u_{2n},...,u_{2n},u,\lambda)}\}.\end{aligned}$$

Taking limit as $n \to \infty$, we have

$$M(u_{2n},...,u_{2n},u,\lambda) = A(u,...,u,Gu,\lambda) \text{ (in the limit)}.$$

Also, we have $A(Fu_{2n},...,Fu_{2n},Gu,\lambda) = A(u,...,u,Gu,\lambda)$ (in the limit).
Thus, we have $A(u,...,u,Gu,\lambda) \leq kA(u,...,u,Gu,\lambda)$
$\Longrightarrow$ $A(u,...,u,Gu,\lambda) = 0$, since $0 < k < 1$
$\Longrightarrow$ $Gu = u$.
Therefore, $u$ is a fixed point of $G$ and consequently $u$ is also a fixed point of $F$ i.e. $Fu = u$.

Let $v$ be another fixed point of $G$ so that $Gv = v$ and also $Fv = v$.
Then, we have
$$\begin{aligned} A(u, ..., u, v, \lambda) &= A(Fu, ..., Fu, Gv, \lambda) \\ &\leq kM(u, ..., u, v, \lambda), \end{aligned}$$
where
$$\begin{aligned} M(u, ..., u, v, \lambda) &= max\{A(u, ..., u, v, \lambda), A(u, ..., u, Fu, \lambda), \\ &\qquad A(v, ..., v, Gv, \lambda), \frac{A(v, ..., v, Fu, \lambda)A(u, ..., u, Gv, \lambda)}{1 + A(u, ..., u, v, \lambda)}\} \\ &= max\{A(u, ..., u, v, \lambda), A(u, ..., u, u, \lambda), \\ &\qquad A(v, ..., v, v, \lambda), \frac{A(v, ..., v, u, \lambda)A(u, ..., u, v, \lambda)}{1 + A(u, ..., u, v, \lambda)}\} \\ &= A(u, ..., u, v, \lambda). \end{aligned}$$
Therefore, we have $A(u, ..., u, v, \lambda) \leq kA(u, ..., u, v, \lambda)$
$\implies A(u, ..., u, v, \lambda) = 0$, since $0 < k < 1$
$\implies u = v$.
Thus, $u$ is the unique fixed point of $G$ and also of $F$. Hence, $u$ is the unique common fixed point of $F$ and $G$.

**Theorem 3.3.** *Suppose that $F$ and $G$ are two self-maps on a complete symmetric extended parametric $A_b$-metric space $(X, A)$ such that for each $u, v \in X$ and $\lambda > 0$, the following condition is satisfied:*

$$A(Fu, ...Fu, Gv, \lambda) \leq a_1 A(u, ..., u, v, \lambda) + a_2 A(u, ..., u, Fu, \lambda) + a_3 A(v, ..., v, Gv, \lambda) + a_4 \frac{A(v, ..., v, Fu, \lambda)A(u, ..., u, Gv, \lambda)}{1 + A(u, ..., u, v, \lambda)}$$

*where $a_1, a_2, a_3, a_4$ are non-negative constants such that $a_1 + a_2 + a_3 + a_4 < 1$. Further, for any $u_0 \in X$, let $\lim_{l,m\to\infty} \Psi(u_l, ..., u_l, u_m) < \frac{1}{k}$ where $u_1 = Fu_0, u_2 = (GF)u_0, u_3 = (FGF)u_0, ...$ and $k = a_1 + a_2 + a_3 + a_4$. Then, $F$ and $G$ have a UCFP.*

*Proof*: We have
$$\begin{aligned} A(Fu, ...Fu, Gv, \lambda) &\leq a_1 A(u, ..., u, v, \lambda) + a_2 A(u, ..., u, Fu, \lambda) + a_3 A(v, ..., v, Gv, \lambda), \\ &\quad + a_4 \frac{A(v, ..., v, Fu, \lambda)A(u, ..., u, Gv, \lambda)}{1 + A(u, ..., u, v, \lambda)} \\ &\leq (a_1 + a_2 + a_3 + a_4).max\{A(u, ..., u, v, \lambda), A(u, ..., u, Fu, \lambda), \\ &\quad A(v, ..., v, Gv, \lambda), \frac{A(v, ..., v, Fu, \lambda)A(u, ..., u, Gv, \lambda)}{1 + A(u, ..., u, v, \lambda)}\} \\ &= kM(u, ...u, v, \lambda), \text{ where } k = a_1 + a_2 + a_3 + a_4 < 1 \end{aligned}$$
and
$$\begin{aligned} M(u, ..., u, v, \lambda) &= max\{A(u, ..., u, v, \lambda), A(u, ..., u, Fu, \lambda), \\ &\quad A(v, ..., v, Gv, \lambda), \frac{A(v, ..., v, Fu, \lambda)A(u, ..., u, Gv, \lambda)}{1 + A(u, ..., u, v, \lambda)}\} \end{aligned}$$
Thus, we see that all the conditions of Theorem 3.2. are satisfied and it therefore follows that $F$ and $G$ have a $UCFP$.

**Corollary 3.4.** *Suppose that $F$ is a self-map on a complete symmetric extended parametric $A_b$-metric space $(X, A)$ such that for each $u, v \in X$ and $\lambda > 0$, the following condition is satisfied:*

$$A(Fu, ...Fu, Fv, \lambda) \leq kM(u, ..., u, v, \lambda), \text{ where } 0 < k < 1$$

*and*
$$M(u, ..., u, v, \lambda) = max\{A(u, ..., u, v, \lambda), A(u, ..., u, Fu, \lambda), A(v, ..., v, Fv, \lambda), \frac{A(v, ..., v, Fu, \lambda)A(u, ..., u, Fv, \lambda)}{1 + A(u, ..., u, v, \lambda)}\}.$$
*Further, for any $u_0 \in X$, let $\lim_{l,m\to\infty} \Psi(F^l u_0, ..., F^l u_0, F^m u_0) < \frac{1}{k}$. Then, $F$ has a unique fixed point.*

**Corollary 3.5.** *Suppose that $F$ is a self-map on a complete symmetric extended parametric $A_b$-metric space $(X, A)$ such that for each $u, v \in X$ and $\lambda > 0$, the following condition is satisfied:*

$$A(Fu, ...Fu, Fv, \lambda) \leq a_1 A(u, ..., u, v, \lambda) + a_2 A(u, ..., u, Fu, \lambda) + a_3 A(v, ..., v, Fv, \lambda) +$$

$$a_4 \frac{A(v, ..., v, Fu, \lambda)A(u, ..., u, Fv, \lambda)}{1 + A(u, ..., u, v, \lambda)}$$

*where $a_1, a_2, a_3, a_4$ are non-negative constants such that $a_1 + a_2 + a_3 + a_4 < 1$. Further, for any $u_0 \in X$, let $\lim_{l,m\to\infty} \Psi(F^l u_0, ..., F^l u_0, F^m u_0) < \frac{1}{k}$, where $k = a_1+a_2+a_3+a_4 < 1$. Then, $F$ has a unique fixed point.*

**3.6. Some Consequences of the Theorems**

1. If $a_2 = a_3 = a_4 = 0$ in Theorem 3.3, we get a generalization of Banach Contraction Principle for two mappings in the setting of an extended parametric $A_b$-metric space.

If we further set $n = 3$, then we get the corresponding result in the setting of an extended parametric $S_b$-metric space.

2. If $a_1 = a_4 = 0$ and $a_2 = a_3 = a$ in Theorem 3.3, we get a generalization of Kannan fixed point theorem[19] for two mappings in the setting of an extended parametric $A_b$-metric space.

If we further set $n = 3$, then we get the corresponding result in the setting of an extended parametric $S_b$-metric space.

3. If $a_2 = a_3 = a_4 = 0$ in Corollary 3.5, then we get the analog on Banach Contraction Principle in the setting of an extended parametric $A_b$-metric space.

If we further set $n = 3$, then we get the corresponding result in the setting of an extended parametric $S_b$-metric space.

4. If $a_1 = a_4 = 0$ and $a_2 = a_3 = a$ in Corollary 3.5, then we get the analog of Kannan fixed point Theorem[19] in the setting of an extended parametric $A_b$-metric space.

If we further set $n = 3$, then we get the corresponding result in the setting of an extended parametric $S_b$-metric space.

5. If $a_2 = a_3 = 0$ in Corollary 3.5, then we get an extension of Jaggi fixed point Theorem[20] in the setting of an extended parametric $A_b$-metric space.

If we further set $n = 3$, then we get the corresponding result in the setting of an extended parametric $S_b$-metric space.

## 4. Application

### 4.1. Solving a system of Integral Equations for Common Solution

Let $\nabla = C[\varsigma, \vartheta]$ be the set of all real valued continuous functions defined on $[\varsigma, \vartheta]$. For any $\partial > 0$ and $\forall$ $\ell_1, \ell_2, ..., \ell_n \in \nabla$, let us define $A : \nabla^n \times (0, \infty) \to [0, \infty)$ by
$A(\ell_1, \ell_2, ..., \ell_n, \partial) = \partial \sup_{\gimel \in [\varsigma, \vartheta]} \Big|\max\Big\{\ell_1(\gimel), \ell_2(\gimel), ..., \ell_{n-1}(\gimel)\Big\} - \ell_n(\gimel)\Big|^2$ and $\Psi : \nabla^n \to [1, \infty)$ by
$\Psi(\ell_1, \ell_2, ..., \ell_n) = \sup_{\gimel \in [\varsigma, \vartheta]} \Big|\max\{\ell_1(\gimel), \ell_2(\gimel), ..., \ell_{n-1}(\gimel)\} - \ell_n(\gimel)\Big| + 1$. Then, it is easy to see that $(\nabla, A)$ is a complete extended parametric $A_b$-metric space. The space is also symmetric because

$$\begin{aligned} A(\ell, ..., \ell, \wp, \partial) &= \partial \sup_{\gimel \in [\varsigma, \vartheta]} \Big|\max\{\ell(\gimel), \ell(\gimel), ..., \ell(\gimel)\} - \wp(\gimel)\Big|^2 \\ &= \partial \sup_{\gimel \in [\varsigma, \vartheta]} \Big|\ell(\gimel) - \wp(\gimel)\Big|^2 \end{aligned}$$

and
$$\begin{aligned} A(\wp, ..., \wp, \ell, \partial) &= \partial \sup_{\gimel \in [\varsigma, \vartheta]} \Big|\max\{\wp(\gimel), \wp(\gimel), ..., \wp(\gimel)\} - \ell(\gimel)\Big|^2 \\ &= \partial \sup_{\gimel \in [\varsigma, \vartheta]} \Big|\ell(\gimel) - \wp(\gimel)\Big|^2, \end{aligned}$$

implying that $A(\ell, ..., \ell, \wp, \partial) = A(\wp, ..., \wp, \ell, \partial), \forall \ell, \wp \in \nabla$ and $\partial > 0$.

**Theorem 4.2.** *Let us consider the following system of integral equations:*
$\ell(\gimel) = \varrho(\gimel) + \lambda \int_\varsigma^\vartheta \Gamma_1(\gimel, \delta, \wp(\delta))d\delta$
$\wp(\gimel) = \varrho(\gimel) + \lambda \int_\varsigma^\vartheta \Gamma_2(\gimel, \delta, \ell(\delta))d\delta,$
*where, (i) $\Gamma_1, \Gamma_2 : [\varsigma, \vartheta] \times [\varsigma, \vartheta] \times \mathbb{R} \to \mathbb{R}$ are continuous,*

*(ii)* $\left|\Gamma_1(\beth,\delta,\wp(\delta))-\Gamma_2(\beth,\delta,\ell(\delta))\right| \leq \dfrac{\sqrt{l}}{|\lambda|(\vartheta-\varsigma)}\left|\wp(\delta)-\ell(\delta)\right|$, $\forall \beth,\delta \in [\varsigma,\vartheta]$ *and* $\ell,\wp \in \nabla$, *for some* $l \in [0,1)$.

*Then, the system of Integral Equations has a unique common solution in* $\nabla$.

*Proof*:Let us define the self maps $\Omega$ and $\Sigma$ on $\nabla$ by

$$\Omega\ell(\beth) = \varrho(\beth) + \lambda \int_\varsigma^\vartheta \Gamma_1(\beth,\delta,\wp(\delta))d\delta \text{ and}$$
$$\Sigma\wp(\beth) = \varrho(\beth) + \lambda \int_\varsigma^\vartheta \Gamma_2(\beth,\delta,\ell(\delta))d\delta.$$

Then, we have

$$\begin{aligned}
A(\Omega\ell,...,\Omega\ell,\Sigma\wp,\partial) &= \partial \sup_{\beth\in[\varsigma,\vartheta]}\left|max\left\{\Omega\ell(\beth),...,\Omega\ell(\beth)\right\}-\Sigma\wp(\beth)\right|^2 \\
&= \partial \sup_{\beth\in[\varsigma,\vartheta]}\left|\Omega\ell(\beth)-\Sigma\wp(\beth)\right|^2 \\
&= \partial \sup_{\beth\in[\varsigma,\vartheta]}\Big|\lambda \int_\varsigma^\vartheta \Big(\Gamma_1\big(\beth,\delta,\wp(\delta)\big)d\delta - \\
&\quad \int_\varsigma^\vartheta \Gamma_2\big(\beth,\delta,\ell(\delta)\big)\Big)d\delta\Big|^2 \\
&\leq \partial|\lambda|^2 \sup_{\beth\in[\varsigma,\vartheta]}\Big(\int_\varsigma^\vartheta\Big|\Gamma_1\big(\beth,\delta,\wp(\delta)\big)- \\
&\quad \Gamma_2\big(\beth,\delta,\ell(\delta)\big)\Big|d\delta\Big)^2 \\
&\leq \partial|\lambda|^2\Big(\int_\varsigma^\vartheta \frac{\sqrt{l}}{|\lambda|(\vartheta-\varsigma)}\big|\wp(\delta)-\ell(\delta)\big|d\delta\Big)^2 \\
&= \partial|\lambda|^2 . \tfrac{l}{|\lambda|^2(\vartheta-\varsigma)^2}\Big(\int_\varsigma^\vartheta\big|\wp(\delta)-\ell(\delta)\big|d\delta\Big)^2 \\
&\leq \frac{l}{(\vartheta-\varsigma)^2}.\partial \sup_{\delta\in[\varsigma,\vartheta]}\left|\ell(\delta)-\wp(\delta)\right|^2\Big(\int_\varsigma^\vartheta d\delta\Big)^2 \\
&= \frac{l}{(\vartheta-\varsigma)^2}.A(\ell,...,\ell,\wp,\partial).(\vartheta-\varsigma)^2 \\
&= lA(\ell,...\ell,\wp,\partial) \\
&\leq l\Big\{max\big(A(\ell,...,\ell,\wp,\partial),A(\ell,...,\ell,\Omega\ell,\partial),A(\wp,...,\wp,\Sigma\wp,\partial), \\
&\quad \tfrac{A(\wp,...,\wp,\Omega\ell,\partial)A(\ell,...,\ell,\Sigma\wp,\partial)}{1+A(\ell,...,\ell,\wp,\partial)}\big)\Big\} \\
&= lM(\ell,...,\ell,\wp,\partial)
\end{aligned}$$

Hence, all the conditions of Theorem 3.2. being satisfied, $\Omega$ and $\Sigma$ have a unique common fixed point $\ell \in C[\varsigma,\vartheta]$ which is the unique common solution of the system of Integral equations.

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**References**

1. K. A. Singh, W. S. Singh, L. Shanjit, T. C. Singh, and K. M. Devi, *On some fixed points of symmetric extended parametric $A_b$-metric space with an application to integral equations*, (communicated).
2. K. A. Singh, M. Singh, and T. C. Singh, *Some fixed point theorems in cone $A_b$-metric space*, Electronic Journal of Mathematical Analysis & Applications, vol. 9, no. 1, (2021).
3. A. Bakhtin I, *The contraction mapping principle in almost metric spaces*, Funct. Anal., Gos. Ped. Inst. Unianowsk, vol. 30, pp. 26–37, (1989).
4. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta mathematica et informatica universitatis ostraviensis, vol. 1, no. 1, pp. 5–11, (1993).
5. S. Beniwal, N. Mani, R. Shukla, and A. Sharma, *Fixed point results for compatible mappings in extended parametric $S_b$-metric spaces*, Mathematics, vol. 12, no. 10, p. 1460, (2024).
6. S. Sedghi, N. Shobe, and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Matematički vesnik, vol. 64, no. 249, pp. 258–266, (2012).

7. N. Souayah and N. Mlaiki, *A fixed point theorem in $S_b$-metric spaces*, J. Math. Computer Sci, vol. 16, no. 315, pp. 131–139, (2016).

8. Y. Rohen, T. Došenović, and S. Radenović, *A note on the paper "A fixed point theorems in $S_b$-metric space"*, *Filomat*, vol. 31, no. 11, pp. 3335–3346, (2017).

9. H. N, K. S, S. P, and A. A.A.N, *A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces*, Abstr. Appl. Anal., vol. 16, no. 690139, (2014).

10. N. Hussain, P. Salimi, and V. Parvaneh, *Fixed point results for various contractions in parametric and fuzzy b-metric spaces*, J. Nonlinear Sci. Appl, vol. 8, no. 5, pp. 719–739, (2015).

11. N. Taş and N. Yılmaz Özgür, *On parametric S-metric spaces and fixed-point type theorems for expansive mappings*, Journal of mathematics, vol. 2016, no. 1, p. 4746732, (2016).

12. N. Tas and N. Y. Ozgur, *Some fixed-point results on parametric $N_b$-metric spaces*, Commun. Korean Math. Soc., 33(2018), No. 3, pp. 943–960, (2018).

13. N. Mlaiki, *Extended $S_b$-metric spaces*, J. Math. Anal, vol. 9, no. 1, pp. 124–135, (2018).

14. M. Abbas, B. Ali, and Y. I. Suleiman, *Generalized coupled common fixed point results in partially ordered A-metric spaces*, Fixed point theory and applications, vol. 2015, no. 1, p. 64, (2015).

15. M. Ughade, D. Turkoglu, S. Singh, and R. Daheriya, *Some fixed point theorems in $A_b$-metric space*, British Journal of Mathematics & Computer Science, vol. 19, no. 6, pp. 1–24, (2016).

16. N. Mani, S. Beniwal, R. Shukla, and M. Pingale, *Fixed point theory in extended parametric $S_b$-metric spaces and its applications*, Symmetry, vol. 15, no. 12, p. 2136, (2023).

17. N. Priyobarta, Y. Rohen, and S. Radenovic, *Fixed point theorems on parametric A-metric space*, American Journal of Applied Mathematics and Statistics, vol. 6, no. 1, pp. 1–5, (2018).

18. N. Mani, A. Somal, S. Beniwal, and R. Shukla, *Solving systems of linear integral equations via fixed point theory in extended parametric $S_b$-metric spaces*, International Journal of Differential Equations, vol. 2025, no. 1, p. 1226661, (2025).

19. R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc., vol. 60, pp. 71–76, (1968).

20. D. Jaggi, *Some unique fixed point theorems*, Indian J. Pure Appl. Math., vol. 8, no. 2, pp. 223–230, (1977).

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