Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.82261

Patterns of Integer Solutions to Non-Homogeneous Ternary Nonic Equation $x^2 + y^2 = 125z^9$

N. Thiruniraiselvi [1], J. Shanthi [2*], M. A. Gopalan [3]

ABSTRACT: This paper aims at determining patterns of non-zero distinct integer solutions to nonhomogeneous ternary nonic Diophantine equation $x^2 + y^2 = 125z^9$. Substitution technique and factorization method are utilized to obtain the same. We are able to solve the given higher degree equation through the methods suggested above for obtaining plenty of non-zero distinct integer solutions.

Keywords: Nonic Diophantine equation,Ternary nonic equation, substitution technique, factorization method, integer solutions.

Contents

1. Introduction

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree Diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power Diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of Cubic, Quintic ,Heptic and Nonic Diophantine equations with multi variables [1-33]. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree nine with three unknowns given by $x^2 + y^2 = 125z^9$.

2. Method of Analysis

The non-homogeneous nonic polynomial equation of degree nine with three unknowns to be solved for its integer solutions is given by

$$x^2 + y^2 = 125z^9 \tag{2.1}$$

Introduction of the transformations

$$x = m\left(m^2 + n^2\right), y = n\left(m^2 + n^2\right) \tag{2.2}$$

in (2.1) leads to the non-homogeneous ternary cubic equation

$$m^2 + n^2 = 5z^3 \tag{2.3}$$

2020 *Mathematics Subject Classification:* 11D99.
Submitted February 28, 2026. Published April 09, 2026

The process of obtaining patterns of integer solutions to (2.1) through solving (2.3) is illustrated below:

Pattern 2.1

$$z = a^2 + b^2 \tag{2.4}$$

Write the integer 5 on the R.H.S. of (2.3) as the product of complex conjugates as given by

$$5 = (2+i)(2-i) \tag{2.5}$$

Substitute (2.4) & (2.5) in (2.3). Applying the method of factorization and equating the positive factors, one has

$$\begin{aligned} m + in &= (2+i)(a+ib)^3 \\ &= (2+i)[f(a,b) + ig(a,b)] \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} f(a,b) &= a^3 - 3ab^2 \\ g(a,b) &= 3a^2b - b^3 \end{aligned} \tag{2.7}$$

On equating the coefficients of corresponding terms on both sides of (2.6),we have

$$\begin{aligned} m &= 2f(a,b) - g(a,b) \\ n &= f(a,b) + 2g(a,b) \end{aligned}$$

In view of (2.2),we get

$$\begin{aligned} x &= 5[2f(a,b) - g(a,b)]\left[\{f(a,b)\}^2 + \{g(a,b)\}^2\right] \\ y &= 5[f(a,b) + 2g(a,b)]\left[\{f(a,b)\}^2 + \{g(a,b)\}^2\right] \end{aligned} \tag{2.8}$$

Thus,(2.4) & (2.8) satisfy (2.1).
A few numerical solutions to (2.1) are given in Table-2.1 below:

Table-2.1-Numerical solutions

a	b	f(a,b)	g(a,b)	x	y	z
1	1	-2	2	5*(-6)*8	5*2*8	2
2	2	-16	16	5*(16)*512	5*16*512	8
2	1	2	11	5*(-7)*125	5*24*125	5
1	2	-11	-2	5*(20)*125	5*(-15)*125	5
3	1	18	26	5*10*1000	5*70*1000	10

Note 2.1

In addition to (2.5), the integer 5 is given by

$$5 = (1+2i)(1-2i)$$

For this choice, we have

$$m = f(a,b) - 2g(a,b)$$
$$n = 2f(a,b) + g(a,b$$

The respective values of x, y are given by

$$x = 5[f(a,b) - 2g(a,b)]\left[\{f(a,b)\}^2 + \{g(a,b)\}^2\right]$$
$$y = 5[2f(a,b) + g(a,b)]\left[\{f(a,b)\}^2 + \{g(a,b)\}^2\right]$$

Note 2.2

Apart from (2.2), one may consider the following transformations

$$x = m\left(m^2 - 3n^2\right)$$
$$y = n\left(3m^2 - n^2\right)$$

leading to (2.3) , but giving a different set of integer solutions.

Pattern 2.2

Write (2.3) as

$$m^2 + n^2 = 5z^3 * 1 \tag{2.9}$$

Consider the integer 1 on the R.H.S. of (2.9) as the product of complex conjugates as presented below:

$$1 = \frac{\left(\mathrm{p}^2 - \mathrm{q}^2 + \mathrm{i2pq}\right)\left(\mathrm{p}^2 - \mathrm{q}^2 - \mathrm{i2pq}\right)}{\left(\mathrm{p}^2 + \mathrm{q}^2\right)^2} \tag{2.10}$$

Assume

$$\mathrm{z} = \left(\mathrm{p}^2 + \mathrm{q}^2\right)^2\left(\mathrm{a}^2 + \mathrm{b}^2\right) \tag{2.11}$$

Substitute (2.5), (2.10) & (2.11) in (2.9). Employing the factorization method and equating the positive factors on both sides,we have

$$\begin{aligned} m + in &= (2+i)\left(p^2+q^2\right)^3[f(a,b) + ig(a,b)]\frac{\left(p^2 - q^2 + i2pq\right)}{\left(p^2+q^2\right)} \\ &= \left(p^2+q^2\right)^2\{[2f(a,b) - g(a,b)] + i[f(a,b) + 2g(a,b)]\}\left(p^2 - q^2 + i2pq\right) \end{aligned} \tag{2.12}$$

On equating the real and imaginary parts in (2.12), we have

$$m = \left(p^2+q^2\right)^2\left\{[2f(a,b) - g(a,b)]\left(p^2 - q^2\right) - 2pq[f(a,b) + 2g(a,b)]\right\}$$
$$n = \left(p^2+q^2\right)^2\left\{[2f(a,b) - g(a,b)](2pq) + \left(p^2 - q^2\right)[f(a,b) + 2g(a,b)]\right\}$$

In view of (2.2) & (2.11), we obtain the corresponding integer solutions to (2.1).

Pattern 2.3

Taking

$$y = 11z^4 \tag{2.13}$$

in (2.1), we get

$$x^2 = z^8(125z - 121) \tag{2.14}$$

After some algebra , it is seen that (2.14) is satisfied by the values of z & x given by

$$\begin{aligned} z_n &= 1 + 4n + 125n^2 \\ x_n &= \left(1 + 4n + 125n^2\right)^4 [2 + 125n] \end{aligned} \tag{2.15}$$

From (2.13), one has

$$y_n = 11\left(1 + 4n + 125n^2\right)^4 \tag{2.16}$$

Thus, the values of x, y, z given by (2.15) & (2.16) satisfy (2.1).

Pattern 2.4

Taking

$$y = 2z^4 \tag{2.17}$$

in (2.1), we get

$$x^2 = z^8(125z - 4) \tag{2.18}$$

After some algebra , it is seen that (2.18) is satisfied by the values of z & x given by

$$\begin{aligned} z_n &= 1 + 22n + 125n^2 \\ x_n &= \left(1 + 22n + 125n^2\right)^4 [11 + 125n] \end{aligned} \tag{2.19}$$

From (2.17), one has

$$y_n = 2\left(1 + 22n + 125n^2\right)^4 \tag{2.20}$$

Thus , the values of x, y, z given by (2.19) & (2.20) satisfy (2.1).

Pattern 2.5

Taking

$$y = 5z^4 \tag{2.21}$$

in (2.1), we get

$$x^2 = z^8(125z - 25) \tag{2.22}$$

After some algebra , it is seen that (2.22) is satisfied by the values of z & x given by

$$\begin{aligned} z_n &= 1 + 20n + 125n^2 \\ x_n &= \left(1 + 20n + 125n^2\right)^4 [10 + 125n] \end{aligned} \tag{2.23}$$

From (2.21), one has

$$y_n = 5\left(1+20n+125n^2\right)^4 \tag{2.24}$$

Thus , the values of x, y, z given by (2.23) & (2.24) satisfy (2.1).

Pattern 2.6

Taking

$$\mathrm{y} = 10\mathrm{z}^4 \tag{2.25}$$

in (2.1), we get

$$x^2 = z^8(125z-100) \tag{2.26}$$

After some algebra , it is seen that (2.26) is satisfied by the values of z & x given by

$$z_n = 1+10n+125n^2 \tag{2.27}$$
$$x_n = \left(1+10n+125n^2\right)^4[5+125n]$$

From (2.25), one has

$$y_n = 10\left(1+10n+125n^2\right)^4 \tag{2.28}$$

Thus , the values of x, y, z given by (2.27) & (2.28) satisfy (2.1).

3. Conclusion

The polynomial equation of degree nine with three unknowns given by $x^2+y^2=125z^9$ has been studied to obtain non-zero integer solution. As nonic equations are plenty, one may attempt to determine the solutions in integers for other choices of nonic Diophantine equations. Solving the higher degree Diophantine equation and finding the non-zero distinct integer solutions are used in various fields like cryptography, Number patterns. The concepts of Diophantine equations are encouraging the young researchers to discover new ideas in various fields like some mentioned above. To conclude, one may search for different types of higher degree Diophantine equations and their solutions.

Acknowledgments

We thank the referee for the suggestions.

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[1]*Associate Professor, Department of Mathematics,*
Sri Ramakrishna College of Engineering,
Affiliated to Anna University,
Chennai, Saradha Nagar,
Perambalur-621 113, Tamil Nadu, India.

E-mail address: drntsmaths@gmail.com

and

[2*]*Assistant Professor, Department of Mathematics,*
Shrimati Indira Gandhi College,
Affiliated to Bharathidasan University,
Tiruchirappalli-620002, Tamil Nadu, India.
E-mail address: shanthitharshi@gmail.com

and

[3]*Professor, Department of Mathematics,*
Shrimati Indira Gandhi College,
Affiliated to Bharathidasan University,
Tiruchirappalli-620002, Tamil Nadu, India.
E-mail address: mayilgopalan@gmail.com