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Characterizations of $e^{\star}$-Open Sets and Nearby Open Sets on Infra Topological Spaces

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ABSTRACT: The study of infra-topological spaces focuses on characterizations of $e^{\star}$-open sets and nearby open sets in infra-topological spaces. The $e^{\star}$-open sets, a variation of open sets, are explored for their unique properties and relationships within the infra-topological framework. Additionally, nearby open sets, which capture the notion of points being close to each other, are investigated to provide a comprehensive understanding of the topological structure. The research aims to contribute to the broader field of topology by extending traditional concepts to infra-topological spaces, offering new perspectives on openness and proximity. The findings not only deepen our understanding of mathematical structures but also open avenues for applications in various scientific and engineering disciplines.

Keywords: e-open set, $e^{\star}$-open sets, infra topological spaces.

Contents

1. Introduction

A seminal work by S. Mashhour et al. [9] laid the foundation for supra topological spaces, wherein they delved into the intricacies of s-continuous functions and s^{*}-continuous functions. The groundbreaking exploration of supra topological spaces marked a significant contribution to the field.

Building upon this foundation, Adel. M. Al. Odhari [1] extended the theoretical landscape by introducing and studying infra topological spaces. Odhari's work included a comprehensive examination of open sets within infra topological spaces, shedding light on their fundamental properties.

The fundamental concepts and classical results of general topology that underpin the present study can be found in standard references such as Brown [2], Dugundji [3], and Munkres [10]. In addition, the notion of nearly open sets, which plays a significant role in the development of generalized open sets, was introduced by Njåstad [11]. These foundational works provide the essential background for the investigation carried out in this paper.

E. Ekici, in a series of noteworthy contributions [4,5,6,7,8], introduced and extensively investigated the properties of e and $e^{\star}$ along with nearby open sets in the context of general topological spaces. Ekici's research provided valuable insights into the behavior of these sets, enriching the understanding of topological structures.

Motivated by these advancements, our present work focuses on extending the study of e and $e^{\star}$ open sets to infra topological structures. We embark on an in-depth investigation, exploring the properties and characteristics of these sets within the specific framework of infra topological spaces. To bolster our findings, we present concrete examples that not only validate our assumptions but also serve as illustrative instances of the nuanced interplay between infra topological structures and the introduced open sets. Through this research, we aim to contribute to the ongoing dialogue surrounding the interplay of different topological spaces and the behavior of distinct open sets within them.

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2. Preliminaries

Definition 2.1 [1] Let $\mathbb{X}$ be any arbitrary set. An infra topological space ($\mathbb{ITS}$) on $\mathbb{X}$ is a collection, infra topology ($\mathbb{IT}$), of subsets of $\mathbb{X}$ such that the following axioms are satisfied:

(i) $\emptyset,\ \mathbb{X} \in \mathbb{IT},.$

(ii) The intersection of the elements of any subcollection of $\mathbb{IT}$ in $\mathbb{X}$.

i.e, If $O_i \in \mathbb{IT},\ 1 \subseteq i \subseteq n,\ \cap O_i \in \mathbb{IT}$.

Terminology, the ordered pair $(\mathbb{X}, \mathbb{IT})$ is called $\mathbb{ITS}$. we simply say $\mathbb{X}$ is a $\mathbb{ITS}$.

Definition 2.2 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. A is called infra open set ($\mathbb{IOS}$) if $A \in \mathbb{IT}$. The complement of $\mathbb{IOS}$ is called infra closed set ($\mathbb{ICS}$).

Definition 2.3 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be $\mathbb{ITS}$. A subset $C \subset \mathbb{X}$ is called $\mathbb{ICS}$ in $\mathbb{X}$ if $\mathbb{X} - C$ is $\mathbb{IOS}$ in $\mathbb{X}$. That is C is $\mathbb{ICS}$ iff $\mathbb{X} - C \in \mathbb{IT}$.

Theorem 2.1 [1] let $(X,\ \mathbb{IT})$ and $(\mathbb{X},\ \mathbb{IT}^*)$ be two $\mathbb{ITS}'$s on set $\mathbb{X}$. Then the intersection $\mathbb{IT}$ and $\mathbb{IT}^*$ is an $\mathbb{ITS}$, while the union $\mathbb{IT}$ and $\mathbb{IT}^*$ not necessarily.

Definition 2.4 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. A point $x \in \mathbb{X}$ is called Infra-Cluster Point ($\mathbb{IC}_{\mathbb{P}}$) of A, if for all $\mathbb{IOS}$ O containing x, then $A \cap (O \backslash \{x\}) \neq \emptyset$.

Definition 2.5 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. The set of all $\mathbb{IC}_{\mathbb{P}}$ of A is called the Infra Derived Set ($\mathbb{IDS}$) of A.

Theorem 2.2 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be $\mathbb{ITS}$. Then:

(i) $\emptyset,\ \mathbb{X} \in \mathbb{IT}$ are $\mathbb{ICS}$.

(ii) Any arbitrary finite intersections of $\mathbb{ICS}'$s is an $\mathbb{ICS}'$s.

Definition 2.6 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. The infra closure ($\mathbb{I.CL}$) of A is a set denoted by $\mathbb{I.CL}(A)$ and given by: $\mathbb{I.CL}(A) = \bigcap\{C_i : A \subset C_i,\ \mathbb{X} - C_i \in \mathbb{IT}\}$.That is, $\mathbb{I.CL}(A)$ is the intersection of all $\mathbb{ICS}$ containing the set A.

Definition 2.7 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. The Infra Interior ($\mathbb{I.INT}$) of A is a set denoted by $\mathbb{I.INT}(A)$ and given by: $\mathbb{I.INT}(A) = \bigcup\{O_i : O_i \subset A,\ O_i \in \mathbb{IT}\}$.That is, $\mathbb{I.INT}(A)$ is the union of all $\mathbb{IOS}$ contained in the set A.

Definition 2.8 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. The Infra Exterior Point ($\mathbb{IEP}$) of A is a set denoted by $\mathbb{IEP}(A)$ and given by: $\mathbb{IEP}(A) = \mathbb{I.INT}(A^c)$.That is, Set of all $\mathbb{I.INT}$ of complement of A.

Definition 2.9 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. The Infra-Boundary Points ($\mathbb{IBP}$) of A is a set denoted by $\mathbb{IBP}(A)$ and given by: $\mathbb{IBP}(A) = \mathbb{X} \backslash \mathbb{I.INT}(A) \cup \mathbb{IEP}(A)$

Theorem 2.3 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A,\ B \subset \mathbb{X}$. The $\mathbb{IDS}$ Axioms satisfies the followings:

(i) $\mathbb{IDS}(\emptyset) = \emptyset$.

(ii) If $A \subset B$ then $\mathbb{IDS}(A) \subset \mathbb{IDS}(B)$

(iii) If $x \in \mathbb{IDS}(A)$ then $x \in \mathbb{IDS}(A \backslash \{x\})$.

(iv) $\mathbb{IDS}(A \cap B) \subset \mathbb{IDS}(A) \cap \mathbb{IDS}(B)$.

(v) $\mathbb{IDS}(A \cup B) = \mathbb{IDS}(A) \cup \mathbb{IDS}(B)$.

Theorem 2.4 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset \mathbb{X}$.The $\mathbb{I}.\mathbb{CL}$ Axioms satisfying the following conditions:

(i) A is $\mathbb{ICS}$ iff $A = \mathbb{I}.\mathbb{CL}(A)$.

(ii) $\mathbb{I}.\mathbb{CL}(\emptyset) = \emptyset$ and $\mathbb{I}.\mathbb{CL}(\mathbb{X}) = \mathbb{X}$.

(iii) $\mathbb{I}.\mathbb{CL}(\mathbb{I}.\mathbb{CL}(A)) = \mathbb{I}.\mathbb{CL}(A)$.

(iv) If $A \subset B$ then $\mathbb{I}.\mathbb{CL}(A) \subset \mathbb{I}.\mathbb{CL}(B)$.

(v) $\mathbb{I}.\mathbb{CL}(A \cap B) \subset \mathbb{I}.\mathbb{CL}(A) \cap \mathbb{I}.\mathbb{CL}(B)$.

Theorem 2.5 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset \mathbb{X}$. The $\mathbb{I}.\mathbb{INT}$ Axioms given by:

(i) A is $\mathbb{IOS}$ iff $A = \mathbb{I}.\mathbb{INT}(A)$.

(ii) $\mathbb{I}.\mathbb{INT}(\mathbb{X}) = \mathbb{X}$ and $\mathbb{I}.\mathbb{INT}(\emptyset) = \emptyset$.

(iii) $\mathbb{I}.\mathbb{INT}(\mathbb{I}.\mathbb{INT} \subset (A)) = \mathbb{I}.\mathbb{INT}(A)$.

(iv) If $A \subset B$ then $\mathbb{I}.\mathbb{INT}(A) \subset \mathbb{I}.\mathbb{INT}(B)$.

(v) $\mathbb{I}.\mathbb{INT}(A \cap B) = \mathbb{I}.\mathbb{INT}(A) \cap \mathbb{I}.\mathbb{INT}(B)$.

Theorem 2.6 [1] Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset \mathbb{X}$. The $\mathbb{IEP}$ Axioms given by:

(i) $\mathbb{IEP}(\mathbb{X}) = \emptyset$ and $\mathbb{IEP}(\emptyset) = X$.

(ii) $\mathbb{IEP}(A) \subset A^c$.

(iii) $\mathbb{IEP}(A \cup B) = \mathbb{IEP}(A) \cap \mathbb{IEP}(B)$.

(iv) If $A \subset B$, then $\mathbb{IEP}(B) \subset \mathbb{IEP}(A)$.

(v) $\mathbb{IEP}(A \cap B) \subset \mathbb{IEP}(A) \cup \mathbb{IEP}(B)$.

Theorem 2.7 [1] Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. The $\mathbb{IB}$ Axioms given by:

(i) $\mathcal{IBP}(X) = \mathcal{IBP}(\emptyset) = \emptyset$.

(ii) $\mathcal{IBP}(A \cap B) = \mathcal{IBP}(A) \cup \mathcal{IBP}(B)$.

Theorem 2.8 [1] Let $(X,\ \mathbb{IT})$ be an (ITS) and $A \subset \mathbb{X}$. Then:

(i) $A \subset \mathcal{I}.\mathcal{CL}(A) \to \mathcal{IDS}(A) \subset \mathcal{IDS}(\mathcal{I}.\mathcal{CL}(A))$.

(ii) $\mathcal{I}.\mathcal{INT}(A) \subset A \to \mathcal{IDS}(\mathcal{I}.\mathcal{INT}(A)) \subset \mathcal{IDS}(A)$.

(iii) If A is $\mathcal{ICS}$, then $\mathcal{IDS}(A) \subset A$.

(iv) $\mathcal{I}.\mathcal{CL}(A) = A \cup \mathcal{IDS}(A)$.

(v) $\mathcal{IB}(A) = \mathcal{I}.\mathcal{CL}(A) \backslash \mathcal{I}.\mathcal{INT}(A)$.

(vi) $\mathcal{I}.\mathcal{CL}(A) = \mathcal{IBP}(A) \cup \mathcal{I}.\mathcal{INT}(A)$.

(vii) $\mathcal{IBP}(A) \subset \mathcal{I}.\mathcal{CL}(A)$.

(viii) $\mathcal{I}.\mathcal{INT}(A) \cap \mathcal{IBP}(A) = \emptyset$.

3. Characterizations of e-Open and Nearby Open Sets Sets

Definition 3.1 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. The Infra δ Interior $(\mathbb{I}.\mathbb{INT}_\delta)$ of A is a set denoted by $\mathbb{I}.\mathbb{INT}_\delta(A)$ and given by: $\mathbb{I}.\mathbb{INT}_\delta(A) = \bigcup\{O_i : O_i \subset A,\ O_i \in \mathbb{I}.\mathbb{ROS}\}$.

Definition 3.2 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. The infra δ closure $(\mathbb{I}.\mathbb{CL}_\delta)$ of A is a set denoted by $\mathbb{I}.\mathbb{CL}_\delta(A)$ and given by: $\mathbb{I}.\mathbb{CL}_\delta(A) = \bigcap\{C_i : A \subset C_i,\ \mathbb{X} - C_i \in \mathbb{I}.\mathbb{RCS}\}$.

Definition 3.3 Let $(\mathbb{X},\ \mathbb{IT})$ be $\mathbb{ITS}$. Then a Set A is said to be Infra

- (i) regular open set($\mathbb{I}.\mathbb{ROS}$) if $A = \mathbb{I}.\mathbb{INT}(\mathbb{I}.\mathbb{CL}(A))$
- (ii) δ-open set ($\mathbb{I}.\delta\mathbb{OS}$) if $A = \mathbb{I}.\mathbb{INT}_\delta(A)$
- (iii) pre open set($\mathbb{IPOS}$) if $A \subseteq \mathbb{I}.\mathbb{INT}(\mathbb{I}.\mathbb{CL}(A))$
- (iv) semi open set($\mathbb{ISOS}$) if $A \subseteq \mathbb{I}.\mathbb{CL}(\mathbb{I}.\mathbb{INT}(A))$
- (v) δ-pre open set($\mathbb{I}.\delta\mathbb{POS}$) if $A \subseteq \mathbb{I}.\mathbb{INT}(\mathbb{I}.\mathbb{CL}_\delta(A))$
- (vi) δ-semi open set($\mathcal{I}.\delta\mathcal{SOS}$) if $A \subseteq \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))$
- (vii) e-open set($\mathcal{I}.e\mathcal{OS}$) if $A \subseteq \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)) \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))$
- (viii) e^*-open set ($\mathcal{I}.e^*\mathcal{OS}$) if $A \subseteq \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)))$
- (ix) a-open set ($\mathcal{I}.a\mathcal{OS}$) if $A \subseteq \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)))$
- (x) β-open set($\mathcal{I}.\beta\mathcal{OS}$) if $A \subseteq \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(A)))$

Definition 3.4 Let $(\mathbb{X},\ \mathbb{IT})$ be $\mathbb{ITS}$. A subset $C \subset X$ is called $\mathbb{I}.\mathbb{RCS}$, (resp.$\mathbb{I}.\delta\mathbb{CS}, \mathbb{IPCS}, \mathbb{ISCS}, \mathbb{I}.\delta\mathbb{PCS}, \mathbb{I}.\delta\mathbb{SCS}, \mathbb{I}.a\mathbb{CS}, \mathbb{I}.e\mathbb{CS}, \mathbb{I}.e^*\mathbb{CS}, \mathbb{I}.\beta\mathcal{CS}$) in X if $\mathbb{X} - C$ is $\mathbb{I}.\mathbb{ROS}$, (resp.$\mathbb{I}.\delta\mathbb{OS}, \mathbb{IPOS}, \mathbb{ISOS}, \mathbb{I}.\delta\mathbb{POS}, \mathbb{I}.\delta\mathbb{SOS}, \mathbb{I}.a\mathbb{OS}, \mathbb{I}.e\mathbb{OS}, \mathbb{I}.e^*\mathbb{OS}, \mathcal{I}.\beta\mathcal{OS}$) in X.

Definition 3.5 Let $(X,\ \mathbb{IT})$ be an and $A \subset \mathbb{X}$. A point $x \in X$ is called Infra

- (i) δ Cluster Point($\mathcal{I}.\delta\mathcal{C}_\mathcal{P}$) if for all $\mathcal{I}.\delta\mathcal{OS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.
- (ii) Pre Cluster Point($\mathcal{IPC}_\mathcal{P}$) if for all $\mathcal{IPOS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.
- (iii) Semi Cluster Point($\mathcal{ISC}_\mathcal{P}$) if for all $\mathcal{ISOS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.
- (iv) δPre Cluster Point($\mathcal{I}.\delta\mathcal{PC}_\mathcal{P}$) if for all $\mathcal{I}.\delta\mathcal{POS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.
- (v) δSemi Cluster Point($\mathcal{I}.\delta\mathcal{SC}_\mathcal{P}$) if for all $\mathcal{I}.\delta\mathcal{SOS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.
- (vi) a Cluster Point($\mathcal{I}.a\mathcal{C}_\mathcal{P}$) if for all $\mathcal{I}.a\mathcal{OS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.
- (vii) e Cluster Point($\mathcal{I}.e\mathcal{C}_\mathcal{P}$) if for all $\mathcal{I}.e\mathcal{OS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.
- (viii) e^* Cluster Point($\mathcal{I}.e^*\mathcal{C}_\mathcal{P}$) if for all $\mathcal{I}.e^*\mathcal{OS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.
- (ix) β Cluster Point($\mathcal{I}.\beta\mathcal{C}_\mathcal{P}$) if for all $\mathcal{I}.\beta\mathcal{OS}$ O containing x, then $A \cap (O\backslash\{x\}) \neq \emptyset$.

Definition 3.6 Let $(X,\ \mathbb{IT})$ be an (ITS) and $A \subset \mathbb{X}$. The set of all $\mathbb{I}.\delta\mathbb{C}_\mathbb{P}$, (resp. $\mathbb{IPC}_\mathbb{P}, \mathbb{ISC}_\mathbb{P}, \mathbb{I}.\delta\mathbb{PC}_\mathbb{P}, \mathbb{I}.\delta\mathbb{SC}_\mathbb{P}, \mathbb{I}.a\mathbb{C}_\mathbb{P}, \mathbb{I}.e\mathbb{C}_\mathbb{P}, \mathbb{I}.e^*\mathbb{C}_\mathbb{P}, \mathbb{I}.\beta\mathcal{C}_\mathcal{P}$) of A is called the Infra δ Derived Set(resp. Infra Pre Derived Set,Infra Semi Derived Set,Infra δPre Derived Set, Infra δSemi Derived Set,Infra a Derived Set,Infra e Derived Set,Infra e^* Derived Set,Infra β Derived Set) of A and is denoted by $\mathbb{I}.\delta\mathbb{DS}$($resp.$(resp.$\mathbb{IPDS}, \mathbb{ISDS}, \mathbb{I}.\delta\mathbb{PDS}, \mathbb{I}.\delta\mathbb{SDS}, \mathbb{I}.a\mathbb{DS}, \mathbb{I}.e\mathbb{DS}, \mathbb{I}.e^*\mathbb{DS}, \mathbb{I}.\beta\mathcal{DS}$)) of A.

Theorem 3.1 Let $(\mathbb{X},\ \mathbb{IT})$ be $\mathbb{ITS}$. Then:

i. $\emptyset,\ X \in \mathbb{IT}$ are $\mathbb{ICS}$.

ii. Any arbitrary finite intersections of $\mathcal{ICS}'$s is an $\mathcal{ICS}'$s.

Proof: (i) Since $X - \emptyset = X \in \mathbb{IT}$ and $X - X = \emptyset \in \mathbb{IT}$ are $\mathbb{ICS}'$s.

(ii) Let $\{C_i\ :\ i \in I\}$ be an arbitrary family of $\mathbb{ICS}'$s such that $C_i \in \mathbb{IT}$ for all $i \in I$. Now, $X - C_i \in \mathbb{IT}$ is $\mathbb{IOS}$ for all $i \in I$. But $X - C_i = C_i^c \in \mathbb{IT}$ then $\bigcap C_i^c = \bigcap(X - C_i) = X - \bigcap C_i \in \mathbb{IT}, \forall i \in I$. Hence $\bigcap C_i \in \mathbb{IT}, \forall i \in I$ is $\mathbb{ICS}$.

Definition 3.7 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. The Infra

(i) Pre Closure ($\mathcal{I}.\mathcal{PCL}$) of A and given by $\mathcal{I}.\mathcal{PCL}(A) = \bigcap\{C_i : A \subset C_i,\ X - C_i \in \mathcal{IPOS}\}$.

(ii) Semi Closure ($\mathcal{I}.\mathcal{SCL}$) of A and given by $\mathcal{I}.\mathcal{SCL} = \bigcap\{C_i : A \subset C_i,\ X - C_i \in \mathcal{ISOS}\}$.

(iii) δPre Closure ($\mathcal{I}.\delta\mathcal{PCL}$) of A and given by $\mathcal{I}.\delta\mathcal{PCL} = \bigcap\{C_i : A \subset C_i,\ X - C_i \in \mathcal{I}.\delta\mathcal{POS}\}$.

(iv) δSemi Closure ($\mathcal{I}.\delta\mathcal{SCL}$) of A and given by $\mathcal{I}.\delta\mathcal{SCL} = \bigcap\{C_i : A \subset C_i,\ X - C_i \in \delta\mathcal{SOS}\}$.

(v) a Closure ($\mathcal{I}.a\mathcal{CL}$) of A and given by $\mathcal{I}.a\mathcal{CL} = \bigcap\{C_i : A \subset C_i,\ X - C_i \in \mathcal{I}.a\mathcal{OS}\}$.

(vi) e Closure ($\mathcal{I}.e\mathcal{CL}$) of A and given by $\mathcal{I}.e\mathcal{CL} = \bigcap\{C_i : A \subset C_i,\ X - C_i \in \mathcal{I}.e\mathcal{OS}\}$.

(vii) e^* Closure ($\mathcal{I}.e^*\mathcal{CL}$) of A and given by $\mathcal{I}.e^*\mathcal{CL} = \bigcap\{C_i : A \subset C_i,\ X - C_i \in \mathcal{I}.e^*\mathcal{OS}\}$.

(viii) β Closure ($\mathcal{I}.\beta\mathcal{CL}$) of A and given by $\mathcal{I}.\beta\mathcal{CL}) = \bigcap\{C_i : A \subset C_i,\ X - C_i \in \mathcal{I}.\beta\mathcal{OS})\}$.

Definition 3.8 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. The Infra

(i) Pre Interior ($\mathcal{I}.\mathcal{PINT}$) of A and given by $\mathcal{I}.\mathcal{PINT}(A) = \bigcup\{C_i : A \supset C_i,\ C_i \in \mathcal{IPOS}\}$.

(ii) Semi Interior ($\mathcal{I}.\mathcal{SINT}$) of A and given by $\mathcal{I}.\mathcal{SINT} = \bigcup\{C_i : A \supset C_i,\ C_i \in \mathcal{ISOS}\}$.

(iii) δPre Interior ($\mathcal{I}.\delta\mathcal{PNT}$) of A and given by $\mathcal{I}.\delta\mathcal{PINT} = \bigcup\{C_i : A \supset C_i,\ C_i \in \mathcal{I}.\delta\mathcal{POS}\}$.

(iv) δSemi Interior ($\mathcal{I}.\delta\mathcal{SINT}$) of A and given by $\mathcal{I}.\delta\mathcal{SINT} = \bigcup\{C_i : A \supset C_i,\ C_i \in \delta\mathcal{SOS}\}$.

(v) a Interior($\mathcal{I}.a\mathcal{INT}$) of A and given by $\mathcal{I}.a\mathcal{INT} = \bigcup\{C_i : A \supset C_i,\ C_i \in \mathcal{I}.a\mathcal{OS}\}$.

(vi) e Interior ($\mathcal{I}.e\mathcal{INT}$) of A and given by $\mathcal{I}.e\mathcal{INT} = \bigcup\{C_i : A \supset C_i,\ C_i \in \mathcal{I}.e\mathcal{OS}\}$.

(vii) e^* Interior ($\mathcal{I}.e^*\mathcal{INT}$) of A and given by $\mathcal{I}.e^*\mathcal{INT} = \bigcup\{C_i : A \supset C_i,\ C_i \in \mathcal{I}.e^*\mathcal{OS}\}$.

(viii) β Interior ($\mathcal{I}.\beta\mathcal{INT}$) of A and given by $\mathcal{I}.\beta\mathcal{INT}) = \bigcup\{C_i : A \supset C_i,\ C_i \in \mathcal{I}.\beta\mathcal{OS})\}$.

Definition 3.9 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. The Infra δ Exterior (resp. Infra Pre Exterior,Infra Semi Exterior,Infra δPre Exterior, Infra δSemi Exterior,Infra a Exterior,Infra e Exterior,Infra e^* Exterior,Infra β Exterior) of A is a set denoted by $\mathbb{I}.\delta\mathbb{EX}$((resp. $\mathbb{IPEX}$, $\mathbb{ISEX}, \mathbb{I}.\delta\mathbb{PEX}, \mathbb{I}.\delta\mathbb{SEX}, \mathcal{I}.a\mathcal{E}\mathbb{X}, \mathcal{I}.e\mathcal{EX}$, $\mathcal{I}.e^*\mathcal{E}\mathbb{X}, \mathcal{I}.\beta\mathcal{E}\mathbb{X}$) of A and given by: $\mathcal{I}.\delta\mathcal{E}\mathbb{X}$(resp. $\mathcal{IPE}\mathbb{X}, \mathcal{ISE}\mathbb{X}, \mathcal{I}.\delta\mathcal{PE}\mathbb{X}, \mathcal{I}.\delta\mathcal{SE}\mathbb{X}, \mathcal{I}.a\mathcal{E}\mathbb{X}, \mathcal{I}.e\mathcal{E}\mathbb{X}, \mathcal{I}.e^*\mathcal{E}\mathbb{X}$, $\mathcal{I}.\beta\mathcal{E}\mathbb{X})(A) = \mathbb{I}.\delta\mathbb{INT}$((resp. $\mathbb{IPIP}, \mathbb{ISIP}, \mathbb{I}.\delta\mathbb{PIP}, \mathbb{I}.\delta\mathbb{SIP}, \mathbb{I}.a\mathbb{INT}, \mathbb{I}.e\mathbb{INT}, \mathbb{I}.e^*\mathbb{INT}$, $\mathbb{I}.\beta\mathcal{INT})(A^c)$.

Definition 3.10 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. The Infra

(i) δ Boundary (briefly,$\mathcal{I}.\delta\mathcal{B}$) of A is described and indicated by $\mathcal{I}.\delta\mathcal{B}(A) = X \backslash \mathcal{I}.\delta\mathcal{INT}(A) \cup \mathcal{I}.\delta\mathcal{EX}(A)$

(ii) Semi Boundary (briefly,$\mathcal{I}.\mathcal{SB}$) of A is described and indicated by $\mathcal{I}.\mathcal{SB}(A) = X \backslash \mathcal{I}.\mathcal{SINT}(A) \cup \mathcal{I}.\mathcal{SEX}(A)$

(iii) pre Boundary (briefly,$\mathcal{I}.\mathcal{PB}$) of A is described and indicated by $\mathcal{I}.\mathcal{PB}(A) = X \backslash \mathcal{I}.\mathcal{PINT}(A) \cup \mathcal{I}.\mathcal{PEX}(A)$

(iv) δ Semi Boundary (briefly,$\mathcal{I}.\delta\mathcal{SB}$) of A is described and indicated by $\mathcal{I}.\delta\mathcal{SB}(A) = X\backslash\mathcal{I}.\delta\mathcal{SINT}(A)\cup\mathcal{I}.\delta\mathcal{SEX}(A)$

(v) δ pre Boundary (briefly,$\mathcal{I}.\delta\mathcal{PB}$) of A is described and indicated by $\mathcal{I}.\delta\mathcal{PB}(A) = X\backslash\mathcal{I}.\delta\mathcal{PINT}(A)\cup\mathcal{I}.\delta\mathcal{PEX}(A)$

(vi) a Boundary (briefly,$\mathcal{I}.a\mathcal{B}$) of A is described and indicated by $\mathcal{I}.a\mathcal{B}(A) = X\backslash\mathcal{I}.a\mathcal{INT}(A)\cup\mathcal{I}.a\mathcal{EX}(A)$

(vii) e Boundary (briefly,$\mathcal{I}.e\mathcal{B}$) of A is described and indicated by $\mathcal{I}.e\mathcal{B}(A) = X\backslash\mathcal{I}.e\mathcal{INT}(A)\cup\mathcal{I}.e\mathcal{EX}(A)$

(viii) e^* Boundary (briefly,$\mathcal{I}.e^*\mathcal{B}$) of A is described and indicated by $\mathcal{I}.e^*\mathcal{B}(A) = X\backslash\mathcal{I}.e^*\mathcal{INT}(A)\cup\mathcal{I}.e^*\mathcal{EX}(A)$

(ix) β Boundary (briefly,$\mathcal{I}.\beta\mathcal{B}$) of A is described and indicated by $\mathcal{I}.\beta\mathcal{B}(A) = X\backslash\mathcal{I}.\beta\mathcal{INT}(A)\cup\mathcal{I}.\beta\mathcal{EX}(A)$

Theorem 3.2 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset X$. Then

(i) $\mathcal{I}.e\mathcal{DS}(\emptyset) = \emptyset$.

(ii) If $A \subset B$ then $\mathcal{I}.e\mathcal{DS}(A) \subset \mathcal{I}.e\mathcal{DS}(B)$

(iii) if $x \in \mathcal{I}.e\mathcal{DS}(A)$ then $x \in \mathcal{I}.e\mathcal{DS}(A\backslash\{x\})$.

(iv) $\mathcal{I}.e\mathcal{DS}(A\cap B) \subset \mathcal{I}.e\mathcal{DS}(A)\cap\mathcal{I}.e\mathcal{DS}(B)$.

(v) $\mathcal{I}.e\mathcal{DS}(A\cup B) = \mathcal{I}.e\mathcal{DS}(A)\cup\mathcal{I}.e\mathcal{DS}(B)$.

Proof:
(i) Suppose that $\mathcal{I}.e\mathcal{DS}(\emptyset) \neq \emptyset \to \exists\ x \in \mathcal{I}.e\mathcal{DS}(A) \ni \emptyset\cap(O\backslash\{x\}) \neq \emptyset$
$x \in \emptyset$ and $x \notin \emptyset$.That is contradiction.
$\mathcal{I}.e\mathcal{DS}(\emptyset) = \emptyset$.
(ii) : Suppose that $A \subset B$. Let $x \in \mathcal{I}.e\mathcal{DS}(A) \to \forall O \ni x, A\cap(O\backslash\{x\}) \neq \emptyset$.
$\forall O \ni x, B\cap(O\backslash\{x\}) \neq \emptyset$.
$x \in \mathcal{I}.e\mathcal{DS}(B)$
$\mathcal{I}.e\mathcal{DS}(A) \subset \mathcal{I}.e\mathcal{DS}(B)$.
(iii) Assume that $x \in \mathcal{I}.e\mathcal{DS}(A) \to \forall O \ni x, A\cap(O\backslash\{x\}) \neq \emptyset$
$\forall O \ni x, A\cap(O\cap\{x\}^c) \neq \emptyset$
$\forall O \ni x, A\cap(O\cap\{x\}^c\cap\{x\}^c) \neq \emptyset$
$\forall O \ni x, A\cap(\{x\}^c\cap O\cap\{x\}^c) \neq \emptyset$
$\forall O \ni x, (A\cap(\{x\}^c)\cap(O\cap\{x\}^c) \neq \emptyset$
$\forall O \ni x,\ (A\backslash\{x\})\cap(O\backslash\{x\}) \neq \emptyset$
$x \in \mathcal{I}.e\mathcal{DS}(A\backslash\{x\})$
(iv) Since $A\cap B \subset A\cap A\cap B \subset B$
$\mathcal{I}.e\mathcal{DS}(A\cap B) \subset \mathcal{I}.e\mathcal{DS}(A)\cap\mathcal{I}.e\mathcal{DS}(A\cap B) \subset \mathcal{I}.e\mathcal{DS}(B))$.
$\mathcal{I}.e\mathcal{DS}(A\cap B) \subset \mathcal{I}.e\mathcal{DS}(A)\cap\mathcal{I}.e\mathcal{DS}(B))$.
(v) Since $A \subset A\cup B$ and $B \subset A\cup B$, then $\mathcal{I}.e\mathcal{DS}(A) \subset \mathcal{I}.e\mathcal{DS}(A\cup B)$ and $\mathcal{I}.e\mathcal{DS}(B) \subset \mathcal{I}.e\mathcal{DS}(A\cup B)$ hence $\mathcal{I}.e\mathcal{DS}(A)\cup\mathcal{I}.e\mathcal{DS}(B) \subset \mathcal{I}.e\mathcal{DS}(A\cup B)$. Conversely,
Suppose that $x \in \mathcal{I}.e\mathcal{DS}(A\cup B) \to \forall O \ni x, (A\cup B)\cap(O\backslash\{x\}) \neq \emptyset$.
$\forall O \ni x, A\cap(O\backslash\{x\}) \neq \emptyset \cup B\cap(O\backslash\{x\}) \neq \emptyset$.
$x \in \mathcal{I}.e\mathcal{DS}(A)\cup\mathcal{I}.e\mathcal{DS}(B)$.Hence,
$\mathcal{I}.e\mathcal{DS}(A\cup B) = \mathcal{I}.e\mathcal{DS}(A)\cup\mathcal{I}.e\mathcal{DS}(B)$.

Theorem 3.3 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset X$. Then

(i) $\mathcal{I}.e^*\mathcal{DS}(\emptyset) = \emptyset$.

(ii) If $A \subset B$ then $\mathcal{I}.e^*\mathcal{DS}(A) \subset \mathcal{I}.e^*\mathcal{DS}(B)$

(iii) if $x \in \mathcal{I}.e^*\mathcal{DS}(A)$ then $x \in \mathcal{I}.e^*\mathcal{DS}(A\backslash\{x\})$.

(iv) $\mathcal{I}.e^{*}\mathcal{DS}(A\cap B)\subset \mathcal{I}.e^{*}\mathcal{DS}(A)\cap \mathcal{I}.e^{*}\mathcal{DS}(B)$.

(v) $\mathcal{I}.e^{*}\mathcal{DS}(A\cup B)=\mathcal{I}.e^{*}\mathcal{DS}(A)\cup \mathcal{I}.e^{*}\mathcal{DS}(B)$.

Proof: It follows Theorem 3.2

Theorem 3.4 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A,\ B\subset X$. Then

(i) $\mathcal{I}.a\mathcal{DS}(\emptyset)=\emptyset$.

(ii) If $A\subset B$ then $\mathcal{I}.a\mathcal{DS}(A)\subset \mathcal{I}.a\mathcal{DS}(B)$

(iii) if $x\in \mathcal{I}.a\mathcal{DS}(A)$ then $x\in \mathcal{I}.a\mathcal{DS}(A\backslash\{x\})$.

(iv) $\mathcal{I}.a\mathcal{DS}(A\cap B)\subset \mathcal{I}.a\mathcal{DS}(A)\cap \mathcal{I}.a\mathcal{DS}(B)$.

(v) $\mathcal{I}.a\mathcal{DS}(A\cup B)=\mathcal{I}.a\mathcal{DS}(A)\cup \mathcal{I}.a\mathcal{DS}(B)$.

Proof: It follows Theorem 3.2

Theorem 3.5 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A,B\subset X$.Then

(i) A is $\mathcal{I}.e\mathcal{CS}$ iff $A=\mathcal{I}.e\mathcal{CL}(A)$.

(ii) $\mathcal{I}.e\mathcal{CL}(\emptyset)=\emptyset$ and $\mathcal{I}.e\mathcal{CL}(X)=X$.

(iii) $\mathcal{I}.e\mathcal{CL}(\mathcal{I}.e\mathcal{CL}(A))=\mathcal{I}.e\mathcal{CL}(A)$.

(iv) If $A\subset B$ then $\mathcal{I}.e\mathcal{CL}(A)\subset \mathcal{I}.e\mathcal{CL}(B)$.

(v) $\mathcal{I}.e\mathcal{CL}(A\cap B)\subset \mathcal{I}.e\mathcal{CL}(A)\cap \mathcal{I}.e\mathcal{CL}(B)$.

Proof.

(i) Suppose that A is $\mathcal{I}.e\mathcal{CS}$. Since $A\subset A$ and $A\cap A=A\rightarrow \mathcal{I}.e\mathcal{CL}(A)\subset A$, Also $A\subset \mathcal{I}.e\mathcal{CL}(A)\rightarrow A=\mathcal{I}.e\mathcal{CL}(A)$. Conversely, Let $A=\mathcal{I}.e\mathcal{CL}(A)$, obviously, $\mathcal{I}.e\mathcal{CL}(A)$ is the smallest $\mathcal{I}.e\mathcal{CS}$. Hence A is $\mathcal{I}.e\mathcal{CS}$.

(ii) Since X and $\emptyset$ are $\mathcal{I}.e\mathcal{CS}$'s, so by (1.) $\mathcal{I}.e\mathcal{CL}(\emptyset)=\emptyset$ and $\mathcal{I}.e\mathcal{CL}(X)=X$.

(iii)Since $\mathcal{I}.e\mathcal{CL}(A)$ is the intersection of all $\mathcal{I}.e\mathcal{CS}$'s are $\mathcal{I}.e\mathcal{CS}$s, then $\mathcal{I}.e\mathcal{CL}(\mathcal{I}.e\mathcal{CL}(A))=\mathcal{I}.e\mathcal{CL}(A)$.

(iv) Consider $A\subset B$. Since $A\subset \mathcal{I}.e\mathcal{CL}(A)$ and $B\subset \mathcal{I}.e\mathcal{CL}(B)$, so $\mathcal{I}.e\mathcal{CL}(A)\subset \mathcal{I}.e\mathcal{CL}(B)$.

(v) Since $(A\cap B\subset A\cap A\cap B\subset B)$ then $\mathcal{I}.e\mathcal{CL}(A\cap B)\subset \mathcal{I}.e\mathcal{CL}(A)$ and $\mathcal{I}.e\mathcal{CL}(A\cap B)\subset \mathcal{I}.e\mathcal{CL}(B)\rightarrow \mathcal{I}.e\mathcal{CL}(A\cap B)\subset \mathcal{I}.e\mathcal{CL}(A)\cap \mathcal{I}.e\mathcal{CL}(B)$.

Theorem 3.6 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A,B\subset X$.Then

(i) A is $\mathcal{I}.e^{*}\mathcal{CS}$ iff $A=\mathcal{I}.e^{*}\mathcal{CL}(A)$.

(ii) $\mathcal{I}.e^{*}\mathcal{CL}(\emptyset)=\emptyset$ and $\mathcal{I}.e^{*}\mathcal{CL}(X)=X$.

(iii) $\mathcal{I}.e^{*}\mathcal{CL}(\mathcal{I}.e^{*}\mathcal{CL}(A))=\mathcal{I}.e^{*}\mathcal{CL}(A)$.

(iv) If $A\subset B$ then $\mathcal{I}.e^{*}\mathcal{CL}(A)\subset \mathcal{I}.e^{*}\mathcal{CL}(B)$.

(v) $\mathcal{I}.e^{*}\mathcal{CL}(A\cap B)\subset \mathcal{I}.e^{*}\mathcal{CL}(A)\cap \mathcal{I}.e^{*}\mathcal{CL}(B)$.

Proof: It follows Theorem 3.5

Theorem 3.7 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A,B\subset X$.Then

(i) A is $\mathcal{I}.a\mathcal{CS}$ iff $A=\mathcal{I}.a\mathcal{CL}(A)$.

(ii) $\mathcal{I}.a\mathcal{CL}(\emptyset)=\emptyset$ and $\mathcal{I}.a\mathcal{CL}(X)=X$.

(iii) $\mathcal{I}.a\mathcal{CL}(\mathcal{I}.a\mathcal{CL}(A))=\mathcal{I}.a\mathcal{CL}(A)$.

(iv) If $A \subset B$ then $\mathcal{I}.a\mathcal{CL}(A) \subset \mathcal{I}.a\mathcal{CL}(B)$.

(v) $\mathcal{I}.a\mathcal{CL}(A \cap B) \subset \mathcal{I}.a\mathcal{CL}(A) \cap \mathcal{I}.a\mathcal{CL}(B)$.

Proof: It follows Theorem 3.5

Theorem 3.8 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset X$. Then

(i) A is $\mathcal{I}.e\mathcal{OS}$ iff $A = \mathcal{I}.e\mathcal{INT}(A)$.

(ii) $\mathcal{I}.e\mathcal{INT}(X) = X$ and $\mathcal{I}.e\mathcal{INT}(\emptyset) = \emptyset$.

(iii) $\mathcal{I}.e\mathcal{INT}(\mathcal{I}.e\mathcal{INT} \subset (A)) = \mathcal{I}.e\mathcal{INT}(A)$.

(iv) If $A \subset B$ then $\mathcal{I}.e\mathcal{INT}(A) \subset \mathcal{I}.e\mathcal{INT}(B)$.

(v) $\mathcal{I}.e\mathcal{INT}(A \cap B) = \mathcal{I}.e\mathcal{INT}(A) \cap \mathcal{I}.e\mathcal{INT}(B)$.

Proof.

(i) Suppose that A is $\mathcal{I}.e\mathcal{OS}$. Since $A \subset A$, then A is $\mathcal{I}.e\mathcal{OS}$ containing itself, so $A \subset \mathcal{I}.e\mathcal{INT}(A)$ and $\mathcal{I}.e\mathcal{INT}(A) \subset A$, that implies $A = \mathcal{I}.e\mathcal{INT}(A)$. Conversely, Let $A = \mathcal{I}.e\mathcal{INT}(A)$, suppose that $A = \mathcal{I}.e\mathcal{INT}(A)$. Since $\mathcal{I}.e\mathcal{INT}(A)$ is $\mathcal{I}.e\mathcal{OS}$, then A is $\mathcal{I}.e\mathcal{OS}$.

(ii) Since $X,\ \ \emptyset$ are $\mathcal{I}.e\mathcal{OS}$'s, by (1), we have $\mathcal{I}.e\mathcal{INT}(X) = X$ and $\mathcal{I}.e\mathcal{INT}(\emptyset) = \emptyset$.

(iii) Since $\mathcal{I}.e\mathcal{INT}(A)$ is $\mathcal{I}.e\mathcal{OS}$. so by (1) $\mathcal{I}.e\mathcal{INT}(\mathcal{I}.e\mathcal{INT}(A)) = \mathcal{I}.e\mathcal{INT}(A)$.

(iv) Suppose that If $A \subset B$. Let $O_i \in \mathcal{I}.e\mathcal{INT}(A) \to O_i \subset A \to O_i \subset B \to O_i \in \mathcal{I}.e\mathcal{INT}(B)$. Therefore $\mathcal{I}.e\mathcal{INT}(A) \subset \mathcal{I}.e\mathcal{INT}(B)$.

(v) Let $O_i \in \mathcal{I}.e\mathcal{INT}(A) \cap \mathcal{I}.e\mathcal{INT}(B) \leftrightarrow O_i \in \mathcal{I}.e\mathcal{INT}(A) \cap O_i \in \mathcal{I}.e\mathcal{INT}(B)$.
$\leftrightarrow \cup O_i, O_i \subset A, \forall i \cap \cup O_i, O_i \subset B, \forall i.$
$\leftrightarrow \cup O_i, O_i \subset A \cap B, \forall i.$
$\leftrightarrow O_i \in \mathcal{I}.e\mathcal{INT}(A \cap B), \forall i.$

Theorem 3.9 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset X$. Then

(i) A is $\mathcal{I}.e^*\mathcal{OS}$ iff $A = \mathcal{I}.e^*\mathcal{INT}(A)$.

(ii) $\mathcal{I}.e^*\mathcal{INT}(X) = X$ and $\mathcal{I}.e^*\mathcal{INT}(\emptyset) = \emptyset$.

(iii) $\mathcal{I}.e^*\mathcal{INT}(\mathcal{I}.e^*\mathcal{INT} \subset (A)) = \mathcal{I}.e^*\mathcal{INT}(A)$.

(iv) If $A \subset B$ then $\mathcal{I}.e^*\mathcal{INT}(A) \subset \mathcal{I}.e^*\mathcal{INT}(B)$.

(v) $\mathcal{I}.e^*\mathcal{INT}(A \cap B) = \mathcal{I}.e^*\mathcal{INT}(A) \cap \mathcal{I}.e^*\mathcal{INT}(B)$.

Proof: It follows Theorem 3.8

Theorem 3.10 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset X$. Then

(i) A is $\mathcal{I}.a\mathcal{OS}$ iff $A = \mathcal{I}.a\mathcal{INT}(A)$.

(ii) $\mathcal{I}.a\mathcal{INT}(X) = X$ and $\mathcal{I}.a\mathcal{INT}(\emptyset) = \emptyset$.

(iii) $\mathcal{I}.a\mathcal{INT}(\mathcal{I}.a\mathcal{INT} \subset (A)) = \mathcal{I}.a\mathcal{INT}(A)$.

(iv) If $A \subset B$ then $\mathcal{I}.a\mathcal{INT}(A) \subset \mathcal{I}.a\mathcal{INT}(B)$.

(v) $\mathcal{I}.a\mathcal{INT}(A \cap B) = \mathcal{I}.a\mathcal{INT}(A) \cap \mathcal{I}.a\mathcal{INT}(B)$.

Proof: It follows Theorem 3.8

Theorem 3.11 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset X$. Then

(i) $\mathcal{I}.e\mathcal{EP}(X) = \emptyset$ and $\mathcal{I}.e\mathcal{EP}(\emptyset) = X$.

(ii) $\mathcal{I}.e\mathcal{EP}(A) \subset A^c$.

(iii) $\mathcal{I}.e\mathcal{EP}(A \cup B) = \mathcal{I}.e\mathcal{EP}(A) \cap \mathcal{I}.e\mathcal{EP}(B)$.

(iv) If $A \subset B$, then $\mathcal{I}.e\mathcal{EP}(B) \subset \mathcal{I}.e\mathcal{EP}(A)$.

(v) $\mathcal{I}.e\mathcal{EP}(A \cap B) \subset \mathcal{I}.e\mathcal{EP}(A) \cup \mathcal{I}.e\mathcal{EP}(B)$.

Proof.
(i):$\mathcal{I}.e\mathcal{EP}(X) = \mathcal{I}.e\mathcal{INT}(X^c) = \mathcal{I}.e\mathcal{INT}(\emptyset) = \emptyset$ and $\mathcal{I}.e\mathcal{EP}(\emptyset) = \mathcal{I}.e\mathcal{INT}(\emptyset^c) = \mathcal{I}.e\mathcal{INT}(X) = X$.
(ii)$\mathcal{I}.e\mathcal{EP}(A) = \mathcal{I}.e\mathcal{INT}(A^c) \subset A^c$.
(iii) $\mathcal{I}.e\mathcal{EP}(A \cup B) = \mathcal{I}.e\mathcal{INT}(A \cup B)^c = \mathcal{I}.e\mathcal{INT}(A^c \cap B^c) = \mathcal{I}.e\mathcal{INT}(A^c) \cap \mathcal{I}.e\mathcal{INT}(B^c) = \mathcal{I}.e\mathcal{EP}(A) \cap \mathcal{I}.e\mathcal{EP}(B)$
(iv) let $A \subset B \to B^c \subset A^c \to \mathcal{I}.e\mathcal{INT}(B^c) \subset \mathcal{I}.e\mathcal{INT}(A^c) \to \mathcal{I}.e\mathcal{EP}(B) \subset \mathcal{I}.e\mathcal{EP}(A)$.
(v) $\mathcal{I}.e\mathcal{EP}(A \cap B) = \mathcal{I}.e\mathcal{INT}(A \cap B)^c = \mathcal{I}.e\mathcal{INT}(A^c \cup B^c) \subset \mathcal{I}.e\mathcal{INT}(A^c) \cup \mathcal{I}.e\mathcal{INT}(B^c) = \mathcal{I}.e\mathcal{EP}(A) \cup \mathcal{I}.e\mathcal{EP}(B)$.

Theorem 3.12 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset X$. Then

(i) $\mathcal{I}.e^*\mathcal{EP}(X) = \emptyset$ and $\mathcal{I}.e^*\mathcal{EP}(\emptyset) = X$.

(ii) $\mathcal{I}.e^*\mathcal{EP}(A) \subset A^c$.

(iii) $\mathcal{I}.e^*\mathcal{EP}(A \cup B) = \mathcal{I}.e^*\mathcal{EP}(A) \cap \mathcal{I}.e^*\mathcal{EP}(B)$.

(iv) If $A \subset B$, then $\mathcal{I}.e^*\mathcal{EP}(B) \subset \mathcal{I}.e^*\mathcal{EP}(A)$.

(v) $\mathcal{I}.e^*\mathcal{EP}(A \cap B) \subset \mathcal{I}.e^*\mathcal{EP}(A) \cup \mathcal{I}.e^*\mathcal{EP}(B)$.

Proof: It follows Theorem 3.11

Theorem 3.13 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A, B \subset X$. Then

(i) $\mathcal{I}.a\mathcal{EP}(X) = \emptyset$ and $\mathcal{I}.a\mathcal{EP}(\emptyset) = X$.

(ii) $\mathcal{I}.a\mathcal{EP}(A) \subset A^c$.

(iii) $\mathcal{I}.a\mathcal{EP}(A \cup B) = \mathcal{I}.a\mathcal{EP}(A) \cap \mathcal{I}.a\mathcal{EP}(B)$.

(iv) If $A \subset B$, then $\mathcal{I}.a\mathcal{EP}(B) \subset \mathcal{I}.a\mathcal{EP}(A)$.

(v) $\mathcal{I}.a\mathcal{EP}(A \cap B) \subset \mathcal{I}.a\mathcal{EP}(A) \cup \mathcal{I}.a\mathcal{EP}(B)$.

Proof: It follows Theorem 3.11

Theorem 3.14 Let $(X,\ \mathbb{IT})$ be an (ITS) and $A \subset \mathbb{X}$.

(i) $\mathcal{I}.e\mathcal{B}(X) = \mathcal{I}.e\mathcal{B}(\emptyset) = \emptyset$.

(ii) $\mathcal{I}.e\mathcal{B}(A \cap B) = \mathcal{I}.e\mathcal{B}(A) \cup \mathcal{I}.e\mathcal{B}(B)$.

Proof.
(i) $\mathcal{I}.e\mathcal{B}(X) = X \backslash \mathcal{I}.e\mathcal{INT}(X) \cup \mathcal{I}.e\mathcal{EP}(X) = X \backslash X \cup \emptyset = X \backslash X = \emptyset$.
$\mathcal{I}.e\mathcal{B}(\emptyset) = X \backslash \mathcal{I}.e\mathcal{INT}(\emptyset) \cup \mathcal{I}.e\mathcal{EP}(\emptyset) = X \backslash \emptyset \cup X = X \backslash X = \emptyset$
(ii) $\mathcal{IB}(A \cap B) = X \backslash \mathcal{I}.e\mathcal{INT}(A \cap B) \cup \mathcal{I}.e\mathcal{EP}(A \cap B)$
$= X \backslash \mathcal{I}.e\mathcal{INT}(A) \cap \mathcal{I}.e\mathcal{INT}(B) \cup \mathcal{I}.e\mathcal{EP}(A \cap B)$
$= X \backslash \mathcal{I}.e\mathcal{INT}(A) \cup X \backslash \mathcal{I}.e\mathcal{INT}(B) \cup \mathcal{I}.e\mathcal{EP}(A \cap B)$
$= X \backslash \mathcal{I}.e\mathcal{INT}(A) \cup X \backslash \mathcal{I}.e\mathcal{INT}(B) \cup \mathcal{I}.e\mathcal{EP}(A) \cup \mathcal{I}.e\mathcal{EP}(B))$
$= \mathcal{I}.e\mathcal{B}(A) \cup \mathcal{I}.e\mathcal{B}(B)$.

Theorem 3.15 Let $(X,\ \mathbb{IT})$ be an (ITS) and $A \subset \mathbb{X}$.

(i) $\mathcal{I}.e^*\mathcal{B}(X) = \mathcal{I}.e^*\mathcal{B}(\emptyset) = \emptyset$.

(ii) $\mathcal{I}.e^*\mathcal{B}(A \cap B) = \mathcal{I}.e^*\mathcal{B}(A) \cup \mathcal{I}.e^*\mathcal{B}(B)$.

Proof: It follows Theorem 3.14

Theorem 3.16 Let $(X,\ \mathbb{IT})$ be an (ITS) and $A \subset \mathbb{X}$.

(i) $\mathcal{I}.a\mathcal{B}(X) = \mathcal{I}.a\mathcal{B}(\emptyset) = \emptyset$.

(ii) $\mathcal{I}.a\mathcal{B}(A \cap B) = \mathcal{I}.a\mathcal{B}(A) \cup \mathcal{I}.a\mathcal{B}(B)$.

Proof: It follows Theorem 3.14

Theorem 3.17 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. Then:

(i) $A \subset \mathcal{I}.e\mathcal{CL}(A) \to \mathcal{I}.e\mathcal{DS}(A) \subset \mathcal{I}.e\mathcal{DS}(\mathcal{I}.e\mathcal{CL}(A))$.

(ii) $\mathcal{I}.e\mathcal{INT}(A) \subset A \to \mathcal{I}.e\mathcal{DS}(\mathcal{I}.e\mathcal{INT}(A)) \subset \mathcal{I}.e\mathcal{DS}(A)$.

(iii) If A is $\mathcal{ICS}$, then $\mathcal{I}.e\mathcal{DS}(A) \subset A$.

(iv) $\mathcal{I}.e\mathcal{CL}(A) = A \cup \mathcal{I}.e\mathcal{DS}(A)$.

(v) $\mathcal{IB}(A) = \mathcal{I}.e\mathcal{CL}(A) \backslash \mathcal{I}.e\mathcal{INT}(A)$.

(vi) $\mathcal{I}.e\mathcal{CL}(A) = \mathcal{IB}(A) \cup \mathcal{I}.e\mathcal{INT}(A)$.

(vii) $\mathcal{IB}(A) \subset \mathcal{I}.e\mathcal{CL}(A)$.

(viii) $\mathcal{I}.e\mathcal{INT}(A) \cap \mathcal{IB}(A) = \emptyset$.

Proof.
(i) Let $A \subset \mathcal{I}.e\mathcal{CL}(A)$. By (ii) $\mathcal{I}.e\mathcal{DS}(A) \subset \mathcal{I}.e\mathcal{DS}(\mathcal{I}.e\mathcal{CL}(A))$.
(ii) Let $\mathcal{I}.e\mathcal{INT}(A) \subset A$. By (ii) $\mathcal{I}.e\mathcal{DS}(\mathcal{I}.e\mathcal{INT}(A)) \subset \mathcal{I}.e\mathcal{DS}(A)$.
(iii) Let A be a $\mathcal{ICS}$ and $x \in \mathcal{I}.e\mathcal{DS}(A)$, then $\forall O \ni x, A \cap (O - (x)) \neq \emptyset$ Hence $x \in A$ and $\mathcal{I}.e\mathcal{DS}(A) \subset A$.
(iv) Since $A \subset \mathcal{I}.e\mathcal{CL}(A)$ and $\mathcal{I}.e\mathcal{DS}(A) \subset \mathcal{I}.e\mathcal{DS}(\mathcal{I}.e\mathcal{CL}(A)) \subset \mathcal{I}.e\mathcal{CL}(A)$. we have $A \cup \mathcal{I}.e\mathcal{DS}(A) \subset \mathcal{I}.e\mathcal{CL}(A)$. Another direction, To show that $\mathcal{I}.e\mathcal{CL}(A) \subset A \cup \mathcal{I}.e\mathcal{DS}(A)$.
Let $x \in \mathcal{I}.e\mathcal{CL}(A)$, but $A \subset \mathcal{I}.e\mathcal{CL}(A)$, then $x \in A$ or $x \notin A$.
(a) If $x \in A$, then $x \in A \cup \mathcal{I}.e\mathcal{DS}(A)$.
(b) If $x \notin A$, Let $x \notin \mathcal{I}.e\mathcal{DS}(A) \to \exists O \ni x, A \cap (O \backslash \{x\}) = \emptyset$, but $x \notin A$, that is contradiction, therefore $x \in \mathcal{I}.e\mathcal{DS}(A)$ and $x \in A \cup \mathcal{I}.e\mathcal{DS}(A)$.
So $\mathcal{I}.e\mathcal{CL}(A) = A \cup \mathcal{I}.e\mathcal{DS}(A)$.
(v) By definition: $\mathcal{IB}(A) = X \backslash \mathcal{I}.e\mathcal{INT}(A) \cup iep(A)$
$= X \backslash \mathcal{I}.e\mathcal{INT}(A) \cap X \backslash iep(A)$
$= X \backslash \mathcal{I}.e\mathcal{INT}(A) \cap \mathcal{I}.e\mathcal{CL} \backslash (A)$
Since $\mathcal{I}.e\mathcal{INT}(A) \subset \mathcal{I}.e\mathcal{CL}(A) \subset X \to \mathcal{I}.e\mathcal{CL}(A) \cap (X \backslash \mathcal{I}.e\mathcal{INT}(A) = \mathcal{I}.e\mathcal{CL}(A) \backslash \mathcal{I}.e\mathcal{INT}(A)$. Then we have $\mathcal{IB}(A) = \mathcal{I}.e\mathcal{CL}(A) \backslash \mathcal{I}.e\mathcal{INT}(A)$.
(vi) By (i) $\mathcal{IB}(A) = \mathcal{I}.e\mathcal{CL}(A) \backslash \mathcal{I}.e\mathcal{INT}(A)$
$\to \mathcal{IB}(A) \cup \mathcal{I}.e\mathcal{INT}(A) = ic(A) \backslash \mathcal{I}.e\mathcal{INT}(A) \cup \mathcal{I}.e\mathcal{INT}(A) = \mathcal{I}.e\mathcal{CL}(A)$.
(vii) By (ii) it is clear that $\mathcal{IB}(A) \subset \mathcal{I}.e\mathcal{CL}(A)$.
(viii) $\mathcal{I}.e\mathcal{INT}(A) \cap \mathcal{IB}(A) = \mathcal{I}.e\mathcal{INT}(A) \cap \mathcal{I}.e\mathcal{CL}(A) \backslash \mathcal{I}.e\mathcal{INT}(A) = \emptyset$.

Theorem 3.18 Let $(\mathbb{X},\ \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. Then:

(i) $A \subset \mathcal{I}.e^*\mathcal{CL}(A) \to \mathcal{I}.e^*\mathcal{DS}(A) \subset \mathcal{I}.e^*\mathcal{DS}(\mathcal{I}.e^*\mathcal{CL}(A))$.

(ii) $\mathcal{I}.e^*\mathcal{INT}(A) \subset A \to \mathcal{I}.e^*\mathcal{DS}(\mathcal{I}.e^*\mathcal{INT}(A)) \subset \mathcal{I}.e^*\mathcal{DS}(A)$.

(iii) If A is $\mathcal{ICS}$, then $\mathcal{I}.e^{*}\mathcal{DS}(A) \subset A$.

(iv) $\mathcal{I}.e^{*}\mathcal{CL}(A) = A \cup \mathcal{I}.e^{*}\mathcal{DS}(A)$.

(v) $\mathcal{IB}(A) = \mathcal{I}.e^{*}\mathcal{CL}(A) \backslash \mathcal{I}.e^{*}\mathcal{INT}(A)$.

(vi) $\mathcal{I}.e^{*}\mathcal{CL}(A) = \mathcal{IB}(A) \cup \mathcal{I}.e^{*}\mathcal{INT}(A)$.

(vii) $\mathcal{IB}(A) \subset \mathcal{I}.e^{*}\mathcal{CL}(A)$.

(viii) $\mathcal{I}.e^{*}\mathcal{INT}(A) \cap \mathcal{IB}(A) = \emptyset$.

Proof: It follows Theorem 3.17

Theorem 3.19 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. Then:

(i) $A \subset \mathcal{I}.a\mathcal{CL}(A) \to \mathcal{I}.a\mathcal{DS}(A) \subset \mathcal{I}.a\mathcal{DS}(\mathcal{I}.a\mathcal{CL}(A))$.

(ii) $\mathcal{I}.a\mathcal{INT}(A) \subset A \to \mathcal{I}.a\mathcal{DS}(\mathcal{I}.a\mathcal{INT}(A)) \subset \mathcal{I}.a\mathcal{DS}(A)$.

(iii) If A is $\mathcal{ICS}$, then $\mathcal{I}.a\mathcal{DS}(A) \subset A$.

(iv) $\mathcal{I}.a\mathcal{CL}(A) = A \cup \mathcal{I}.a\mathcal{DS}(A)$.

(v) $\mathcal{IB}(A) = \mathcal{I}.a\mathcal{CL}(A) \backslash \mathcal{I}.a\mathcal{INT}(A)$.

(vi) $\mathcal{I}.a\mathcal{CL}(A) = \mathcal{IB}(A) \cup \mathcal{I}.a\mathcal{INT}(A)$.

(vii) $\mathcal{IB}(A) \subset \mathcal{I}.a\mathcal{CL}(A)$.

(viii) $\mathcal{I}.a\mathcal{INT}(A) \cap \mathcal{IB}(A) = \emptyset$.

Proof: It follows Theorem 3.17

Theorem 3.20 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. Then:

(i) $\mathcal{I}.\delta\mathcal{PCL}(A) \supseteq A \cup \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_{\delta}(A))$ and $\mathcal{I}.\delta\mathcal{PNT}(A) \subseteq A \cap \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_{\delta}(A))$

(ii) $\mathcal{I}.\delta\mathcal{SCL}(A) \supseteq A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_{\delta}(A))$ and $\mathcal{I}.\delta\mathcal{SINT}(A) \subseteq A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_{\delta}(A))$.

Proof: We will prove only the first statement of (i) and the others is similar. Since $\mathcal{I}.\delta\mathcal{PCL}(A)$ is $\mathcal{I}.\delta\mathcal{PCS}$, we have $\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_{\delta}(A)) \subseteq \mathcal{I}.\mathcal{CL}(\mathcal{I}.\delta\mathcal{PCL}(A)) \subseteq \mathcal{I}.\delta\mathcal{PCL}(A)$. Thus $A \cup \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_{\delta}(A)) \subseteq \mathcal{I}.\delta\mathcal{PCL}(A)$.

Proposition 3.1 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. Then:

(i) If A is an $\mathcal{I}.e\mathcal{OS}$ and $\mathcal{I}.\mathcal{INT}_{\delta}(A) = \phi$, then A is an $\mathcal{I}.\delta\mathcal{POS}$.

(ii) If A is an $\mathcal{I}.e\mathcal{OS}$ and $\mathcal{I}.\mathcal{CL}_{\delta}(A) = \phi$, then A is an $\mathcal{I}.\delta\mathcal{SOS}$.

(iii) If A is an $\mathcal{I}.e\mathcal{OS}$ and $\mathcal{I}.\delta\mathcal{CS}$, then A is an $\mathcal{I}.\delta\mathcal{SOS}$.

(iv) If A is an $\mathcal{I}.\delta\mathcal{SOS}$ and $\mathcal{I}.\delta\mathcal{CS}$, then A is an $\mathcal{I}.e\mathcal{OS}$.

Proof: (i) Let A be an $\mathcal{I}.e\mathcal{OS}$, that is $A \subseteq \mathcal{I}.\mathcal{CLI}.\mathcal{INT}_{\delta}(A) \cup \mathcal{I}.\mathcal{INTI}.\mathcal{CL}_{\delta}(A) = \phi \cup \mathcal{I}.\mathcal{INTI}.\mathcal{CL}_{\delta}(A) = \mathcal{I}.\mathcal{INTI}.\mathcal{CL}_{\delta}(A)$. Hence A is an $\mathcal{IPOS}$.

(ii) Follows from (i).

(iii) Let A be an $\mathcal{I}.e\mathcal{OS}$ and $\mathcal{I}.\delta\mathcal{CS}$, that is $A \subseteq \mathcal{I}.\mathcal{CLI}.\mathcal{INT}_{\delta}(A) \cup \mathcal{I}.\mathcal{INTI}.\mathcal{CL}_{\delta}(A) = \mathcal{I}.\mathcal{CLI}.\mathcal{INT}_{\delta}(A) \cup \mathcal{I}.\mathcal{INT}(A) = cl\mathcal{I}.\mathcal{INT}_{\delta}(A)$. Hence A is an $\mathcal{I}.\delta\mathcal{SOS}$.

(iv) Let A be an $\mathcal{I}.\delta\mathcal{SOS}$ and $\mathcal{I}.\delta\mathcal{CS}$, that is $A \subseteq \mathcal{I}.\mathcal{CLI}.\mathcal{INT}_{\delta}(A) \subseteq \mathcal{I}.\mathcal{CLI}.\mathcal{INT}_{\delta}(A) \vee \mathcal{I}.\mathcal{INTI}.\mathcal{CL}_{\delta}(A)$. Hence A is an $\mathcal{I}.e\mathcal{OS}$.

Theorem 3.21 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. Then, A is an $\mathbb{I}.e\mathbb{OS}$ if and only if $A = \mathbb{I}.\delta\mathbb{PINT}(A) \cup \mathbb{I}.\delta\mathbb{SINT}(A)$.

Proof: Let A be an $\mathcal{I}.e\mathcal{OS}$. Then $A \subseteq \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)) \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))$. By Theorem 3.20, we have $\mathcal{I}.\delta\mathcal{PINT}(A) \cup \mathcal{I}.\delta\mathcal{SINT}(A) = (A \cap \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))) \cup (A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) = A \cap (\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)) \cup \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) = A$.

Conversely, if $A = \mathcal{I}.\delta\mathcal{PINT}(A) \cup \mathcal{I}.\delta\mathcal{SINT}(A)$ then, by Theorem 3.20 $A = \mathcal{I}.\delta\mathcal{PINT}(A) \cup \mathcal{I}.\delta\mathcal{SINT}(A) = (A \cap \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))) \cup (A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) = A \cap (\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)) \cup \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) \subseteq \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)) \cup \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))$ and hence A is an $\mathcal{I}.e\mathcal{OS}$.

Proposition 3.2 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. Then:

(i) $\mathcal{I}.e\mathcal{CL}(\overline{A}) = \overline{\mathcal{I}.e\mathcal{INT}(A)}$, $\mathcal{I}.e\mathcal{INT}(\overline{A}) = \overline{\mathcal{I}.e\mathcal{CL}(A)}$.

(ii) $\mathcal{I}.e\mathcal{CL}(A \vee B) \geq \mathcal{I}.e\mathcal{CL}(A) \vee \mathcal{I}.e\mathcal{CL}(B)$, $\mathcal{I}.e\mathcal{INT}(A \vee B) \geq \mathcal{I}.e\mathcal{INT}(A) \vee \mathcal{I}.e\mathcal{INT}(B)$.

(iii) $\mathcal{I}.e\mathcal{CL}(A \wedge B) \subseteq \mathcal{I}.e\mathcal{CL}(A) \wedge \mathcal{I}.e\mathcal{CL}(B)$, $\mathcal{I}.e\mathcal{INT}(A \wedge B) \subseteq \mathcal{I}.e\mathcal{INT}(A) \wedge \mathcal{I}.e\mathcal{INT}(B)$.

Proposition 3.3 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset X$. Then:

(i) $\mathcal{I}.e\mathcal{CL}(A) \geq \mathcal{I}.\mathcal{CL}\mathcal{I}.\mathcal{INT}_\delta(A) \wedge \mathcal{I}.\mathcal{INT}\mathcal{I}.\mathcal{CL}_\delta(A)$.

(ii) $\mathcal{I}.e\mathcal{INT}(A) \subseteq \mathcal{I}.\mathcal{CL}\mathcal{I}.\mathcal{INT}_\delta(A) \vee \mathcal{I}.\mathcal{INT}\mathcal{I}.\mathcal{CL}_\delta(A)$.

Proof: *(i) $\mathcal{I}.e\mathcal{CL}(A)$ is an $\mathcal{I}.e\mathcal{CS}$ and $A \subseteq \mathcal{I}.e\mathcal{CL}(A)$, then $\mathcal{I}.e\mathcal{CL}(A) \geq \mathcal{I}.\mathcal{CL}\mathcal{I}.\mathcal{INT}_\delta\mathcal{I}.e\mathcal{CL}(A) \wedge \mathcal{I}.\mathcal{INT}\mathcal{I}.\mathcal{CL}_\delta\mathcal{I}.e\mathcal{CL}(A) \geq \mathcal{I}.\mathcal{CL}\mathcal{I}.\mathcal{INT}_\delta(A) \wedge \mathcal{I}.\mathcal{INT}\mathcal{I}.\mathcal{CL}_\delta(A)$.*

(ii) Follows from (i) by taking the complementation.

Theorem 3.22 Let $(\mathbb{X}, \mathbb{IT})$ be an $\mathbb{ITS}$ and $A \subset \mathbb{X}$. Then: $\mathbb{I}.e\mathbb{CL}(A) = \mathbb{I}.\delta\mathbb{PCL}(A) \cap \mathbb{I}.\delta\mathbb{SCL}(A)$.

Proof: *It is obvious that, $\mathcal{I}.e\mathcal{CL}(A) \subseteq \mathcal{I}.\delta\mathcal{PCL}(A) \cap \mathcal{I}.\delta\mathcal{SCL}(A)$. Conversely, from Definition we have $\mathcal{I}.e\mathcal{CL}(A) \supseteq \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.e\mathcal{CL}(A))) \cap \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(\mathcal{I}.e\mathcal{CL}(A))) \supseteq \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)) \cap \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))$. Since $\mathcal{I}.e\mathcal{CL}(A)$ is $\mathcal{I}.e\mathcal{OS}$, by Theorem 3.20, we have $\mathcal{I}.\delta\mathcal{PCL}(A) \cap \mathcal{I}.\delta\mathcal{SCL}(A) = (A \cup \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) \cup (A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))) = A \cup (\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)) \cap \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))) = A \subseteq \mathcal{I}.e\mathcal{CL}(A)$.*

Lemma 3.1 The following hold for a subset A of a space X:

(1) $\mathcal{I}.\delta\mathcal{SINT}(A) = A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))$ and $\mathcal{I}.\delta\mathcal{SCL}(A) = A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))$

(2) $\mathcal{I}.\delta\mathcal{PCL}(A) = A \cup \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))$

(3) $\mathcal{I}.\delta\mathcal{SCL}(\mathcal{I}.\delta\mathcal{SINT}(A)) = \mathcal{I}.\delta\mathcal{SINT}(A) \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)))$ and $\mathcal{I}.\delta\mathcal{SINT}(\mathcal{I}.\delta\mathcal{SCL}(A)) = \mathcal{I}.\delta\mathcal{SCL}(A) \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)))$

(4) $\mathcal{I}.\mathcal{CL}_\delta(\mathcal{I}.\delta\mathcal{SINT}(A)) = \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))$

(5) $\mathcal{I}.\delta\mathcal{SCL}(\mathcal{I}.\mathcal{INT}_\delta(A)) = \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)))$

Lemma 3.2 The following hold for a subset A of a space X:

(1) $\mathcal{I}.e^*\mathcal{CL}(A)$ is $\mathcal{I}.e^*\mathcal{OS}$

(2) $X \backslash \mathcal{I}.e^*\mathcal{CL}(A) = \mathcal{I}.e^*\mathcal{INT}(X \backslash A)$

Theorem 3.23 The following hold for a subset A of a space X:

(i) A is $\mathcal{I}.e^*\mathcal{OS}$ if and only if $A = A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)))$

(ii) A is $\mathcal{I}.e^*\mathcal{CS}$ if and only if $A = A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)))$

(iii) $\mathcal{I}.e^*\mathcal{CL}(A) = A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)))$

(iv) $\mathcal{I}.e^*\mathcal{INT}(A) = A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)))$

Proof. (i) : Let A be $\mathcal{I}.a\mathcal{OS}$. Then $A \subset \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)))$. We obtain $A \subset A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)))$. Conversely, let $A = A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)))$. We have $A = A \cap \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A))) \subset \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(A)))$ and hence, A is $\mathcal{I}.e^*\mathcal{OS}$.

(iii):Since $\mathcal{I}.e^*\mathcal{CL}(A)$ is $\mathcal{I}.e^*\mathcal{CS}$, $\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) \subset \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.e^*\mathcal{CL}(A)))) \subset \mathcal{I}.e^*\mathcal{CL}(A)$. Hence, $A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) \subset \mathcal{I}.e^*\mathcal{CL}(A)$.

Conversely, since: $\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)))))) =$

$$= \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A \cup \mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{CL}_\delta(\mathcal{I}.\mathcal{INT}_\delta(A))))))$$

$$= \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A) \cup \mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{CL}_\delta(\mathcal{I}.\mathcal{INT}_\delta(A))))))$$
$$= \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A) \cup \mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{CL}_\delta(\mathcal{I}.\mathcal{INT}_\delta(A)))))$$
$$= \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{CL}_\delta(\mathcal{I}.\mathcal{INT}_\delta(A)))))$$
$$= \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) \subset A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) ,$$

then $A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)))$ is $\mathcal{I}.e^*\mathcal{CS}$ containing A and hence:

$$\mathcal{I}.e^*\mathcal{CL}(A) \subset A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A))) .$$

Thus, we obtain $\mathcal{I}.e^*\mathcal{CL}(A) = A \cup \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(A)))$.

(ii) follows from (i) and (iv) follows from (iii).

Theorem 3.24 Let N be a subset of an $\mathbb{ITS}$ $\mathbb{X}$. The following are equivalent:

(i) N is $\mathcal{I}.\mathcal{ROS}$,

(ii) N is $\mathcal{I}.a\mathcal{OS}$ and $\mathcal{I}.e^*\mathcal{CS}$,

(iii) N is $\mathcal{I}.\delta\mathcal{POS}$ and $\mathcal{I}.\delta\mathcal{SCS}$.

Proof. (i) $\Rightarrow$ (ii) : Obvious.

(ii) $\Rightarrow$(i) : Let N be $\mathcal{I}.a\mathcal{OS}$ and $\mathcal{I}.e^*\mathcal{CS}$. We have $N \subset \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(N)))$ and $\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(N))) \subset N$ and hence $N = \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(N)))$. Thus, N is $\mathcal{I}.\mathcal{ROS}$.

(i) $\Leftrightarrow$ (iii) : Let N be $\mathcal{I}.\delta\mathcal{POS}$ and $\mathcal{I}.\delta\mathcal{SCS}$. Then $N \subset \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(N))$ and $\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(N)) \subset N$. Thus, $N = \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(N)) = \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(N))$ and hence N is $\mathcal{I}.\mathcal{ROS}$. The converse is similar.

Theorem 3.25 Let N be a subset of an $\mathbb{ITS}$ $\mathbb{X}$. The following are equivalent:

(i) N is $\mathcal{I}.\delta\mathcal{SOS}$,

(ii) N is $\mathcal{I}.e^*\mathcal{OS}$ and $\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\delta\mathcal{FR}(N)) = \emptyset$.

Proof. (i) $\Rightarrow$ (ii) : Let N be $\mathcal{I}.\delta\mathcal{SOS}$. We have $\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(N)) \subset \mathcal{I}.\mathcal{CL}_\delta(N) \subset \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(N))$. Since $\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\delta\mathcal{FR}(N)) = \mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{CL}_\delta(N) \cap (X\backslash\mathcal{I}.\mathcal{INT}_\delta(N))) = \mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{CL}_\delta(N))\backslash\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(N))$, then $\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\delta\mathcal{FR}(N)) = \emptyset$.

(ii) $\Rightarrow$ (i) : Let N be $\mathcal{I}.e^*\mathcal{OS}$ and $\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\delta\mathcal{FR}(N)) = \emptyset$. Then $N \subset \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(N))) \subset \mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(N))$. Thus, N is $\mathcal{I}.\delta\mathcal{SOS}$.

Theorem 3.26 Let X be a topological space. Then $I.aO(X) = \mathcal{I}.\delta\mathcal{SOS}(X) \cap \mathcal{I}.\delta\mathcal{POS}(X)$.

Proof. Let $N \in \mathcal{I}.a\mathcal{O}(X)$. Then $N \in \mathcal{I}.\delta\mathcal{SO}(X)$ and $N \in \mathcal{I}.\delta\mathcal{PO}(X)$. Thus, $\mathcal{I}.a\mathcal{O}(X) \subset \mathcal{I}.\delta\mathcal{SO}(X) \cap \mathcal{I}.\delta\mathcal{PO}(X)$.

Conversely, let $N \in \mathcal{I}.\ \delta\mathcal{SO}(X) \cap \mathcal{I}.\delta\mathcal{PO}(X)$. Then $N \in \mathcal{I}.\delta\mathcal{SO}(X)$ and $N \in \mathcal{I}.\delta\mathcal{PO}(X)$. Since $N \in \mathcal{I}.\delta\mathcal{SO}(X)$, then , $\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\delta\mathcal{FR}(N)) = \emptyset$. Since $\mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\delta\mathcal{FR}(N)) = \mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{CL}_\delta(N) \cap (X\backslash\mathcal{I}.\mathcal{INT}_\delta(N))) = \mathcal{I}.\mathcal{INT}_\delta(\mathcal{I}.\mathcal{CL}_\delta(N))\backslash\mathcal{I}.\mathcal{CL}_\delta(\mathcal{I}.\mathcal{INT}_\delta(N))$, then $\mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(N)) \subset \mathcal{I}.\mathcal{CL}\mathcal{I}.\mathcal{INT}_\delta(N))$. Since $N \in \mathcal{I}.\delta\mathcal{PO}(X)$, we have $N \subset \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}_\delta(N)) \subset \mathcal{I}.\mathcal{INT}(\mathcal{I}.\mathcal{CL}(\mathcal{I}.\mathcal{INT}_\delta(N)))$. Thus, $N \in \mathcal{I}.a\mathcal{O}(X)$.

4. Interrelations

Example 4.1 Let $\mathbb{X}$ be a set $\mathbb{X} = \{a, b, c, d\}, \mathbb{IT} = \{\phi, \mathbb{X}, \{a\}, \{b\}, \{a, c\}\}$ Then

(i) $\{a\}$ is $\mathcal{IOS}$ and $\mathcal{ISOS}$ but not $\mathcal{I}.\delta\mathcal{OS}$ and $\mathcal{I}.\delta\mathcal{SOS}$

(ii) $\{a, b\}$ is $\mathcal{I}.\delta\mathcal{POS}$ but not $\mathcal{IOS}$

(iii) $\{c\}$ is $\mathcal{I}.\delta\mathcal{POS}$ and $\mathcal{I}.e\mathcal{OS}$ but not $\mathcal{IPOS}$ and $\mathcal{I}.\delta\mathcal{SOS}$

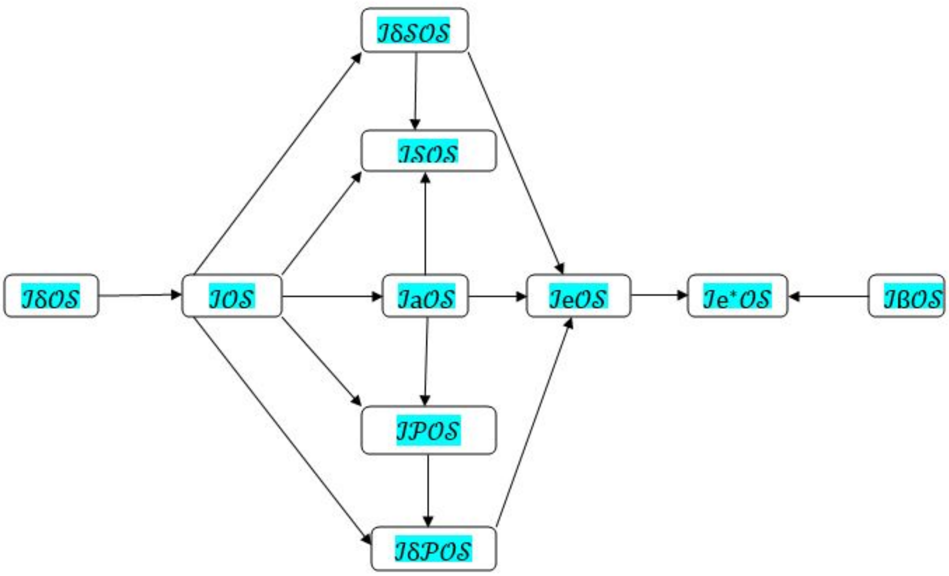
(iv) $\{b, d\}$ is $\mathcal{I}.e\mathcal{OS}$ but not $\mathcal{I}.\delta\mathcal{POS}$

(v) $\{a, b, c\}$ is $\mathcal{I}.e\mathcal{OS}$ but not $\mathcal{IOS}$

(vi) $\{c, d\}$ is $\mathcal{I}.e^*\mathcal{OS}$ but not $\mathcal{I}.e\mathcal{OS}$

(vii) $\{b, c, d\}$ is $\mathcal{I}.e^*\mathcal{OS}$ but not $\mathcal{I}.\beta\mathcal{OS}$

Example 4.2 Let $\mathbb{X}$ be a set $\mathbb{X} = \{a, b, c, d\}, \mathbb{IT} = \{\phi, \mathbb{X}, \{b\}, \{c\}.\{b, c, d\}\}$ Then $\{b, d\}$ is $\mathbb{I}.\delta\mathbb{SOS}$ but not $\mathbb{IOS}$



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